A GENERALIZED KONTSEVICH-VISHIK TRACE FOR FOURIER INTEGRAL OPERATORS AND THE LAURENT EXPANSION OF ζ-FUNCTIONS

TOBIAS HARTUNG AND SIMON SCOTT

ABSTRACT. Based on Guillemin's work on gauged Lagrangian distributions, we will introduce the notion of a poly-log-homogeneous distribution as an approach to ζ-functions for a class of Fourier Integral Operators which includes cases of amplitudes with asymptotic expansion $\sum_{k\in\mathbb{N}} a_{m_k}$ where each a_{m_k} is log-homogeneous with degree of homogeneity m_k but violating $\Re(m_k) \to -\infty$. We will calculate the Laurent expansion for the ζ-function and give formulae for the coefficients in terms of the phase function and amplitude as well as investigate generalizations to the Kontsevich-Vishik quasi-trace. Using stationary phase approximation, series representations for the Laurent coefficients and values of ζ-functions will be stated explicitly. Additionally, we will introduce an approximation method (mollification) for ζ-functions of Fourier Integral Operators whose symbols have singularities at zero by ζ-functions of Fourier Integral Operators with regular symbols.

Contents

Introduction

In [4], Guillemin showed the existence of ζ-functions of gauged Lagrangian distributions, investigated their residues, and used the residues to study the commutator structure of certain algebras of Fourier Integral Operators. Having extended the residue trace (cf. [20]) to Fourier Integral Operators many special cases have been studied; e.g. the class of Toeplitz operators (cf. [2]), wave traces (cf. e.g. [6, 21]), and operators with log-terms (cf. e.g. [14]). However, many questions about ζ-functions are still to be answered. For instance, is there a natural extension of the Kontsevich-Vishik (quasi-)trace (cf. [12])? Other questions may revolve around ζ-determinants or other traces induced by the ζ-function.

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For such questions, knowing the Laurent expansion would be very helpful. Furthermore, it would be quite interesting to know in itself how the Laurent expansion of ζ-functions of Fourier Integral Operators relates to the special case of pseudo-differential operators (cf. [16]). Hence, taking derivatives, i.e. being able to handle log-terms, will be crucial. We will, therefore, assume a generalized approach and define the notion of a gauged poly-log-homogeneous distribution which is based on Guillemin's approach in [4]. It is interesting to note that all the cases above are covered and some other cases (including some relaxations which might be advantageous in explicit calculations) can be considered, as well.

We will, however, start with the rather restrictive notion of gauged poly-log-homogeneous distributions which only contain holomorphic families A such that the degrees of homogeneity d in the expansion are of the form

$$\forall z \in \mathbb{C}: \ d(z) = d(0) + z.$$

As it turns out, this will be sufficient as the most general families we will consider (these are holomorphic families A in an open, connected subset of $\mathbb{C}$ where the degrees of homogeneity are non-constant holomorphic functions) are germ equivalent to this special form and, hence, all local properties are shared, that is, in particular, the Laurent expansion.

In sections 1-3 we will calculate those Laurent expansions, extend them to more general poly-log-homogeneous distributions, and apply them to Fourier Integral Operators whose amplitudes have no singularities. This will yield the following Laurent expansion (in a neighborhood of zero).

THEOREM *Let $(A(z))_{z\in\mathbb{C}}$ be a family of Fourier Integral Operators with phase function ϑ and amplitudes $a(z)(x,y,\xi) = a_0(z)(x,y,\xi) + \sum_{\iota\in I} a_\iota(z)(x,y,\xi)$ holomorphic in z such that each $a(z) \in C^\infty\left(X \times X \times \mathbb{R}^N\right)$, the $a_0(z)(x,x,\xi)$ are integrable in a neighborhood of $\{z \in \mathbb{C};\ \Re(z) \le 0\}$, and the $a_\iota(z)$ are homogeneous in ξ with degree of homogeneity $d_\iota + z$, $I \subseteq \mathbb{N}$, $I_0 := \{\iota \in I;\ d_\iota = -N\}$, and $\Delta(X)$ is the diagonal in X^2, i.e. $\Delta(X) = \{(x,y) \in X^2;\ x = y\}$.*

Then, there exists $c \in \mathbb{R}$ such that $\operatorname{tr} A(z)$ is well-defined for $\Re(z) < c$ and $z \mapsto \operatorname{tr} A(z)$ has a meromorphic extension $\zeta(A)$ to $\mathbb{C}$. Furthermore, $\zeta(A)$ has the Laurent series (locally)

$$\begin{aligned}
\zeta(A)(z) = {} & \sum_{\iota\in I_0} \frac{-\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{z} \\
& + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I_0} \frac{-\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n+1} a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(n+1)!} z^n \\
& + \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!} z^n \\
& + \sum_{n\in\mathbb{N}_0} \frac{\int_{\Delta(X)\times(\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta} \partial^n a_0(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N})}}{n!} z^n \\
& + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I\setminus I_0} \sum_{j=0}^{n} \frac{(-1)^{j+1} j! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n-j} a_\iota(0)\ \mathrm{dvol}}{n!(N+d_\iota)^{j+1}} z^n
\end{aligned} \tag{1}$$

in a neighborhood of zero. In particular, we obtain the residue trace

$$-\sum_{\iota\in I_0}\int_{X\times\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}a_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi) \tag{2}$$

and the residue density[1]

$$-\sum_{\iota\in I_0}\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}a_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ d\mathrm{vol}_X(x),$$

as well as the generalized Kontsevich-Vishik density

$$\begin{aligned}&\int_{\mathbb{R}^N} e^{i\vartheta(x,x,\xi)}a_0(0)(x,x,\xi)d\xi\ d\mathrm{vol}_X(x)\\ &+\sum_{\iota\in I\setminus I_0}\int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)}a_\iota(0)(x,x,\xi)d\xi\ d\mathrm{vol}_X(x)\\ &+\sum_{\iota\in I\setminus I_0}\frac{-\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}a_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)}{N+d_\iota}\ d\mathrm{vol}_X(x)\end{aligned} \tag{3}$$

or, in short,

$$\mathrm{tr}_{\mathrm{KV}}(A(0)) = \zeta(\mathfrak{fp}_0 A)(0) \tag{4}$$

where $\mathfrak{fp}_0 A$ has the amplitude $a-\sum_{\iota\in I_0} a_\iota$, i.e. we split off those a_ι that have critical degree of homogeneity.

In particular, the generalized local Kontsevich-Vishik density at zero is given by the evaluation of "ζ minus pole at zero minus 'residue' of the derivative of the pole inducing term of the expansion" at zero.

Note that these formulae are local representations. We will have a closer look at the Kontsevich-Vishik generalization, as well as global properties, in section 6.

Using the Laurent expansion, we can reproduce many well-known facts about ζ-functions of pseudo-differential operators and Fourier Integral Operators like (2.21) in [12], (9) in [15], (0.12), (0.14), (0.17), (0.18), and (2.20) in [16], as well as

$$\mathrm{tr}\, e^{-t|\Delta|} = \frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi t)^{\frac{N}{2}}}\sum_{\gamma\in\Gamma}\exp\left(-\frac{\|\gamma\|^2_{\ell_2(N)}}{4t}\right)$$

on the flat torus $\mathbb{R}^N/\Gamma$ where Δ is the Dirichlet Laplacian, and

$$\zeta\left(s\mapsto\sqrt{|\Delta|}^{s+\alpha}\right)(z) = 2\zeta_R(-z-\alpha),$$

or

$$\zeta\left(s\mapsto\left(h+\sqrt{|\Delta|}\right)^{s+\alpha}\right)(z) = 2\zeta_H(-z-\alpha;h) - h^{z+\alpha}$$

on $\mathbb{R}/_{2\pi\mathbb{Z}}$ where ζ_R denotes the Riemann-ζ-function and ζ_H the Riemann-Hurwitz-ζ-function. We will, then, introduce an approximation method, which we call mollification, to extend the results to Fourier Integral Operators with asymptotic expansions which have singularities at zero, that is, justify the calculations for $\zeta\left(s\mapsto\sqrt{|\Delta|}^{s+\alpha}\right)$.

[1] cf. section 4.8.4.2 in [17] for pseudo-differential analogue

Furthermore, we will have a closer look at the coefficients. For polyhomogeneous amplitudes, we will obtain the residue trace (as Guillemin has shown to exist). For poly-log-homogeneous amplitudes we will find a generalization of the Kontsevich-Vishik (quasi-)trace and we can generalize Lesch's main statements about the residue trace and the Kontsevich-Vishik (quasi-)trace for pseudo-differential operators in [14] to Fourier Integral Operators. We will show that both (the residue trace and the generalized Kontsevich-Vishik (quasi-)trace) induce globally well-defined densities on the underlying manifold (provided that we started with globally defined kernels). We will see that the Laurent coefficients vanish if and only if the corresponding term $e^{i\vartheta}a$ in the Schwartz kernel is a divergence on $X\times\partial B_{\mathbb{R}^N}$

Finally, we will use stationary phase approximation to treat the integrals

$$I(x,y,r)=\int_{\partial B_{\mathbb{R}^N}}e^{ir\vartheta(x,y,\eta)}a(x,y,\eta)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)$$

which appear as coefficients in the Laurent expansion for $r=1$. The stationary phase approximation also allows us to calculate the kernel singularity structure of certain Fourier Integral Operators by integrating $I(x,y,r)$ over $r\in\mathbb{R}_{>0}$. This yields many "exotic" algebras of Fourier Integral Operators which happen to be subsets of the Hilbert-Schmidt operators and ζ-functions in such algebras have no poles. The kernel singularity structure also allows us to produce analogues of Boutet de Monvel's result that the residue trace is the trace of the logarithmic coefficient for a certain class of Fourier Integral Operators (equations (3) and (4) in [1]).

Additionally to Boutet de Monvel's result, we can also calculate the Kontsevich-Vishik (quasi-)trace. In the case of [1] (one dimensional Fourier integrals on the half-line bundle with phase function satisfying $\vartheta(x,x,r)=0$), we will see that the generalized Kontsevich-Vishik trace reduces to the pseudo-differential form. More precisely, let A have the amplitude $a\sim\sum_{j\in\mathbb{N}_0}a_{d-j}$, each a_{d-j} homogeneous of degree $d-j$, $d\in\mathbb{C}\setminus\mathbb{Z}_{\geq-1}$, and $N\in\mathbb{N}_{0,>\Re(d)+1}$. Then,

$$\operatorname{tr}_{KV}A=\int_X\int_{\mathbb{R}_{>0}}a(x,x,r)-\sum_{j=0}^{N}a_{d-j}(x,x,r)\ dr\ d\mathrm{vol}_X(x)$$

independent of N. In general, this cannot be expected. However, there are some cases in which we can prove such a statement.

Theorem *Let A be a Fourier Integral Operator with kernel*

$$k(x,y)=\int_{\mathbb{R}^N}e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi$$

whose phase function ϑ satisfies $\forall x\in X\ \forall\xi\in\mathbb{R}^N:\ \vartheta(x,x,\xi)=0$, and whose amplitude has an asymptotic expansion $a\sim\sum_{\iota\in\mathbb{N}}a_\iota$ where each a_ι is log*-homogeneous with degree of homogeneity d_ι and logarithmic order l_ι, and $\Re(d_\iota)\to-\infty$. Let $N_0\in\mathbb{N}$ such that $\forall\iota\in\mathbb{N}_{>N_0}:\ \Re(d_\iota)<-N$ and let*

$$k^{\mathrm{sing}}(x,y)=\int_{\mathbb{R}^N}e^{i\vartheta(x,y,\xi)}\sum_{\iota=1}^{N_0}a_\iota(x,y,\xi)d\xi$$

denote the singular part of the kernel.

Then, the regularized kernel $k^{\mathrm{reg}}:=k-k^{\mathrm{sing}}$ is continuous along the diagonal and independent of the particular choice of N_0 (along the diagonal). Furthermore, the generalized Kontsevich-Vishik density is given by

$$k^{\mathrm{reg}}(x,x)d\mathrm{vol}_X(x)=\int_{\mathbb{R}^N}a(x,x,\xi)-\sum_{\iota=1}^{N_0}a_\iota(x,x,\xi)d\xi d\mathrm{vol}_X(x).$$

Note that $k^{\mathrm{reg}}(x,x)d\mathrm{vol}_X(x)$ need not patch together globally. This is only the case if we explicitly start with kernels that patch together and if there are no singular terms of critical order ($d_\iota = -N$) or at least all those contributions to the constant Laurent coefficient regularize to zero; i.e. "the generalized Kontsevich-Vishik density is globally defined if and only if the generalized Kontsevich-Vishik (quasi-)trace is tracial and independent of gauge."

Furthermore, reduction to the pseudo-differential form is highly non-trivial and false in general. Consider, for instance,

$$\int_X \int_{\mathbb{R}} e^{i\Theta(x,x)r} r^{-n} dr d\mathrm{vol}_X(x) = \int_X \frac{-i\pi(-2\pi i\Theta(x,x))^{n-1}\operatorname{sgn}(\Theta(x,x))}{(n-1)!} d\mathrm{vol}_X(x).$$

If $\Theta(x,x) = 1$ and $n = 4$, then this term reduces to $\frac{4\pi^4 \mathrm{vol}(X)}{3}$. In other words, such a term would violate independence of N.

Regarding the Laurent coefficients $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$, we can use a partition of unity such that $a^0(x,y,\cdot)$ has no stationary point in its support and each $a^s(x,y,\cdot)$ has exactly one stationary point $\hat\xi^s(x,y) \in \partial B_{\mathbb{R}^N}$ of $\vartheta(x,y,\cdot)$ in the support of $a^s(x,y,\cdot)$. Then,

$$\begin{aligned}
&\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
=&\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a^0(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&+ \sum_{s=1}^{S} e^{i\hat\vartheta^s(x,y)} \sum_{j\in\mathbb{N}_0} \frac{(2\pi)^{\frac{N-1}{2}} \left|\det \Theta^s(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta^s(x,y)}}{j!(2i)^j} \Delta^j_{\partial B,\Theta^s} a^s\left(x,y,\hat\xi^s(x,y)\right)
\end{aligned}$$

holds with $\hat\vartheta^s(x,y) = \vartheta\left(x,y,\hat\xi^s(x,y)\right)$, $\Theta^s(x,y) = \partial^2_{\partial B}\vartheta\left(x,y,\hat\xi^s(x,y)\right)$, $\operatorname{sgn}\Theta^s(x,y)$ the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta^s(x,y)$, and $\Delta_{\partial B,\Theta^s(x,y)} = \left\langle \Theta^s(x,y)^{-1}\partial_{\partial B}, \partial_{\partial B}\right\rangle = -\operatorname{div}_{\partial B_{\mathbb{R}^N}} \Theta^s(x,y)^{-1} \operatorname{grad}_{\partial B_{\mathbb{R}^N}}$.

The full kernel singularity structure will include another summation over $\iota \in I$. Let $\tilde I := I \cup \{0\}$ and

$$h^s_{j,\iota}(x,y) := \frac{(2\pi)^{\frac{N-1}{2}} \left|\det \Theta^s(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta^s(x,y)}}{j!(2i)^j} \Delta^j_{\partial B,\Theta^s} a^s_\iota\left(x,y,\hat\xi^s(x,y)\right).$$

Then, we will show

$$\int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi = \sum_{\iota\in\tilde I} \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a^0_\iota(x,y,\xi) d\xi + \sum_{s=1}^{S} \sum_{j\in\mathbb{N}_0} h^s_{j,\iota}(x,y) g^s_{j,\iota}(x,y)$$

with

$$g^s_{j,\iota}(x,y) := \partial^{l_\iota}\left(z \mapsto \Gamma(q+1+z)\, i^{q+1+z} \left(\hat\vartheta^s(x,y) + i0\right)^{-q-1-z}\right)(0)$$

for $q = d_\iota + \frac{N+1}{2} - j \in \mathbb{C} \setminus (-\mathbb{N}_0)$ and

$$g^s_{j,\iota}(x,y) := \partial^{l_\iota}\left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i\ (-q)!} \int_{c+i\mathbb{R}} \frac{(-\sigma)^{-q}\left(c_{\ln} + \ln\sigma\right)}{\left(-i\hat\vartheta^s(x,y) + 0 - \sigma\right)^{z+1}} d\sigma\right)(0)$$

for $q = d_\iota + \frac{N+1}{2} - j \in -\mathbb{N}_0$, $c \in \mathbb{R}_{>0}$, and some constant $c_{\ln} \in \mathbb{C}$.

Using these results, we will prove the following two results.

Theorem *Let A be a Fourier Integral Operator with phase function ϑ satisfying $\partial_3^2\left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi)\in GL\left(\mathbb{R}^{N-1}\right)$ whenever $\partial_3\vartheta(x,y,\xi)=0$ (in a neighborhood of the diagonal) and $\hat\xi^s$ $\left(s\in\mathbb{N}_{\le S}\right)$ the stationary points. Furthermore, let*

$$\forall x\in X\ \ \forall s\in\mathbb{N}_{\le S}:\ \ \vartheta\left(x,x,\hat\xi^s(x,x)\right)\ne 0.$$

Then,

$$\left(\ X\ni x\mapsto k(x,x)\in\mathbb{C}\ \right)\in C(X)$$

and

$$\operatorname{tr}A=\int_X k(x,x)d\mathrm{vol}_X(x)$$

is well-defined, i.e. A is a Hilbert-Schmidt operator. Furthermore, ζ-functions of such operators have no poles (since the trace integral always exists).

Corollary *Let Γ be a co-compact discrete group on a manifold X acting continuously*[2] *and freely*[3] *on $X/_\Gamma$, $\tilde k$ a $\Gamma\times\Gamma$-invariant*[4] *Schwartz kernel on X, and $k=\sum_{\gamma\in\Gamma}k(x,\gamma y)$ its projection to $X/_\Gamma$. Suppose $\tilde k$ is pseudo-differential, i.e. $\tilde k(x,y)=\int_{\mathbb{R}^N}e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}}a(x,y,\xi)d\xi$ and*

$$k(x,y)=\sum_{\gamma\in\Gamma}\underbrace{\int_{\mathbb{R}^N}e^{i\langle x-\gamma y,\xi\rangle_{\mathbb{R}^N}}a(x,\gamma y,\xi)d\xi}_{=:k_\gamma(x,y)}.$$

Then, k_{id} is the kernel of a pseudo-differential operator and the k_γ, for $\gamma\in\Gamma\setminus\{\mathrm{id}\}$, are continuous.

Before diving into the calculation of the Laurent expansion, let us have a look at some examples highlighting some of the technical tweaks poly-log-homogeneous distributions allow us to use directly due to their generalized form. In particular, we would like to point out that we can replace the sphere $\partial B_{\mathbb{R}^N}$ by a family of manifolds M_x which might be advantageous if we want to calculate residue traces, for instance, since (at least in some cases) we are reduced to the pseudo-differential case (cf. concluding remark of the introduction).

Example Let us consider a quotient manifold $X=Y/_\Gamma$ where Γ is a co-compact, torsion-free, discrete lattice in the isometries of Y and the Laplacian on Y has the symbol $\left\|G^{-\frac12}(x)\xi\right\|_{\ell_2(N)}^2$ (for the sake of simplicity). Then, the ζ-regularized wave trace of the Laplacian is given by

$$\sum_{\gamma\in\Gamma}(2\pi)^{-N}\int_X\int_{\mathbb{R}^N}e^{it\left\|G^{-\frac12}(x)\xi\right\|_{\ell_2(N)}}e^{i\langle x-\gamma x,\xi\rangle_{\mathbb{R}^N}}d\xi dx.$$

For $x\in X$, let $M_x:=\left\{\left\|G^{-\frac12}(x)\xi\right\|_{\ell_2(N)}^{-1}\xi\in\mathbb{R}^N;\ \xi\in\partial B_{\mathbb{R}^N}\right\}$. Then,

$$\begin{aligned}&(2\pi)^{-N}\int_X\int_{\mathbb{R}^N}e^{it\left\|G^{-\frac12}(x)\xi\right\|_{\ell_2(N)}}e^{i\langle x-\gamma x,\xi\rangle_{\mathbb{R}^N}}d\xi dx\\ &=\int_X\int_{\mathbb{R}_{>0}}\int_{rM_x}\frac{e^{it\left\|G^{-\frac12}(x)\tilde\mu\right\|_{\ell_2(N)}+i\langle x-\gamma x,\tilde\mu\rangle_{\mathbb{R}^N}}\left\|G^{-\frac12}(x)\tilde\mu\right\|_{\ell_2(N)}}{(2\pi)^N\left\|G^{-1}(x)\tilde\mu\right\|_{\ell_2(N)}}d\mathrm{vol}_{rM_x}(\tilde\mu)drdx\end{aligned}$$

---

[2] $\Gamma\times X/_\Gamma\ni(\gamma,x)\mapsto\gamma x\in X/_\Gamma$ is continuous

[3] $\forall\gamma\in\Gamma:\ (\exists x\in X/_\Gamma:\gamma x=x)\ \Rightarrow\ \gamma=\mathrm{id}$

[4] $\forall\gamma\in\Gamma\ \forall x,y\in X:\ \tilde k(x,y)=\tilde k(\gamma x,\gamma y)$

$$
\begin{aligned}
&=(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}}\int_{M_x} e^{ir\left(t+\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}\right)}\frac{\left\|G^{-\frac{1}{2}}(x)\mu\right\|_{\ell_2(N)}}{\|G^{-1}(x)\mu\|_{\ell_2(N)}}r^{N-1}\mathrm{dvol}_{M_x}(\mu)drdx\\
&=(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_{M_x} e^{ir\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}^{-1}\mathrm{dvol}_{M_x}(\mu)drdx
\end{aligned}
$$

follows from Fubini's theorem[5] applied to $f(\xi)=\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}$ on $\mathbb{R}^N\smallsetminus\{0\}$, i.e. $\mathrm{grad}\,f(\xi)=\frac{G^{-1}(x)\xi}{\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}$. Note that the integrals $\int_X\int_{M_x}$ are integrals over the sphere bundle of X with inverted metric. However, they can be written as integrals over the sphere bundle and as integrals over $X\times\partial B_{\mathbb{R}^N}$.

In particular, if $g^{ij}(x)=\mathfrak{g}(x)^2\delta^{ij}$ for some $\mathfrak{g}>0$ (e.g. on a hyperbolic manifold), then $d\mathrm{vol}_X(x)=\mathfrak{g}(x)^{-N}dx$, $\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}=\mathfrak{g}(x)\left\|G^{-\frac{1}{2}}(x)\mu\right\|_{\ell_2(N)}=\mathfrak{g}(x)$, and $M_x=\mathfrak{g}(x)^{-1}\partial B_{\mathbb{R}^N}$ imply

$$
\begin{aligned}
&(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_{M_x} e^{ir\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}^{-1}\mathrm{dvol}_{M_x}(\mu)drdx\\
=&(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_{M_x} e^{ir\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}}\mathfrak{g}(x)^{-1}\mathrm{dvol}_{M_x}(\mu)drdx\\
=&(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\mathfrak{g}(x)^{-1}\int_{\mathfrak{g}(x)^{-1}\partial B_{\mathbb{R}^N}} e^{ir\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}}\mathrm{dvol}_{\mathfrak{g}(x)^{-1}\partial B_{\mathbb{R}^N}}(\mu)drdx\\
=&(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\mathfrak{g}(x)^{-1}\int_{\partial B_{\mathbb{R}^N}} e^{ir\mathfrak{g}(x)^{-1}\langle x-\gamma x,\eta\rangle_{\mathbb{R}^N}}\mathfrak{g}(x)^{1-N}\mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta)drdx\\
=&(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_{\partial B_{\mathbb{R}^N}} e^{ir\mathfrak{g}(x)^{-1}\langle x-\gamma x,\eta\rangle_{\mathbb{R}^N}}\mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta)dr\mathrm{dvol}_X(x).
\end{aligned}
$$

For $\gamma=\mathrm{id}$, this reduces to

$$
\frac{(N-1)!\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})\mathrm{vol}_X(X)}{(-2\pi it)^N}.
$$

For $\gamma\neq\mathrm{id}$, we can use stationary phase approximation and obtain

$$
\begin{aligned}
&(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_{\partial B_{\mathbb{R}^N}} e^{ir\mathfrak{g}(x)^{-1}\langle x-\gamma x,\eta\rangle_{\mathbb{R}^N}}\mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta)dr\mathrm{dvol}_X(x)\\
=&\sum_{\pm}\int_X\int_{\mathbb{R}_{>0}}\frac{e^{irt}r^{N-1}\left(\frac{\pi}{2\|x-\gamma x\|_{\ell_2(N)}}\right)^{\frac{N-1}{2}}e^{\frac{i\pi}{4}(1-N)}}{(2\pi)^N}e^{\pm ir\mathfrak{g}(x)^{-1}\|x-\gamma x\|_{\ell_2(N)}}dr\mathrm{dvol}_X(x)\\
=&\sum_{\pm}\int_X\frac{\left(\frac{\pi}{2\|x-\gamma x\|_{\ell_2(N)}}\right)^{\frac{N-1}{2}}e^{\frac{i\pi}{4}(1-N)}}{(2\pi)^N}\int_{\mathbb{R}_{>0}} e^{irt\pm ir\mathfrak{g}(x)^{-1}\|x-\gamma x\|_{\ell_2(N)}}r^{N-1}dr\mathrm{dvol}_X(x)
\end{aligned}
$$

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Theorem (Fubini) *Let $\Omega\subseteq\mathbb{R}^n$ be open, $\varphi\in C_c(\Omega)$, $f\in C^1(\Omega,\mathbb{R})$, $\forall x\in\Omega:$ $\mathrm{grad}\,f(x)\neq0$, and $M_r:=[\{r\}]f=\{x\in\Omega;\ f(x)=r\}$. Then,*

$$
\int_\Omega\varphi(x)dx=\int_{\mathbb{R}}\int_{M_r}\varphi(\xi)\left\|\mathrm{grad}\,f(\xi)\right\|_{\ell_2(n)}^{-1}\mathrm{dvol}_{M_r}(\xi)dr.
$$

$$= \sum_{\pm} \int_X \frac{\left(\frac{\pi}{2\|x-\gamma x\|_{\ell_2(N)}}\right)^{\frac{N-1}{2}} e^{\frac{i\pi}{4}(1-N)}}{(2\pi)^N} \frac{(N-1)!}{\left(-i\left(t \pm \mathfrak{g}(x)^{-1} \|x-\gamma x\|_{\ell_2(N)}\right)\right)^N} \mathrm{dvol}_X(x)$$

$$= \sum_{\pm} \frac{(N-1)! \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{\frac{i\pi}{4}(1-N)}}{(-2\pi i)^N} \int_X \frac{\|x-\gamma x\|_{\ell_2(N)}^{\frac{1-N}{2}}}{\left(t \pm \mathfrak{g}(x)^{-1} \|x-\gamma x\|_{\ell_2(N)}\right)^N} \mathrm{dvol}_X(x).$$

■

Remark Replacing $\partial B_{\mathbb{R}^N}$ by M_x becomes even more interesting if we want to calculate the Laurent coefficients

$$\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} \partial^{n+l_\iota+1} \tilde{a}_\iota(0)(x,x,\xi)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}(x,\xi)$$

which are now integrals

$$\int_X \int_{M_x} e^{i\vartheta(x,x,\xi)} \partial^{n+l_\iota+1} \tilde{a}_\iota(0)(x,x,\xi)\ \mathrm{dvol}_{M_x}(\xi) \mathrm{dvol}_X(x).$$

In cases such as the example above, the integration over M_x is now without a phase function because $M_x \ni \xi \mapsto \vartheta(x,x,\xi)$ is a constant ϑ_x, leaving us with integrals of the form

$$e^{i\vartheta_x} \int_{M_x} a_x(\xi)\ \mathrm{dvol}_{M_x}(\xi)$$

where a_x is homogeneous of some degree d. For $M_x = T_x[\partial B_{\mathbb{R}^n}]$ with $T_x \in GL(\mathbb{R}^n)$, this is equivalent to

$$e^{i\vartheta_x} \int_{M_x} a_x(\xi)\ \mathrm{dvol}_{M_x}(\xi) = e^{i\vartheta_x} \int_{\partial B_{\mathbb{R}^n}} a_x(\xi) \left\|T_x^{-1}\xi\right\|^{-n-d} \mathrm{dvol}_{\partial B_{\mathbb{R}^n}}(\xi).$$

In particular, for the case of the residue trace, we have $d = -n$, i.e.

$$e^{i\vartheta_x} \int_{M_x} a_x(\xi)\ \mathrm{dvol}_{M_x}(\xi) = e^{i\vartheta_x} \int_{\partial B_{\mathbb{R}^n}} a_x(\xi) \mathrm{dvol}_{\partial B_{\mathbb{R}^n}}(\xi),$$

which shows that we have reduced the pointwise residue of the Fourier Integral Operator to the pointwise residue of a suitably chosen pseudo-differential operator and a rotation in the complex plane ϑ_x. In fact, the symbol of that pseudo-differential operator can be chosen to be the amplitude of the Fourier Integral Operator itself.

■

1. Gauged poly-log-homogeneous distributions

In this section, we consider distributions of the form

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z)(\xi) \mathrm{dvol}_{\mathbb{R}_{\geq 1}\times M}(\xi)$$

where M is an orientable,[6] compact, finite dimensional manifold without boundary and α is a holomorphic family given by an expansion[7]

$$\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$$

where $I \subseteq \mathbb{N}$, $\alpha_0(z) \in L_1(\mathbb{R}_{\geq 1} \times M)$ in an open neighborhood of $\{z \in \mathbb{C};\ \Re(z) \leq 0\}$ and each of the $\alpha_\iota(z)$ is log-homogeneous with degree of homogeneity $d_\iota + z \in \mathbb{C}$ and logarithmic order $l_\iota \in \mathbb{N}_0$, that is,

$$\exists \tilde{\alpha}_\iota \in \mathbb{C}^M\ \ \forall r \in \mathbb{R}_{\geq 1}\ \ \forall \nu \in M:\ \ \alpha_\iota(z)(r,\nu) = r^{d_\iota + z}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\nu).$$

We will furthermore assume the following.

- The family $(\Re(d_\iota))_{\iota\in I}$ is bounded from above. (Note, we do not require $\Re(d_\iota) \to -\infty$. $\forall \iota \in I:\ \Re(d_\iota) = 42$ is entirely possible.)
- The map $I \ni \iota \mapsto (d_\iota, l_\iota)$ is injective.
- There are only finitely many ι satisfying $d_\iota = d$ for any given $d \in \mathbb{C}$.
- The family $((d_\iota - \delta)^{-1})_{\iota\in I}$ is in $\ell_2(I)$ for any $\delta \in \mathbb{C} \setminus \{d_\iota;\ \iota \in I\}$.
- Each $\sum_{\iota\in I} \tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$.

Any such family α will be called a gauged poly-log-homogeneous distribution. Note that the generic case (that is, applications to Fourier Integral Operators with amplitudes of the form $a \sim \sum_{j\in\mathbb{N}_0} a_{m-j}$) implies that I is a finite set and all these conditions are, therefore, satisfied.

Example Let $A(z)$ be a pseudo-differential operator on an N-dimensional manifold X whose amplitude has an asymptotic expansion $a(z) \sim \sum_{j\in\mathbb{N}} a_j(z)$ where each $a_j(z)$ is homogeneous of degree $m - j + z$. Then, we may want to evaluate the meromorphic extension of

$$\begin{aligned} \operatorname{tr} A(z) &= \int_X \int_{\mathbb{R}^N} a(z)(x,x,\xi) d\xi d\mathrm{vol}_X(x) \\ &= \int_X \int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} a(z)(x,x,\xi) d\xi d\mathrm{vol}_X(x) \\ &\quad + \int_X \int_{B_{\mathbb{R}^N}(0,1)} a(z)(x,x,\xi) d\xi d\mathrm{vol}_X(x) \end{aligned}$$

at zero. The poly-log-homogeneous distribution here is

$$(*) \qquad \int_X \int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} a(z)(x,x,\xi) d\xi d\mathrm{vol}_X(x).$$

At this point, we have many possibilities to write it $(*)$ in the form

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}(\xi).$$

[6]Replacing $\alpha(z)(r,\xi)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}(r,\xi)$ by some family $d\omega(z)(r,\xi)$ allows us to also treat non-orientable manifolds but we will not need this in the following and choose orientability for the sake of simplicity.

[7]This is not meant to be an asymptotic expansion but an actual identity. However, for a classical symbol a with asymptotic expansion $\sum_{j\in\mathbb{N}} a_j$ where a_j is homogeneous of degree $m - j$ for some $m \in \mathbb{C}$, it is possible to choose a finite set $I = \{0, 1, \ldots, J\}$ and α_0 will correspond to $a - \sum_{j=0}^J a_{m-j}$.

This is completely analogous to the Kontsevich-Vishik trace, i.e. splitting off finitely many terms with large degrees of homogeneity while the rest is integrable. The only difference is that those terms (that have been split off) might not regularize to zero anymore.

The easiest choice is $M := \partial B_{\mathbb{R}^N}$ and $I := \{j \in \mathbb{N};\ \Re(m) - j \geq -N\}$. This ensures that

$$\int_X a(z)(x,x,\xi) - \sum_{j\in I} a_j(z)(x,x,\xi) d\mathrm{vol}_X(x)$$

is integrable in $\mathbb{R}_{\geq 1} \times \partial B_{\mathbb{R}^N}$. Furthermore, having a finite I ensures that all of the conditions above are satisfied and α can be defined by

$$\alpha_0(z)(r,\nu) := \int_X a(z)(x,x,r\nu) - \sum_{j\in I} a_j(z)(x,x,r\nu) d\mathrm{vol}_X(x)$$

and

$$\alpha_j(z)(r,\nu) := \int_X a_j(z)(x,x,r\nu) d\mathrm{vol}_X(x) = r^{m-j+z} \underbrace{\int_X a_j(z)(x,x,\nu) d\mathrm{vol}_X(x)}_{=:\tilde{\alpha}_j(z)(\nu)}$$

for $j \in I$.

■

Remark Note that these distributions are strongly connected to traces of Fourier Integral Operators, as well. In fact, Guillemin's argument in [4] relies heavily on the fact that the inner products $\langle u(z), f\rangle$ at question are integrals of the form

$$\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)$$

where α is a gauged polyhomogeneous distribution; cf. equation (2.15) in [4].

■

If the conditions above are satisfied, we obtain formally

$$\begin{aligned}
\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} =& \underbrace{\int_{\mathbb{R}_{\geq 1}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} \\
=&\tau_0(z) + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}} \int_M \alpha_\iota(z)(\varrho,\nu) \varrho^{\dim M} d\mathrm{vol}_M(\nu) d\varrho \\
=&\tau_0(z) + \sum_{\iota\in I} \underbrace{\int_{\mathbb{R}_{\geq 1}} \varrho^{\dim M + d_\iota + z} (\ln\varrho)^{l_\iota} d\varrho}_{=:c_\iota(z)} \underbrace{\int_M \tilde{\alpha}_\iota(z) d\mathrm{vol}_M}_{=:\operatorname{res}\alpha_\iota(z)\in\mathbb{C}} \\
=&\tau_0(z) + \sum_{\iota\in I} c_\iota(z) \operatorname{res}\alpha_\iota(z)
\end{aligned}$$

which now needs to be justified.

Lemma 1.1. $c_\iota(z) = (-1)^{l_\iota+1} l_\iota! \left(\dim M + d_\iota + z + 1\right)^{-(l_\iota+1)}$

Proof. Let Γ_{ui} be the upper incomplete Γ-function given by the meromorphic extension of

$$\Gamma_{ui}(s,x) := \int_x^\infty t^{s-1} e^{-t} dt \qquad (\Re(s) > 0,\ x \in \mathbb{R}_{\geq 0}).$$

Γ_{ui} satisfies $\Gamma_{ui}(s,0) = \Gamma(s)$ where Γ denotes the (usual) Γ-function, $\Gamma(s,\infty) = 0$, and $\partial_2\Gamma_{ui}(s,x) = -x^{s-1}e^{-x}$. Then, we obtain

$$\left(\mathbb{R}_{>0} \ni y \mapsto \frac{-\Gamma_{ui}(l+1, -(d+1)\ln y)}{(-(d+1))^{l+1}}\right)'(x) = \frac{-\partial_2\Gamma_{ui}(l+1,-(d+1)\ln x)\frac{-(d+1)}{x}}{(-(d+1))^{l+1}}$$

$$\begin{aligned}&=\frac{(-(d+1)\ln x)^l e^{(d+1)\ln x}}{(-(d+1))^l x}\\&=\frac{(\ln x)^l x^{d+1}}{x}\\&=x^d(\ln x)^l.\end{aligned}$$

Hence, for $d<-1$,

$$\int_{\mathbb{R}_{\geq 1}} x^d(\ln x)^l dx = \frac{(-1)^{l+1}l!}{(d+1)^{l+1}}$$

which yields

$$c_\iota(z)=\int_{\mathbb{R}_{\geq 1}} \varrho^{\dim M+d_\iota+z}(\ln\varrho)^{l_\iota}\,d\varrho=\frac{(-1)^{l_\iota+1}l_\iota!}{(\dim M+d_\iota+z+1)^{l_\iota+1}}$$

in a neighborhood of $\mathbb{R}_{<-\dim M-d_\iota-1}$ (because any real analytic function can be extended locally to a holomorphic function) and, thence, by meromorphic extension everywhere in $\mathbb{C}\setminus\{-\dim M-d_\iota-z-1\}$.

□

Since the $\operatorname{res}\alpha_\iota$ are holomorphic functions, we now know that $\sum_{\iota\in I}c_\iota\operatorname{res}\alpha_\iota$ is a meromorphic function with isolated poles only (if it converges), because $((d_\iota+\delta)^{-1})_{\iota\in I}\in\ell_2(I)$ implies that there may be at most finitely many d_ι in any compact subset of $\mathbb{C}$.

Lemma 1.2. *For every $z\in\mathbb{C}\setminus\{-\dim M-d_\iota-1;\ \iota\in I\}$, $\sum_{\iota\in I}c_\iota(z)\operatorname{res}\alpha_\iota(z)$ converges absolutely.*

Proof. By assumption, $(c_\iota(z))_{\iota\in I}\in\ell_2(I)$ and $\sum_{\iota\in I}\tilde\alpha_\iota(z)$ converges unconditionally in $L_1(M)$. This allows us to utilize the following theorem.

(Theorem 4.2.1 in [10]) Let $p\in\mathbb{R}_{\geq 1}$, $q=\begin{cases}2 & ,\ p\in[1,2]\\ p & ,\ p\in\mathbb{R}_{>2}\end{cases}$, and $\sum_{j\in\mathbb{N}}x_j$ converges unconditionally in L_p. Then, $\sum_{j\in\mathbb{N}}\|x_j\|_{L_p}^q$ converges.

Hence,

$$\begin{aligned}\sum_{\iota\in I}|c_\iota(z)\operatorname{res}\alpha_\iota(z)|&\leq\sum_{\iota\in I}|c_\iota(z)|\,\|\tilde\alpha_\iota(z)\|_{L_1(M)}\\&=\left\|\left(|c_\iota(z)|\,\|\tilde\alpha_\iota(z)\|_{L_1(M)}\right)_{\iota\in I}\right\|_{\ell_1(I)}\\&=\|(|c_\iota(z)|)_{\iota\in I}\|_{\ell_2(I)}\left\|\left(\|\tilde\alpha_\iota(z)\|_{L_1(M)}\right)_{\iota\in I}\right\|_{\ell_2(I)}\\&=\|(c_\iota(z))_{\iota\in I}\|_{\ell_2(I)}\sqrt{\sum_{\iota\in I}\|\tilde\alpha_\iota(z)\|_{L_1(M)}^2}<\infty.\end{aligned}$$

□

Definition 1.3. *Let α be a gauged poly-log-homogeneous distribution. Then, we define the ζ-function of α to be the meromorphic extension of*

$$\zeta(\alpha)(z):=\int_{\mathbb{R}_{\geq 1}\times M}\alpha(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M},$$

i.e.

$$\zeta(\alpha)(z)=\tau_0(z)+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{l_\iota+1}}.$$

Now, that we know $\zeta(\alpha)$ exists as a meromorphic function, we will calculate its Laurent expansion.

Definition 1.4. *Let f be a meromorphic function defined by its Laurent expansion $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ at $z_0\in\mathbb{C}$ without essential singularity at z_0, that is, $\exists N\in\mathbb{Z}\ \forall n\in\mathbb{Z}_{\le N}:\ a_n=0$. Then, we define the order of the initial Laurent coefficient $\operatorname{oilc}_{z_0}(f)$ of f at z_0 to be*

$$\operatorname{oilc}_{z_0}(f) := \min\{n\in\mathbb{Z};\ a_n\neq 0\}$$

and the initial Laurent coefficient $\operatorname{ilc}_{z_0}(f)$ of f at z_0

$$\operatorname{ilc}_{z_0}(f) := a_{\operatorname{oilc}_{z_0}(f)}.$$

Lemma 1.5. *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ and $\beta=\beta_0+\sum_{\iota\in I'}\beta_\iota$ be two gauged poly-*log-*homogeneous distributions with $\alpha(0)=\beta(0)$ and $\operatorname{res}\alpha_j(0)\neq 0$ if l_j is the maximal logarithmic order with $d_j=-\dim M-1$. Then, $\operatorname{oilc}_0(\zeta(\alpha))=\operatorname{oilc}_0(\zeta(\beta))$ and $\operatorname{ilc}_0(\zeta(\alpha))=\operatorname{ilc}_0(\zeta(\beta))$.*

In other words, $\operatorname{oilc}_0(\zeta(\alpha))$ and $\operatorname{ilc}_0(\zeta(\alpha))$ depend on $\alpha(0)$ only and are, thus, independent of the gauge.

Proof. Since $\alpha(0)=\beta(0)$, we obtain that $z\mapsto\gamma(z):=\frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution again. Furthermore,

$$\operatorname{oilc}_0(\zeta(\gamma))\ge\min\{\operatorname{oilc}_0(\zeta(\alpha)),\operatorname{oilc}_0(\zeta(\beta))\}=:-l=-l_j-1$$

holds because each pair (d_ι,l_ι) in the expansion of γ appears in at least one of the expansions of α or β. This implies that $z\mapsto z^l\zeta(\gamma)(z)=z^{l-1}\left(\zeta(\alpha)(z)-\zeta(\beta)(z)\right)$ is holomorphic at zero (equality holds for $\Re(z)$ sufficiently small and, thence, in general by meromorphic extension). Hence, the highest order poles of $\zeta(\alpha)$ and $\zeta(\beta)$ at zero must cancel out which directly implies $\operatorname{oilc}_0(\zeta(\alpha))=\operatorname{oilc}_0(\zeta(\beta))$ and $\operatorname{ilc}_0(\zeta(\alpha))=\operatorname{ilc}_0(\zeta(\beta))$.

□

Lemma 1.6. *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ and $\beta=\beta_0+\sum_{\iota\in I'}\beta_\iota$ be two gauged poly-*log-*homogeneous distributions with $\alpha(0)=\beta(0)$ and $\forall\iota\in I\cup I':\ d_\iota\neq-\dim M-1$. Then, $\zeta(\alpha)(0)=\zeta(\beta)(0)$.*

Proof. Again, since $\alpha(0)=\beta(0)$, we obtain that $z\mapsto\gamma(z):=\frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution and $\operatorname{oilc}_0(\zeta(\gamma))\ge 0$. Hence

$$\zeta(\alpha)(0)-\zeta(\beta)(0)=\operatorname{res}_0\left(z\mapsto\frac{\zeta(\alpha)(z)-\zeta(\beta)(z)}{z}\right)=\operatorname{res}_0\zeta(\gamma)=0$$

where res_0 denotes the residue of a meromorphic function at zero.

□

Definition 1.7. *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log-*homogeneous distribution and $I_{z_0}:=\{\iota\in I;\ d_\iota=-\dim M-1-z_0\}$. Then, we define*

$$\mathfrak{fp}_{z_0}(\alpha):=\alpha-\sum_{\iota\in I_{z_0}}\alpha_\iota=\alpha_0+\sum_{\iota\in I\setminus I_{z_0}}\alpha_\iota.$$

Corollary 1.8. *$\zeta(\mathfrak{fp}_0\alpha)(0)$ is independent of the chosen gauge.*

Definition 1.9. *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log-*homogeneous distribution and $\operatorname{res}\alpha_\iota\neq 0$ for some $\iota\in I_0$. Then, we say $\zeta(\alpha)$ has a structural singularity at zero.*

Remark Note that the pole structure of $\zeta(\alpha)$ does not only depend on the $\operatorname{res}\alpha_\iota$ but also on derivatives of α. A structural singularity is a property of $\alpha(0)$ in the sense that it cannot be removed under change of gauge. More precisely, choosing β such that $\alpha(0) = \beta(0)$ does not imply that the principal part of the Laurent expansion of $\zeta(\alpha) - \zeta(\beta)$ vanishes. However, if all $\operatorname{res}\alpha_\iota$ vanish ($\iota \in I_0$), then there exists a β with $\alpha(0) = \beta(0)$ such that $\zeta(\beta)$ is holomorphic in a neighborhood of zero (e.g. β being $\mathcal{M}$-gauged; see below). Having a non-vanishing $\operatorname{res}\alpha_\iota$ for some $\iota \in I_0$, on the other hand, implies that every $\zeta(\beta)$ with $\alpha(0) = \beta(0)$ has a pole at zero.

■

Definition 1.10. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution. If all $\tilde\alpha_\iota$ are independent of the complex argument, i.e. $\alpha_\iota(z)(r,\nu) = r^{d_\iota+z}(\ln r)^{l_\iota}\tilde\alpha_\iota(0)(\nu) = r^z\alpha_\iota(0)(r,\nu)$, then we call this choice of gauge an $\mathcal{M}$-gauge (or Mellin-gauge).*

Remark The $\mathcal{M}$-gauge for Fourier Integral Operators can always be chosen locally.

■

Corollary 1.11. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution.*

(i) *If α is $\mathcal{M}$-gauged, then all $\operatorname{res}\alpha_\iota$ are constants.*
(ii) *If $\operatorname{res}\alpha_\iota(0) = 0$ for $\iota \in I$, then the corresponding pole in $\zeta(\alpha)$ can be removed by re-gauging.*
(iii) *If $\operatorname{res}\alpha_\iota(0) \neq 0$ for $\iota \in I_0$, then the corresponding pole in $\zeta(\alpha)$ in independent from the gauge. In particular, $\operatorname{res}\alpha_\iota(0)$ does not depend on the gauge.*

Proof. (i) trivial.

(ii) The corresponding pole contributes the term $\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{l_\iota+1}}$ to the expansion of $\zeta(\alpha)$. Choosing an $\mathcal{M}$-gauge yields

$$\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{l_\iota+1}} = \frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(0)}{(\dim M+d_\iota+z+1)^{l_\iota+1}} = 0$$

by holomorphic extension.

(iii) Lemma 1.5 shows that $\operatorname{oilc}_0\zeta(\alpha_\iota)$ and $\operatorname{ilc}_0(\zeta(\alpha_\iota))$ are independent of the gauge. Since, $\operatorname{res}\alpha_\iota(0) \neq 0$, we obtain $\operatorname{oilc}_0\zeta(\alpha_\iota) = -l_\iota - 1$ and

$$\operatorname{res}\alpha_\iota(0) = \frac{\operatorname{ilc}_0\zeta(\alpha_\iota)}{(-1)^{l_\iota+1}l_\iota!}.$$

□

Remark Suppose we have a gauged distribution α such that

$$\forall z\in\mathbb{C}\ \forall (r,\xi)\in\mathbb{R}_{\geq 1}\times M:\ \alpha(z)(r,\xi) = r^z\alpha(0)(r,\xi)$$

is satisfied and we artificially continue α by zero to $\mathbb{R}_{>0}\times M$. Then,

$$\begin{aligned}\int_{\mathbb{R}_{>0}\times M}\alpha(z)(r,\xi)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}(r,\xi) &= \int_{\mathbb{R}_{>0}} r^{\dim M+z}\underbrace{\int_M \alpha(0)(r,\xi)d\mathrm{vol}_M(\xi)}_{=:A(r)}dr\\ &= \mathcal{M}(A)(\dim M+z+1)\end{aligned}$$

holds where $\mathcal{M}$ denotes the Mellin transform

$$\mathcal{M}f(z) = \int_{\mathbb{R}_{>0}} t^{z-1}f(t)dt$$

for $f: \mathbb{R}_{>0} \to \mathbb{R}$ measurable whenever the integral exists. Hence, the name "$\mathcal{M}$-gauge".

■

Proposition 1.12 (Laurent expansion of $\zeta(\mathfrak{fp}_0\alpha)$)**.** *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly*-log-*homogeneous distribution with $I_0 = \varnothing$. Then,*

$$\zeta(\alpha)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n\alpha)(0)}{n!} z^n$$

holds in a sufficiently small neighborhood of zero.

Let $\beta = \beta_0 + \sum_{\iota\in I'}\beta_\iota$ be a gauged poly-log-*homogeneous distribution without structural singularities at zero, i.e. $\forall \iota \in I_0' :\ \operatorname{res}\beta_\iota = 0$. Then, there exists a gauge $\hat{\beta}$ such that*

$$\zeta\left(\hat{\beta}\right)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n \mathfrak{fp}_0\beta)(0)}{n!} z^n$$

holds in a sufficiently small neighborhood of zero.

Proof. The first assertion is a direct consequence of the facts that the n^{th} Laurent coefficient of a holomorphic function f is given by $\frac{\partial^n f(0)}{n!}$ and

$$\partial^n\zeta(\alpha) = \partial^n \int_{\mathbb{R}_{\ge 1}\times M} \alpha \ \mathrm{dvol}_{\mathbb{R}_{\ge 1}\times M} = \int_{\mathbb{R}_{\ge 1}\times M} \partial^n\alpha \ \mathrm{dvol}_{\mathbb{R}_{\ge 1}\times M} = \zeta(\partial^n\alpha).$$

Now,

$$\zeta\left(\hat{\beta}\right)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n \mathfrak{fp}_0\beta)(0)}{n!} z^n$$

follows from the fact that we may choose an $\mathcal{M}$-gauge for β_ι with $\iota \in I_0'$ which yields $\zeta\left(\hat{\beta}\right) = \zeta(\mathfrak{fp}_0\beta)$.

□

$\mathcal{M}$-gauging will, furthermore, yield the following theorem which can be very handy with respect to actual computations. In particular, the fact that we can remove the influence of higher order derivatives of α_ι with critical degree of homogeneity will imply that the generalized Kontsevich-Vishik density (which we will define in section 6) is globally defined, i.e. for $\mathcal{M}$-gauged families with polyhomogeneous amplitudes the residue trace density and the generalized Kontsevich-Vishik density both exist globally (provided the kernel patches together).

Theorem 1.13. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly*-log-*homogeneous distribution. Then, there exists a gauge $\hat{\alpha}$ such that*

$$\zeta(\hat{\alpha})(z) = \sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(0)}{z^{l_\iota+1}} + \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n \mathfrak{fp}_0\alpha)(0)}{n!} z^n$$

holds in a sufficiently small neighborhood of zero.

Proof. This follows directly from Proposition 1.12 using an $\mathcal{M}$-gauge for α_ι with $\iota \in I_0$.

□

Remark In general, there will be correction terms arising from the Laurent expansion of $\operatorname{res}\alpha_\iota$. Incorporating these yields

$$\zeta(\alpha)(z) = \sum_{\iota\in I_0} \left(\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(0)}{z^{l_\iota+1}} + \sum_{n=1}^{l_\iota} \frac{(-1)^{l_\iota+1} l_\iota! \partial^n \operatorname{res}\alpha_\iota(0)}{n!} z^{n-l_\iota-1} \right)$$

$$+\sum_{n\in\mathbb{N}_0}\left(\frac{\zeta(\partial^n\mathfrak{fp}_0\alpha)(0)}{n!}+\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\partial^{n+l_\iota+1}\operatorname{res}\alpha_\iota(0)}{(n+l_\iota+1)!}\right)z^n.$$

■

Corollary 1.14. *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ and $\beta=\beta_0+\sum_{\iota\in I}\beta_\iota$ be two gauged poly-*log-*homogeneous distributions with $\alpha(0)=\beta(0)$ and such that the degrees of homogeneity and logarithmic orders of α_ι and β_ι coincide. Then,*

$$\begin{aligned}\zeta(\alpha)(z)-\zeta(\beta)(z)=&\sum_{\iota\in I_0}\sum_{n=1}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\partial^n\operatorname{res}(\alpha_\iota-\beta_\iota)(0)}{n!}z^{n-l_\iota-1}\\&+\sum_{n\in\mathbb{N}_0}\frac{\zeta(\partial^n\mathfrak{fp}_0(\alpha-\beta))(0)}{n!}z^n\\&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\partial^{n+l_\iota+1}\operatorname{res}(\alpha_\iota-\beta_\iota)(0)}{(n+l_\iota+1)!}z^n\end{aligned}$$

holds in a sufficiently small neighborhood of zero.

In section 3, we will see that Corollary 1.14 applied to pseudo-differential operators implies many well-known formulae, e.g. (2.21) in [12], (9) in [15], and (2.20) in [16].

Example Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ and $\beta=\beta_0+\sum_{\iota\in I}\beta_\iota$ be two gauged poly-homogeneous distributions with $\alpha(0)=\beta(0)$ and such that the degrees of homogeneity of α_ι and β_ι coincide. Then, $\#I_0\le 1$ and (because) all l_ι are zero. Hence,

$$\zeta(\alpha)(z)=\sum_{\iota\in I_0}\frac{-\operatorname{res}\alpha_\iota(0)}{z}+\sum_{n\in\mathbb{N}_0}\left(\frac{\zeta(\partial^n\mathfrak{fp}_0\alpha)(0)}{n!}-\sum_{\iota\in I_0}\frac{\partial^{n+1}\operatorname{res}\alpha_\iota(0)}{(n+1)!}\right)z^n$$

and

$$\zeta(\alpha)(z)-\zeta(\beta)(z)=\sum_{n\in\mathbb{N}_0}\left(\frac{\zeta(\partial^n\mathfrak{fp}_0(\alpha-\beta))(0)}{n!}-\sum_{\iota\in I_0}\frac{\partial^{n+1}\operatorname{res}(\alpha_\iota-\beta_\iota)(0)}{(n+1)!}\right)z^n$$

holds in a sufficiently small neighborhood of zero. This shows that the residue trace $-\sum_{\iota\in I_0}\operatorname{res}\alpha_\iota(0)$ is well-defined and independent of the gauge for poly-homogeneous distributions. Higher orders of the Laurent expansion depend on the gauge.

Furthermore, $\zeta(\alpha)-\zeta(\beta)$ is holomorphic in a neighborhood of zero and

$$\begin{aligned}(\zeta(\alpha)-\zeta(\beta))(0)=&\zeta(\mathfrak{fp}_0(\alpha-\beta))(0)-\sum_{\iota\in I_0}\partial\operatorname{res}(\alpha_\iota-\beta_\iota)(0)\\=&\underbrace{\zeta(\mathfrak{fp}_0\alpha)(0)-\zeta(\mathfrak{fp}_0\beta)(0)}_{=0}-\sum_{\iota\in I_0}\partial\operatorname{res}(\alpha_\iota-\beta_\iota)(0)\\=&-\sum_{\iota\in I_0}\partial\operatorname{res}(\alpha_\iota-\beta_\iota)(0).\end{aligned}$$

Defining $\gamma_\iota(z):=\frac{\alpha_\iota(z)-\beta_\iota(z)}{z}$ and $\gamma(z):=\frac{\alpha(z)-\beta(z)}{z}$ we, thus, obtain

$$(\zeta(\alpha)-\zeta(\beta))(0)=-\sum_{\iota\in I_0}\partial\operatorname{res}(\alpha_\iota-\beta_\iota)(0)=-\sum_{\iota\in I_0}\operatorname{res}\gamma_\iota(0)=\operatorname{res}_0\zeta(\gamma).$$

Since $\operatorname{res}\gamma_\iota(0)\neq 0$ implies that it is independent of gauge, we obtain that $\operatorname{res}_0\zeta(\gamma)$ is independent of gauge which directly yields

$$(\zeta(\alpha)-\zeta(\beta))(0)=\operatorname{res}_0\zeta(\gamma)=\operatorname{res}_0\zeta(\partial(\alpha-\beta)).$$

In other words, $(\zeta(\alpha)-\zeta(\beta))(0)$ is a trace residue.

■

Theorem 1.15 (Laurent expansion of $\zeta(\alpha)$)**.** *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution. Then,*

$$\begin{aligned}\zeta(\alpha)(z) = &\sum_{\iota\in I_0}\sum_{n=0}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M \partial^n\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}}\\ &+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\ge1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\ge1}\times M}}{n!}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!(\dim M+d_\iota+1)^{l_\iota+j+1}}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}z^n\end{aligned}$$

holds in a sufficiently small neighborhood of zero.

In particular, if α is poly-homogeneous, we obtain

$$\begin{aligned}\zeta(\alpha)(z) = &\sum_{\iota\in I_0}\frac{-\int_M\alpha_\iota(0)d\mathrm{vol}_M}{z}+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\ge1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\ge1}\times M}}{n!}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{j+1}j!\int_M\partial^{n-j}\alpha_\iota(0)d\mathrm{vol}_M}{n!(\dim M+d_\iota+1)^{j+1}}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{-\int_M\partial^{n+1}\alpha_\iota(0)d\mathrm{vol}_M}{(n+1)!}z^n\end{aligned}$$

in a sufficiently small neighborhood of zero.

Proof. Note that having a gauged log-homogeneous distribution

$$\beta(z)(r,\xi) = r^{d+z}(\ln r)^l\tilde{\beta}(z)(\xi)$$

the residue $\operatorname{res}\beta = \int_M\tilde{\beta}\ d\mathrm{vol}_M$ does not depend on the logarithmic order. Hence, we may assume without loss of generality that $l = 0$ and we had a gauged homogeneous distribution in the first place, i.e. replace β by

$$\hat{\beta}(z)(r,\xi) = r^{d+z}\tilde{\beta}(z)(\xi)$$

Then, we observe

$$\partial^n\beta(z)(r,\xi) = \sum_{j=0}^{n}\binom{n}{j}r^{d+z}(\ln r)^{l+j}\partial^{n-j}\tilde{\beta}(z)(\xi)$$

and

$$\partial^n\tilde{\beta}(z)(\xi) = \partial^n\left(x\mapsto r^{-d-x}\hat{\beta}(x)(\xi)\right)(z) = \sum_{j=0}^{n}\binom{n}{j}r^{-d-z}(-\ln r)^j\partial^{n-j}\hat{\beta}(z)(r,\xi)$$

for every $n\in\mathbb{N}_0$, $r\in\mathbb{R}_{\ge1}$, and $\xi\in M$. In particular, for $r = 1$, we deduce

$$\partial^n\tilde{\beta}(z) = \partial^n\hat{\beta}(z)|_M,$$

i.e.

$$\partial^n\operatorname{res}\beta = \partial^n\int_M\tilde{\beta}\ d\mathrm{vol}_M = \int_M\partial^n\tilde{\beta}\ d\mathrm{vol}_M = \int_M\partial^n\hat{\beta}\ d\mathrm{vol}_M.$$

Especially, for β homogeneous, we have $\hat{\beta} = \beta$ and, therefore,

$$\partial^n\operatorname{res}\beta = \int_M\partial^n\tilde{\beta}\ d\mathrm{vol}_M = \int_M\partial^n\hat{\beta}\ d\mathrm{vol}_M = \int_M\partial^n\beta\ d\mathrm{vol}_M.$$

Hence,

$$\zeta(\partial^n \mathfrak{f}\mathfrak{p}_0\alpha)(z) = \int_{\mathbb{R}_{\geq 1}\times M} \partial^n\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M \partial^{n-j}\tilde{\alpha}_\iota(z)d\mathrm{vol}_M}{(\dim M + d_\iota + z + 1)^{l_\iota+j+1}}.$$

This directly yields

$$\begin{aligned}
\zeta(\alpha)(z) &= \sum_{\iota\in I_0}\left(\frac{(-1)^{l_\iota+1}l_\iota!\int_M\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{z^{l_\iota+1}} + \sum_{n=1}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^n\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}}\right)\\
&\quad + \sum_{n\in\mathbb{N}_0}\left(\frac{\zeta(\partial^n\mathfrak{f}\mathfrak{p}_0\alpha)(0)}{n!} + \sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}\right)z^n\\
&= \sum_{\iota\in I_0}\sum_{n=0}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^n\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}} + \sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq 1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}{n!}z^n\\
&\quad + \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!(\dim M + d_\iota + 1)^{l_\iota+j+1}}z^n\\
&\quad + \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}z^n\\
&= \sum_{\iota\in I_0}\sum_{n=0}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^n\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}} + \sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq 1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}{n!}z^n\\
&\quad + \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!(\dim M + d_\iota + 1)^{l_\iota+j+1}}z^n\\
&\quad + \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}z^n.
\end{aligned}$$

.

□

Definition 1.16. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution such that $\zeta(\alpha)$ is holomorphic in a neighborhood of zero. Then, we define the generalized ζ-determinant*

$$\det_\zeta(\alpha) := \exp\left(\zeta(\alpha)'(0)\right).$$

Remark This generalized ζ-determinant reduces to the ζ-determinants as studied by Kontsevich and Vishik in [12, 13]. In other words, we do not expect it to be multiplicative if α corresponds to a general Fourier Integral Operator. Though an interesting question, we will not study classes of families of Fourier Integral Operators satisfying the multiplicative property, here.

■

Knowing the Laurent expansion of $\zeta(\alpha)$ we know that

$$\begin{aligned}
\zeta(\alpha)'(0) &= \int_{\mathbb{R}_{\geq 1}\times M}\alpha_0'(0)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\
&\quad + \sum_{\iota\in I\setminus I_0}\sum_{j=0}^{1}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{1-j}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{l_\iota+j+1}}\\
&\quad + \sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{l_\iota+2}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{(l_\iota+1)!}
\end{aligned}$$

holds. In particular, if $I_0 = \varnothing$,

$$\zeta(\alpha)'(0) = \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I}\sum_{j=0}^{1} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M \partial^{1-j}\tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{l_\iota+j+1}}.$$

If α were poly-homogeneous we obtained

$$\begin{aligned}
\zeta(\alpha)'(0) =& \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} \\
&+ \sum_{\iota\in I\setminus I_0}\sum_{j=0}^{1} \frac{(-1)^{j+1}\int_M \partial^{1-j}\alpha_\iota(0) d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{j+1}} - \sum_{\iota\in I_0}\int_M \alpha_\iota''(0) d\mathrm{vol}_M \\
=& \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I\setminus I_0} \frac{-\operatorname{res}(\alpha_\iota')(0)}{\dim M + d_\iota + 1} \\
&+ \sum_{\iota\in I\setminus I_0} \frac{\operatorname{res}\alpha_\iota(0)}{(\dim M + d_\iota + 1)^2} - \sum_{\iota\in I_0} \operatorname{res}(\alpha_\iota'')(0)
\end{aligned}$$

If we were to choose an $\mathcal{M}$-gauge we would find $\partial\tilde{\alpha}_\iota = 0$ and may assume $I_0 = \varnothing$ ($\zeta(\alpha)$ cannot have a structural singularity and non-structural singularities do not appear within the ζ-function of an $\mathcal{M}$-gauged poly-log-homogeneous distribution), i.e.

$$\begin{aligned}
\zeta(\alpha)'(0) =& \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I\setminus I_0} \frac{(-1)^{l_\iota}(l_\iota+1)!\int_M \tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{l_\iota+2}} \\
=& \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I} \frac{(-1)^{l_\iota}(l_\iota+1)!\operatorname{res}\alpha_\iota(0)}{(\dim M + d_\iota + 1)^{l_\iota+2}}
\end{aligned}$$

and, for α additionally poly-homogeneous,

$$\zeta(\alpha)'(0) = \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I} \frac{\operatorname{res}\alpha_\iota(0)}{(\dim M + d_\iota + 1)^2}.$$

Remark Note that $\zeta(\alpha)'(0)$ depends on the first $1 + \max(\{l_\iota + 1;\ \iota \in I_0\} \cup \{0\})$ derivatives of α. Hence, the generalized ζ-determinant does so, too, and is, thus, not independent of the gauge.

■

2. Remarks on more general gauged poly-log-homogeneous distributions

The results obtained for gauged poly-log-homogeneous distributions can largely be generalized. In fact, the degree of homogeneity $d_\iota(z)$ can be chosen arbitrarily as long as it is not germ equivalent to a critical constant. In this section, we will investigate these direct generalizations and consider distributions of the form

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}(\xi)$$

where M is an orientable, compact, finite dimensional manifold without boundary and the holomorphic family α is given by an expansion

$$\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$$

where $I \subseteq \mathbb{N}$, $\alpha_0(z) \in L_1(\mathbb{R}_{\geq 1} \times M)$ in an open neighborhood of $\{z \in \mathbb{C};\ \Re(z) \leq 0\}$ and each of the $\alpha_\iota(z)$ is log-homogeneous with degree of homogeneity $d_\iota(z) \in \mathbb{C}$ and logarithmic order $l_\iota \in \mathbb{N}_0$, that is,

$$\exists \tilde{\alpha}_\iota \in \mathbb{C}^M\ \forall r \in \mathbb{R}_{\geq 1}\ \forall \nu \in M:\ \alpha_\iota(z)(r,\nu) = r^{d_\iota(z)}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\nu).$$

We will furthermore assume (for now) that every d_ι is an entire function,

$$\forall z \in \mathbb{C}:\ (d_\iota(z) = -\dim M - 1\ \Rightarrow\ d_\iota'(z) \neq 0),$$

the family $(\Re(d_\iota(z)))_{\iota \in I}$ is bounded from above for every $z \in \mathbb{C}$, $\sup_{\iota \in I}\Re(d_\iota(z)) \to -\infty$ $(\Re(z) \to -\infty)$, the maps $I \ni \iota \mapsto (d_\iota(z), l_\iota)$ are injective, there are only finitely many ι satisfying $d_\iota(z) = d$ for any given $d, z \in \mathbb{C}$, the families $((d_\iota(z)+\delta)^{-1})_{\iota \in I}$ are in $\ell_2(I)$ for any $z \in \mathbb{C}$ and $\delta \in \mathbb{C} \setminus \{d_\iota(z);\ \iota \in I\}$, and each $\sum_{\iota \in I}\tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$. Any such family α will be called a gauged poly-log-homogeneous distribution with holomorphic order.

If the conditions above are satisfied, we obtain

$$\begin{aligned}\int_{\mathbb{R}_{\geq 1}\times M}\alpha(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} &= \underbrace{\int_{\mathbb{R}_{\geq 1}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}} + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\ &= \tau_0(z) + \sum_{\iota \in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}}\end{aligned}$$

which converges absolutely. For $d_\iota(0) \neq -\dim M - 1$, we observe

$$\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}} = \frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(\frac{(\dim M+1+d_\iota(0)+z)^{l_\iota+1}}{(\dim M+1+d_\iota(z))^{l_\iota+1}}\alpha_\iota\right)(z)}{(\dim M + 1 + d_\iota(0) + z)^{l_\iota+1}}$$

in a neighborhood of zero. Hence, let

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}(\ln r)^{l_\iota}\underbrace{\frac{(\dim M + 1 + d_\iota(0)+z)^{l_\iota+1}}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}}\tilde{\alpha}_\iota(z)(\xi)}_{=:\tilde{\beta}_\iota(z)(\xi)}.$$

For $d_\iota(0) = -\dim M - 1$, there exists an entire function δ_ι such that

$$\dim M + 1 + d_\iota(z) = d_\iota'(0)z + \delta_\iota(z)z^2$$

and, since $d_\iota'(0) \neq 0$, we obtain that $z \mapsto d_\iota'(0) + \delta_\iota(z)z$ has no zeros in a neighborhood of zero. Then, we observe

$$\begin{aligned}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}} &= \frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(d_\iota'(0)z + \delta_\iota(z)z^2)^{l_\iota+1}}\\ &= \frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{z^{l_\iota+1}\left(d_\iota'(0)+\delta_\iota(z)z\right)^{l_\iota+1}}\\ &= \frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(\frac{\alpha_\iota(z)}{(d_\iota'(0)+\delta_\iota(z)z)^{l_\iota+1}}\right)}{z^{l_\iota+1}}\\ &= \frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(\frac{\alpha_\iota(z)}{(d_\iota'(0)+\delta_\iota(z)z)^{l_\iota+1}}\right)}{(\dim M + 1 + d_\iota(0) + z)^{l_\iota+1}}\end{aligned}$$

and define

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}(\ln r)^{l_\iota}\underbrace{\frac{\tilde{\alpha}_\iota(z)(\xi)}{\left(d_\iota'(0)+\delta_\iota(z)z\right)^{l_\iota+1}}}_{=:\tilde{\beta}_\iota(z)(\xi)}.$$

Thus, we obtain the following observation.

Observation 2.1. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution with holomorphic order. Then, the ζ-function $\zeta(\alpha)$ is germ equivalent to $\zeta(\beta)$ with β as defined above. Thus, $\zeta(\alpha)$ inherits all local properties from $\zeta(\beta)$, i.e. all local properties of ζ-functions associated with gauged poly-*log*-homogeneous distributions.*

In particular, if $\operatorname{res}\alpha_\iota(0) \neq 0$ with $d_\iota(0) = -\dim M - 1$ and l_ι maximal, then the initial Laurent coefficient of $\zeta(\alpha)$ is

$$\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(0)}{d_\iota'(0)^{l_\iota+1}}$$

and the $\zeta(\alpha)$ has the Laurent expansion

$$\begin{aligned}\zeta(\alpha)(z) = &\sum_{\iota\in I_0}\sum_{n=0}^{l_\iota} \frac{(-1)^{l_\iota+1} l_\iota! \int_M \partial^n \tilde\beta_\iota(0) d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}} \\ &+ \sum_{n\in\mathbb{N}_0} \frac{\int_{\mathbb{R}_{\ge1}\times M} \partial^n \alpha_0(0) d\mathrm{vol}_{\mathbb{R}_{\ge1}\times M}}{n!} z^n \\ &+ \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_M \partial^{n-j}\tilde\beta_\iota(0) d\mathrm{vol}_M}{n!(\dim M + d_\iota + 1)^{l_\iota+j+1}} z^n \\ &+ \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_M \partial^{n+l_\iota+1}\tilde\beta_\iota(0) d\mathrm{vol}_M}{(n+l_\iota+1)!} z^n\end{aligned}$$

in a sufficiently small neighborhood of zero.

Proof. Note that zero is either a pole of $\zeta(\alpha)$ or a regular value, that is, we can choose a neighborhood uniformly for all ι with $d_\iota(0) \neq -\dim M - 1$. Since there are only finitely many ι with $d_\iota(0) = -\dim M - 1$, we obtain germ equivalence of the series representations and, since the Laurent expansion was solely determined from the series representation, the observation follows.

□

We may generalize this even further. Suppose α is meromorphic in $\mathbb{C}$, that is, holomorphic in $\Omega \subseteq_{\text{open}} \mathbb{C}$ such that $\mathbb{C}\setminus\Omega$ is a set of isolated points in $\mathbb{C}$. Let $0\in\Omega$ and let α satisfy all properties of being a gauged poly-log-homogeneous distribution with holomorphic order but on Ω instead of $\mathbb{C}$. Then, we call α a meromorphic gauged poly-log-homogeneous distribution with respect to zero. Since $0 \in \Omega$, we directly obtain that α is locally a gauged poly-log-homogeneous distribution and still all local properties are preserved just as they are in Observation 2.1.

Now, we can even drop the assumption

$$\forall z\in\mathbb{C}:\ (d_\iota(z) = -\dim M - 1 \ \Rightarrow\ d_\iota'(z) \neq 0)$$

in the definition of a meromorphic gauged poly-log-homogeneous distribution with respect to zero (in exchange for an increased logarithmic order). Instead, let

$$d_\iota(z) = -\dim M - 1 + \delta_\iota(z) z^{m_\iota}$$

with $\delta_\iota(0) \neq 0$ and call any such α a generalized meromorphic gauged poly-log-homogeneous distribution with respect to zero. Then,

$$\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}}$$

$$
\begin{aligned}
&=\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\delta_\iota(z)z^{m_\iota})^{l_\iota+1}}\\
&=\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(\delta_\iota^{-l_\iota-1}\alpha_\iota\right)(z)}{z^{m_\iota(l_\iota+1)}}\\
&=\frac{(-1)^{m_\iota(l_\iota+1)}(m_\iota(l_\iota+1)-1)!\operatorname{res}\left((-1)^{m_\iota(l_\iota+1)+l_\iota+1}\frac{l_\iota!}{(m_\iota(l_\iota+1)-1)!}\delta_\iota^{-l_\iota-1}\alpha_\iota\right)(z)}{z^{m_\iota(l_\iota+1)}}\\
&=\frac{(-1)^{m_\iota(l_\iota+1)}(m_\iota(l_\iota+1)-1)!\operatorname{res}\left((-1)^{m_\iota(l_\iota+1)+l_\iota+1}\frac{l_\iota!}{(m_\iota(l_\iota+1)-1)!}\delta_\iota^{-l_\iota-1}\alpha_\iota\right)(z)}{(\dim M+1+d_\iota(0)+z)^{m_\iota(l_\iota+1)}}
\end{aligned}
$$

shows that choosing

$$\beta_\iota(z)(r,\xi):=r^{d_\iota(0)+z}(\ln r)^{m_\iota(l_\iota+1)-1}\underbrace{\frac{(-1)^{m_\iota(l_\iota+1)+l_\iota+1}l_\iota!}{(m_\iota(l_\iota+1)-1)!}\delta_\iota(z)^{-l_\iota-1}\tilde\alpha_\iota(z)(\xi)}_{=:\tilde\beta_\iota(z)(\xi)}$$

for $\iota\in I$ with $d_\iota(0)=-\dim M-1$ also yields germ equivalence and, again, all local properties are preserved.

Hence, we can state the following Definition and Theorem.

Definition 2.2. *Let $\Omega\subseteq_{\text{open}}\mathbb{C}$, $\Omega_0\subseteq_{\text{open}}\Omega$, $0\in\Omega$, and $\alpha=(\alpha(z))_{z\in\Omega}$ a holomorphic family of the form*

$$\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$$

where

- *$I\subseteq\mathbb{N}$,*
- *$\forall z\in\Omega:\ \alpha_0(z)\in L_1(\mathbb{R}_{\ge1}\times M)$,*
- *$\forall z\in\Omega_0:\ \alpha(z)\in L_1(\mathbb{R}_{\ge1}\times M)$,*
- *each of the $\alpha_\iota(z)$ is* log*-homogeneous with degree of homogeneity $d_\iota(z)\in\mathbb{C}$ and logarithmic order $l_\iota\in\mathbb{N}_0$, that is,*

  $$\exists\tilde\alpha_\iota\in\mathbb{C}^M\ \ \forall r\in\mathbb{R}_{\ge1}\ \ \forall\nu\in M:\ \alpha_\iota(z)(r,\nu)=r^{d_\iota(z)}(\ln r)^{l_\iota}\tilde\alpha_\iota(z)(\nu),$$
- *each d_ι is holomorphic in Ω,*
- *none of the d_ι is germ equivalent to $-\dim M-1$ at zero (i.e. none of the d_ι is the constant $-\dim M-1$),*
- *the maps $I\ni\iota\mapsto(d_\iota(z),l_\iota)$ are injective,*
- *there are only finitely many ι satisfying $d_\iota(z)=d$ for any given $d\in\mathbb{C}$ and $z\in\Omega$,*
- *the families $((d_\iota(z)+\delta)^{-1})_{\iota\in I}$ are in $\ell_2(I)$ for any $z\in\Omega$ and $\delta\in\mathbb{C}\setminus\{d_\iota(z);\ \iota\in I\}$,*
- *and each $\sum_{\iota\in I}\tilde\alpha_\iota(z)$ converges unconditionally in $L_1(M)$.*

*If every connected component of Ω has non-empty intersection with Ω_0, then we call α a generalized gauged poly-*log*-homogeneous distribution and*

$$\zeta(\alpha):=\int_{\mathbb{R}_{\ge1}\times M}\alpha_0 d\mathrm{vol}_{\mathbb{R}_{\ge1}\times M}+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota}{(\dim M+1+d_\iota)^{l_\iota+1}}$$

the associated ζ-function of α.

*Otherwise (in particular, if $\Omega_0=\varnothing$), we call α an abstract generalized gauged poly-*log*-homogeneous distribution and*

$$\zeta(\alpha):=\int_{\mathbb{R}_{\ge1}\times M}\alpha_0 d\mathrm{vol}_{\mathbb{R}_{\ge1}\times M}+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota}{(\dim M+1+d_\iota)^{l_\iota+1}}$$

the associated ζ-function of α.

Remark Because abstract generalized gauged poly-log-homogeneous distributions have empty Ω_0 on some connected component of Ω, we will still obtain the Laurent expansion and all other local properties derived from the series expansion we used to define the ζ-function here but applications to Fourier Integral Operators might lose all properties that are obtained from meromorphic extension of the classical trace, e.g. traciality.

■

Theorem 2.3. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ and $\beta = \beta_0 + \sum_{\iota\in I}\beta_\iota$ be (abstract) generalized gauged poly-log-homogeneous distributions with $\beta_0 = \alpha_0$,*

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}(\ln r)^{l_\iota}\underbrace{\frac{(\dim M + 1 + d_\iota(0)+z)^{l_\iota+1}}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}}\tilde{\alpha}_\iota(z)(\xi)}_{=\tilde{\beta}_\iota(z)(\xi)}$$

for $\iota\in I$ with $d_\iota(0) \neq -\dim M - 1$, and

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}(\ln r)^{m_\iota(l_\iota+1)-1}\underbrace{\frac{(-1)^{m_\iota(l_\iota+1)+l_\iota+1}l_\iota!}{(m_\iota(l_\iota+1)-1)!}\delta_\iota(z)^{-l_\iota-1}\tilde{\alpha}_\iota(z)(\xi)}_{=\tilde{\beta}_\iota(z)(\xi)}$$

for $\iota\in I$ with $d_\iota(z) = -\dim M - 1 + \delta_\iota(z)z^{m_\iota}$ in a neighborhood of zero and δ_ι holomorphic such that $\delta_\iota(0)\neq 0$.

Then, the ζ-function $\zeta(\alpha)$ is germ equivalent to $\zeta(\beta)$ at zero. In particular, $\zeta(\alpha)$ has the Laurent expansion

$$\begin{aligned}\zeta(\alpha)(z) = &\sum_{\iota\in I_0}\sum_{n=0}^{m_\iota(l_\iota+1)-1}\frac{(-1)^{m_\iota(l_\iota+1)}(m_\iota(l_\iota+1)-1)!\int_M\partial^n\tilde{\beta}_\iota(0)d\mathrm{vol}_M}{n!\ z^{m_\iota(l_\iota+1)-n}}\\ &+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\ge1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\ge1}\times M}}{n!}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde{\beta}_\iota(0)d\mathrm{vol}_M}{n!(\dim M+d_\iota+1)^{l_\iota+j+1}}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{m_\iota(l_\iota+1)}(m_\iota(l_\iota+1)-1)!\int_M\partial^{n+m_\iota(l_\iota+1)}\tilde{\beta}_\iota(0)d\mathrm{vol}_M}{(n+m_\iota(l_\iota+1))!}z^n\end{aligned}$$

in a sufficiently small neighborhood of zero.

3. Application to gauged Lagrangian distributions

If we consider a dual pair $\langle u(z), f\rangle$ where $u:\ \mathbb{C}\to I(X\times X,\Lambda)$ is a gauged Lagrangian distribution and $f\in I(X\times X,\hat{\Lambda})$ (cf. [4] and chapter 25 in [7]) such that Λ and $\hat{\Lambda}$ intersect cleanly at γ, then Theorem 21.2.10 in [7] yields homogeneous symplectic coordinates (x,ξ) near γ such that $\gamma = (1,0,\dots,0)$, $\Lambda = \{(0,\xi)\}$, and $\hat{\Lambda} = \{(0,\hat{x},\check{\xi},0)\}$ where $x = (\check{x},\hat{x})$, $\check{x} = (x_1,\dots,x_k)$, $\hat{x} = (x_{k+1},\dots,x_{\dim X})$, $\xi = (\check{\xi},\hat{\xi})$, $\check{\xi} = (\xi_1,\dots,\xi_k)$, $\hat{\xi} = (\xi_{k+1},\dots,\xi_{\dim X})$, and $k = \dim\Lambda\cap\hat{\Lambda}$.

Since f can be written as $f = P^t\delta_0$ for some pseudo-differential operator P, we obtain $\langle u(z), f\rangle = \langle Pu(z),\delta_0\rangle$ and, using the coordinates above, $Pu(z)$ is an oscillatory integral of the form

$$\int_{\mathbb{R}^k} e^{i\sum_{j=1}^k x_j\xi_j}a(z)\left(x_{k+1},\dots,x_{\dim X},\xi_1,\dots,\xi_k\right)d(\xi_1,\dots,\xi_k),$$

i.e.

$$\langle u(z), f\rangle = \int_{\mathbb{R}^k} a(z)\,(0,\xi)\,d\xi.$$

As pointed out by Guillemin in the proof of Theorem 2.1 in [4], this is a gauged poly-log-homogeneous distribution, i.e. the formalism developed above is applicable.

In order to treat

$$\langle u(z), \delta_{\mathrm{diag}}\rangle = \int_X \int_{\mathbb{R}^N} e^{i\vartheta(x,\xi)} a(z)(x,\xi)\ d\xi\ d\mathrm{vol}_X = \int_{\mathbb{R}^k} \alpha(z)(\xi) d\xi,$$

we will split off the integral

$$\tilde{\tau}_0(z) := \int_{B_{\mathbb{R}^N}(0,1)} \alpha(z)(\xi)\ d\xi$$

which defines a holomorphic function and we are left with

$$\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)$$

which is a distribution as considered in section 1. In other words, if A is a gauged Fourier Integral Operator with phase function ϑ and amplitude a on X, then

$$\begin{aligned}\zeta(A)(z) = &\underbrace{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} a(z)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)}_{=:\tau_0(A)(z)} \\ &+ \int_X \int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} a(z)(x,x,\xi)\ d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)\ d\mathrm{vol}_X(x)\end{aligned}$$

exists and inherits all properties described in section 1.

Theorem 3.1. *If a is poly-log-homogeneous and A_ι the gauged Fourier Integral Operator with phase ϑ and amplitude a_ι then*

$$\operatorname{res} A_\iota(z) = \int_{\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} \tilde{a}_\iota(z)(x,x,\xi)\ d\mathrm{vol}_X(x)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$$

and

$$\begin{aligned}&\zeta(A)(z) \\ &= \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n a(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)}{n!} z^n \\ &\quad + \sum_{\iota\in I_0} \sum_{n=0}^{l_\iota} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^n \tilde{a}_\iota(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!} z^{n-l_\iota-1} \\ &\quad + \sum_{n\in\mathbb{N}_0} \frac{\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} \partial^n a_0(0)(x,x,\xi)\ d\mathrm{vol}_X(x)\ d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)}{n!} z^n \\ &\quad + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I\setminus I_0} \sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n-j} \tilde{a}_\iota(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n \\ &\quad + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n+l_\iota+1} \tilde{a}_\iota(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(n+l_\iota+1)!} z^n\end{aligned}$$

in a neighborhood of zero where $\Delta(X) := \{(x,y)\in X^2;\ x=y\}$.

For a poly-homogeneous a this reduces to

$$\begin{aligned}\zeta(A)(z) = \sum_{n\in\mathbb{N}_0} & \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!} z^n \\ & - \sum_{\iota\in I_0} \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}} z^{-1} \\ & + \sum_{n\in\mathbb{N}_0} \frac{\int_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta}\partial^n a_0(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}}{n!} z^n \\ & + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I\setminus I_0} \sum_{j=0}^{n} \frac{(-1)^{j+1} j! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\partial^{n-j} a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{j+1}} z^n \\ & + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I_0} \frac{-\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\partial^{n+1} a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(n+1)!} z^n,\end{aligned}$$

i.e.

$$\begin{aligned}\zeta(A)(z) = & -\sum_{\iota\in I_0} \operatorname{res} A_\iota(0) z^{-1} - \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0} \frac{\operatorname{res}\partial^{n+1} A_\iota(0)}{(n+1)!} z^n \\ & + \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!} z^n \\ & + \sum_{n\in\mathbb{N}_0} \frac{\int_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta}\partial^n a_0(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}}{n!} z^n \\ & + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I\setminus I_0} \sum_{j=0}^{n} \frac{(-1)^{j+1} j! \operatorname{res}\partial^{n-j} A_\iota(0)}{n!(N+d_\iota)^{j+1}} z^n\end{aligned}$$

where $\partial^n A_\iota$ is the gauged Fourier Integral Operator with phase ϑ and amplitude $\partial^n a_\iota$.

From this last formula and the knowledge that $\operatorname{res} A_\iota(0)$ is independent of the gauge we obtain the following well-known result (cf. [4]).

Theorem 3.2. *Let A and B be poly-homogeneous Fourier Integral Operators. Let G_1 and G_2 be gauged Fourier Integral Operators with $G_1(0) = AB$ and $G_2(0) = BA$. Then,*

$$\operatorname{res}_0 \zeta(G_1) = \operatorname{res}_0 \zeta(G_2),$$

i.e. the residue of the ζ-function is tracial and $A \mapsto \operatorname{res}_0 \zeta\left(\hat{A}\right)$ is a well-defined trace where $\hat{A}$ is any choice of gauge for A.

Proof. This is a direct consequence of the following two facts.

(i) $\operatorname{res}_0 \zeta(G_1) = -\sum_{\iota\in I_0} \operatorname{res}(G_1)_\iota(0)$ is independent of the gauge.

(ii) $\zeta\left(\hat{A}B\right) = \zeta\left(B\hat{A}\right)$ for any gauge $\hat{A}$ of A because it is true for $\mathfrak{R}(z)$ sufficiently small.

□

Similarly, for $I_0(AB) = \varnothing$, we obtain that $\zeta(AB)(0) = \zeta(BA)(0)$ where we used that $\zeta(\mathfrak{fp}_0\alpha)(0)$ is independent of gauge. In other words, we may also generalize the Kontsevich-Vishik (quasi-)trace to $\zeta(\mathfrak{fp}_0 A)(0)$ where $\mathfrak{fp}_0 A$ is the gauged Fourier

Integral Operator with phase ϑ and amplitude $a - \sum_{\iota\in I_0} a_\iota$. In particular, we may also consider the regularized generalized determinant

$$\det_{\mathfrak{fp}}(A) := \exp\zeta(\mathfrak{fp}_0 A)'(0)$$

where

$$\begin{aligned}
&\zeta(\mathfrak{fp}_0 A)(z) \\
&= \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)}\partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!} z^n \\
&\quad + \sum_{n\in\mathbb{N}_0} \frac{\int_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta}\partial^n a_0(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})}}{n!} z^n \\
&\quad + \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\partial^{n-j}\tilde a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n,
\end{aligned}$$

i.e.

$$\begin{aligned}
\zeta(\mathfrak{fp}_0 A)'(0) = &\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} a'(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x) \\
&+ \int_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta} a_0'(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} \\
&+ \sum_{\iota\in I\setminus I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\tilde a_\iota'(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(N+d_\iota)^{l_\iota+1}} \\
&+ \sum_{\iota\in I\setminus I_0} \frac{(-1)^{l_\iota+2}(l_\iota+1)! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\tilde a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(N+d_\iota)^{l_\iota+2}}
\end{aligned}$$

which reduces to

$$\begin{aligned}
\zeta(\mathfrak{fp}_0 A)'(0) = &\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} a'(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x) \\
&+ \int_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta} a_0'(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} \\
&- \sum_{\iota\in I\setminus I_0} \frac{\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\tilde a_\iota'(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{N+d_\iota} \\
&+ \sum_{\iota\in I\setminus I_0} \frac{\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\tilde a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(N+d_\iota)^2} \\
=&\tau_0(\partial A)(0) + \int_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta} a_0'(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} \\
&- \sum_{\iota\in I\setminus I_0} \frac{\operatorname{res}(\partial A_\iota)(0)}{N+d_\iota} + \sum_{\iota\in I\setminus I_0} \frac{\operatorname{res} A_\iota(0)}{(N+d_\iota)^2}
\end{aligned}$$

for poly-homogeneous A. This will further reduce nicely if we choose an $\mathcal{M}$-gauge for the A_ι and no gauge for a_0 on $X \times B_{\mathbb{R}^N}(0,1)$ at all; namely, we obtain

$$\zeta(\mathfrak{fp}_0 A)'(0) = \sum_{\iota\in I\setminus I_0} \frac{\operatorname{res} A_\iota(0)}{(N+d_\iota)^2}$$

in that case. To be fair, this would be a gauge in a generalized sense for Fourier Integral Operators because such a gauge may not yield a $C^\infty\left(X\times X\times\mathbb{R}^N\right)$-amplitude

$a(z)$. Hence, we would have to gauge the $X \times B_{\mathbb{R}^N}(0,1)$ part, as well, and the correction term can easily be estimated by

$$\begin{aligned}|\tau_0(A)'(0)| =& \left| \int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} a'(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x) \right| \\ \leq& \mathrm{vol}_X(X)\mathrm{vol}_{\mathbb{R}^N}\left(B_{\mathbb{R}^N}(0,1)\right) \|a'(0)\|_{L_\infty(\Delta(X)\times B_{\mathbb{R}^N}(0,1))} \\ \leq& \mathrm{vol}_X(X)\mathrm{vol}_{\mathbb{R}^N}\left(B_{\mathbb{R}^N}(0,1)\right) \|a'(0)\|_{L_\infty(X\times X\times B_{\mathbb{R}^N})}.\end{aligned}$$

An important class of gauges are multiplicative gauges.

Definition 3.3. *Let A be a Fourier Integral Operator and G a gauged Fourier Integral Operator with $G(0) = 1$ such that each $G(z)$ and all derivatives are composable with A. Then, we call $AG(\cdot)$ a multiplicative gauge of A.*

Remark If we consider a canonical relation Γ and the corresponding algebra of Fourier Integral Operators $\mathcal{A}_\Gamma$, then we may be inclined to search for multiplicative gauges in $\mathcal{A}_\Gamma$. Unfortunately, the identity will not be an element of $\mathcal{A}_\Gamma$, in general (otherwise, Γ would need to contain the (graph of the) identity on $T^*X \setminus 0$ which would imply that all pseudo-differential operators are in $\mathcal{A}_\Gamma$, as well). An appropriate candidate of an algebra to consider if looking for a multiplicative gauge should, therefore, be the unitalization $\mathcal{A}_\Gamma \oplus \mathbb{C}$ of $\mathcal{A}_\Gamma$. If $\mathcal{A}_\Gamma$ is unital already, taking the direct sum with $\mathbb{C}$ will not change anything at all. Note that we interpret the element $(a,\lambda) \in \mathcal{A}_\Gamma \oplus \mathbb{C}$ to be $a + \lambda$ which directly yields the following structure.

- $(a,0) = a \in \mathcal{A}_\Gamma$, $(0,1) = 1$
- $\forall \lambda \in \mathbb{C}:\ \lambda(a,\mu) + (b,\nu) = (\lambda a, \lambda\mu) + (b,\nu) = (\lambda a + b, \lambda\mu + \nu)$
- $(a,\lambda)(b,\mu) = (a+\lambda)(b+\mu) = ab + a\mu + \lambda b + \lambda\mu = (ab + \mu a + \lambda b, \lambda\mu)$

Since derivatives should exists within the algebra and we might be interested in using a functional calculus, it may be necessary to also include an $L(L_2(X))$ closure of $\mathcal{A}_\Gamma \oplus \mathbb{C}$.

We may, however, gauge with properly supported pseudo-differential operators $G(z)$ (cf. section 18.4 in [18]).

■

Let P be a gauged pseudo-differential operator. Then, we may also consider

$$\langle P(z)u, f\rangle$$

as a gauge. This is due to Theorems 18.2.7 and 18.2.8 in [7]. In particular, if f is a Lagrangian distribution, then it can be represented in the form $\int e^{i\langle x,\xi\rangle} a_f(x,\xi) d\xi$ which is nothing other than $P_f\delta_0$ where P_f is the pseudo-differential operator with amplitude a_f. Hence,

$$\langle P(z)u, f\rangle = \langle P_f' P(z)u, \delta_0\rangle.$$

For traces, though, a multiplicative gauge yields

$$\zeta(A)(z) = \langle \mathfrak{g}(z) \circ k_A, \delta_{\mathrm{diag}}\rangle$$

where $\mathfrak{g}(z) \circ k_A$ is the kernel of $G(z)A$ and $\forall \varphi \in C(X):\ \delta_{\mathrm{diag}}(\varphi) = \int_X \varphi(x,x)dx$ (i.e. δ_{diag} is the kernel of the identity).

Example Suppose u is an $\mathcal{M}$-gauged log-homogeneous distribution. We, thus, obtain

$$u(0)(x) = \tau_0(u(0))(x) + \int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle} v(0)(\xi) = \tilde{\tau}_0(u(0))(x) + (P_u\delta_0)(x)$$

where P_u is a pseudo-differential operator with amplitude $p_u(x,\xi) = v(\xi)$ for $\xi \in \mathbb{R}^n \setminus B_{\mathbb{R}^N}$. Furthermore, the complex power H^z with $H := \sqrt{|\Delta|}$ has the amplitude

$p_z(x,\xi) = (2\pi)^{-N} \|\xi\|_{\ell_2(N)}^z$ where $|\Delta|$ is the (non-negative) Dirichlet Laplacian because $|\Delta|^{-1} = \mathcal{F}^{-1} \|m\|_{\ell_2(N)}^{-2} \mathcal{F}$ where m is the maximal multiplication operator with the argument on $L_2(\mathbb{R}^N)$

$$D(m) := \left\{ f \in L_2(\mathbb{R}^N);\ \left(\mathbb{R}^n \ni \xi \mapsto \xi f(\xi) \in \mathbb{C}^N\right) \in L_2(\mathbb{R}^N;\mathbb{C}^N) \right\},$$
$$m : D(m) \subseteq L_2(\mathbb{R}^N) \to L_2(\mathbb{R}^N;\mathbb{C}^N);\ f \mapsto (\xi \mapsto \xi f(\xi)).$$

$(-\Delta)^{-1}$ is well-known to be a compact operator. Hence, let $r-1$ be its spectral radius. Then, the holomorphic functional calculus yields

$$\begin{aligned}
H^z = \left(|\Delta|^{-1}\right)^{-\frac{z}{2}} &= \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \left(\lambda - (-\Delta)^{-1}\right)^{-1} d\lambda \\
&= \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \left((-\Delta)^{-1}\right)^j d\lambda \\
&= \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \left(\mathcal{F}^{-1} \|m\|_{\ell_2(N)}^{-2} \mathcal{F}\right)^j d\lambda \\
&= \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \mathcal{F}^{-1} \left(\|m\|_{\ell_2(N)}^{-2}\right)^j \mathcal{F} d\lambda \\
&= \mathcal{F}^{-1} \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \left(\|m\|_{\ell_2(N)}^{-2}\right)^j d\lambda \mathcal{F} \\
&= \mathcal{F}^{-1} \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \left(\lambda - \|m\|_{\ell_2(N)}^{-2}\right)^{-1} d\lambda \mathcal{F} \\
&= \mathcal{F}^{-1} \left(\|m\|_{\ell_2(N)}^{-2}\right)^{-\frac{z}{2}} \mathcal{F} \\
&= \mathcal{F}^{-1} \|m\|_{\ell_2(N)}^{z} \mathcal{F}.
\end{aligned}$$

Using the composition formula for pseudo-differential operators, we obtain that $(2\pi)^N H^z P_u$ has the amplitude

$$\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} \partial_2^\alpha \left((2\pi)^N p_z\right)(x,\xi) \underbrace{(-i\partial_1)^\alpha p_u(x,\xi)}_{=0 \ \Leftarrow\ \alpha\neq 0} = \|\xi\|_{\ell_2(N)}^z v(0)(\xi) = v(z)(\xi).$$

In other words,

$$u(z) \equiv (2\pi)^N H^z u(0)$$

modulo whatever happens on $B_{\mathbb{R}^N}$.

■

Example Let A be a poly-log-homogeneous Fourier Integral Operator and u a poly-log-homogeneous distribution with $I_0(A) = I_0(u) = \varnothing$. Suppose G and P are exponential multiplicative gauges, that is,

$$G'(z) = G(z)G_0 \qquad \text{and} \qquad P'(z) = P(z)P_0,$$

for A and u, respectively. Then

$$\zeta(GA)(z) = \sum_{n\in\mathbb{N}_0} \frac{\partial^n \zeta(GA)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n GA)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(GG_0^n A)(0)}{n!} z^n$$

and

$$\zeta(Pu)(z) = \sum_{n\in\mathbb{N}_0} \frac{\partial^n \zeta(Pu)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n Pu)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(PP_0^n u)(0)}{n!} z^n$$

hold in sufficiently small neighborhoods of zero. Using

$\zeta(GG_0^k A)(z)$

$$= \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n \sigma(GG_0^k A)(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)}{n!} z^n$$
$$+ \sum_{n\in\mathbb{N}_0} \frac{\int_{\Delta(X)\times\left(\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}\right)} e^{i\vartheta} \partial^n \sigma(GG_0^k A)_0(0)\ d\mathrm{vol}_{\Delta(X)\times\left(\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}\right)}}{n!} z^n$$
$$+ \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I} \sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n-j} \tilde\sigma(GG_0^k A)_\iota(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n,$$

where $\sigma(G_0^k A)$ denotes the amplitude of $G_0^k A$, we obtain

$$\begin{aligned}
\zeta(GA)(z) &= \sum_{k\in\mathbb{N}_0} \frac{\zeta(GG_0^k A)(0)}{k!} z^k \\
&= \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \Bigg(\int_{\Delta(X)\times B_{\mathbb{R}^N}} e^{i\vartheta} \sigma(G_0^k A)\ d\mathrm{vol}_{\Delta(X)\times B_{\mathbb{R}^N}} \\
&\quad + \int_{\Delta(X)\times\left(\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}\right)} e^{i\vartheta} \sigma(G_0^k A)_0\ d\mathrm{vol}_{\Delta(X)\times\left(\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}\right)} \\
&\quad + \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota!\, \mathrm{res}(G_0^k A)_\iota}{(N+d_\iota)^{l_\iota+1}} \Bigg) z^k
\end{aligned}$$

in a sufficiently small neighborhood of zero. For $\zeta(PP_0^n u)(0)$, we will denote the gauged poly-log-homogeneous distribution associated with $PP_0^k u$ by $\alpha\left(PP_0^k u\right)$ and use

$$\begin{aligned}
\zeta(PP_0^k u)(z) &= \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \int_{B_{\mathbb{R}^N}(0,1)} \partial^n \alpha\left(PP_0^k u\right)(0) d\mathrm{vol}_{B_{\mathbb{R}^N}(0,1)} z^n \\
&\quad + \sum_{n\in\mathbb{N}_0} \frac{\int_{\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}} \partial^n \alpha\left(PP_0^k u\right)_0(0) d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}}}{n!} z^n \\
&\quad + \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I} \sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_{\partial B_{\mathbb{R}^N}} \partial^{n-j} \tilde\alpha\left(PP_0^k u\right)_\iota(0) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n
\end{aligned}$$

to obtain

$$\begin{aligned}
\zeta(Pu)(z) &= \sum_{k\in\mathbb{N}_0} \frac{\zeta(PP_0^k u)(0)}{k!} z^k \\
&= \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \Bigg(\int_{B_{\mathbb{R}^N}(0,1)} \alpha\left(P_0^k u\right) d\mathrm{vol}_{B_{\mathbb{R}^N}(0,1)} \\
&\quad + \int_{\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}} \alpha\left(P_0^k u\right)_0 d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}} \\
&\quad + \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota!\, \mathrm{res}\, \alpha\left(P_0^k u\right)_\iota}{(N+d_\iota)^{l_\iota+1}} \Bigg) z^k.
\end{aligned}$$

■

Example If we consider a multiplicatively gauged $A(z) = BQ^z$ where Q may be non-invertible but is an element of an admissible algebra of Fourier Integral Operators with holomorphic functional calculus, e.g. a pseudo-differential operator of order 1 (order $q > 0$ can be obtained using the results of section 2) and spectral cut (the following is to be interpreted in this setting), then $Q^0 = 1 - 1_{\{0\}}(Q)$ where

$$1_{\{0\}}(Q) := \frac{1}{2\pi i}\int_{\partial B(0,\varepsilon)} (\lambda - Q)^{-1}\, d\lambda$$

with ε sufficiently small such that $B(0,\varepsilon) \cap \sigma(Q) = \{0\}$. Thus, assuming $I_0 = \varnothing$ (that is, the Kontsevich-Vishik trace is well-defined and coincides with $\zeta(A)(0)$), we obtain (abusing the notation tr because ζ is gauge invariant)

$$\zeta(A)(0) = \operatorname{tr}\left(BQ^0\right) = \operatorname{tr}(B) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)$$

and

$$\forall k \in \mathbb{N}:\ \zeta(\partial^k A)(0) = \operatorname{tr}\left(B(\ln Q)^k Q^0\right) = \operatorname{tr}\left(B(\ln Q)^k\right) - \operatorname{tr}\left(B(\ln Q)^k 1_{\{0\}}(Q)\right)$$

where we note that there still is a dependence on the spectral cut used to define the operators Q^z and $\ln Q$. These generalize the formulae (0.17) and (0.18) in [16] (note that the factors $(-1)^k$ are due to sign convention Q^z vs. Q^{-z}).

Proposition 3.4. *Let $A(z) = BQ^z$ be poly-homogeneous, $\mathfrak{fp}\zeta$ the finite part of ζ, and $\operatorname{tr}_{\mathfrak{fp}}$ the finite part of the trace integral (cf. [12], [13], [14], and [16]). Furthermore, let c_k be the coefficient of $\frac{z^k}{k!}$ in the Laurent coefficient with $k \in \mathbb{N}_0$.*

Then, we obtain

$$\begin{aligned} c_k =&\zeta\left(\partial^k \mathfrak{fp}_0 A\right)(0) + \sum_{\iota\in I_0}\int_X\int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)}\partial^k a_\iota(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)\\ &- \sum_{\iota\in I_0}\frac{1}{k+1}\operatorname{res}\left(\partial^{k+1}A_\iota\right)(0)\\ =&\mathfrak{fp}\zeta\left(\partial^k A\right)(0) - \frac{1}{k+1}\operatorname{res}\left(\partial^{k+1}A\right)(0)\\ =&\operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k Q^0\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1}Q^0\right). \end{aligned}$$

In particular,

$$c_0 = \operatorname{tr}_{\mathfrak{fp}}(B) - \operatorname{res}(B\ln Q) - \operatorname{tr}_{\mathfrak{fp}}\left(B1_{\{0\}}(Q)\right)$$

and

$$\forall k \in \mathbb{N}:\ c_k = \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1}\right) - \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k 1_{\{0\}}(Q)\right)$$

generalize equations (0.12) and (0.14) in [16] (keeping in mind the factors $(-1)^k$ due to sign convention).

If Q is invertible, then $1_{\{0\}}(Q) = 0$, and for another admissible and invertible operator Q', we obtain

$$(*)\qquad c_0(Q) - c_0(Q') = -\operatorname{res}\left(B\left(\ln Q - \ln Q'\right)\right)$$

which is a generalization of equation (2.21) in [12] and (9) in [15]. Furthermore, we obtain for $A(z) = [B, CQ^z]$ with invertible Q, that $\zeta(A) = 0$, i.e. $c_0 = 0$ and

$$\operatorname{tr}_{\mathfrak{fp}}([B,C]) = \operatorname{res}([B, C\ln Q])$$

a generalization of (2.20) in [16].

Applying our ζ-calculus and the considerations above to complex powers also allows us to reproduce the variation formula for the multiplicative anomaly (2.18) in [12] using effectively the same proof. However, it should be noted that this approach now also works in algebras of Fourier Integral Operators provided they contain complex powers.

$$\begin{aligned}&\partial_t\partial_s\left(\zeta\left(z\mapsto(A_tB)^z\right)(s)-\zeta\left(z\mapsto A_t^z\right)(s)-\zeta\left(z\mapsto B^z\right)(s)\right)\\=&\partial_s\left(\partial_t\zeta\left(z\mapsto(A_tB)^z\right)(s)-\partial_t\zeta\left(z\mapsto A_t^z\right)(s)\right)\end{aligned}$$

can be evaluated using a suitable contour Γ and $C\in\{B,1\}$ which yields

$$\begin{aligned}\partial_t\zeta\left(z\mapsto(A_tC)^z\right)=&\zeta\left(z\mapsto\partial_t\frac{1}{2\pi i}\int_\Gamma\lambda^z(\lambda-A_tC)^{-1}d\lambda\right)\\=&\zeta\left(z\mapsto\frac{1}{2\pi i}\int_\Gamma\lambda^z(A_t'C)(\lambda-A_tC)^{-2}d\lambda\right)\\=&\zeta\left(z\mapsto(A_t'C)\frac{1}{2\pi i}\int_\Gamma\lambda^z\left(-\partial_\lambda(\lambda-A_tC)^{-1}\right)d\lambda\right)\\=&\zeta\left(z\mapsto(A_t'C)\frac{1}{2\pi i}\int_\Gamma(\partial_\lambda\lambda^z)(\lambda-A_tC)^{-1}d\lambda\right)\\=&\zeta\left(z\mapsto(A_t'C)\frac{1}{2\pi i}\int_\Gamma z\lambda^{z-1}(\lambda-A_tC)^{-1}d\lambda\right)\\=&\zeta\left(z\mapsto z(A_t'C)(A_tC)^{-1}(A_tC)^z\right)\\=&\zeta\left(z\mapsto zA_t'A_t^{-1}(A_tC)^z\right).\end{aligned}$$

Taking the other derivative, we obtain

$$\begin{aligned}\partial_s\partial_t\zeta\left(z\mapsto(A_tC)^z\right)(s)=&\partial_s\zeta\left(z\mapsto zA_t'A_t^{-1}(A_tC)^z\right)(s)\\=&\zeta\left(z\mapsto\partial_z\left(zA_t'A_t^{-1}(A_tC)^z\right)\right)(s)\\=&\zeta\left(z\mapsto A_t'A_t^{-1}(A_tC)^z+z\partial_z\left(A_t'A_t^{-1}(A_tC)^z\right)\right)(s)\\=&(1+s\partial_s)\zeta\left(z\mapsto A_t'A_t^{-1}(A_tC)^z\right)(s).\end{aligned}$$

However, by assumption $\zeta\left(z\mapsto A_t'A_t^{-1}(A_tC)^z\right)$ is holomorphic near zero, i.e. its derivative $\zeta\left(z\mapsto A_t'A_t^{-1}(A_tC)^z\right)'$ is holomorphic near zero, and

$$s\partial_s\zeta\left(z\mapsto A_t'A_t^{-1}(A_tC)^z\right)(s)\to0\qquad(s\to0).$$

In other words,

$$\begin{aligned}&\partial_t\partial_s\left(\zeta\left(z\mapsto(A_tB)^z\right)(s)-\zeta\left(z\mapsto A_t^z\right)(s)-\zeta\left(z\mapsto B^z\right)(s)\right)\\=&\zeta\left(z\mapsto A_t'A_t^{-1}(A_tB)^z\right)(s)-\zeta\left(z\mapsto A_t'A_t^{-1}A_t^z\right)(s)\end{aligned}$$

which, according to $(*)$ above, yields

$$\begin{aligned}\partial_t\ln F(A_t,B)=&\partial_t\partial_s\left(\zeta\left(z\mapsto(A_tB)^z\right)(s)-\zeta\left(z\mapsto A_t^z\right)(s)-\zeta\left(z\mapsto B^z\right)(s)\right)\\=&\zeta\left(z\mapsto A_t'A_t^{-1}(A_tB)^z\right)(s)-\zeta\left(z\mapsto A_t'A_t^{-1}A_t^z\right)(s)\\=&-\operatorname{res}\left(A_t'A_t^{-1}\left(\frac{\ln(A_tB)}{\operatorname{order}A_tB}-\frac{\ln A_t}{\operatorname{order}A_t}\right)\right)\end{aligned}$$

with the multiplicative anomaly

$$F(A,B):=\frac{\exp\left(\zeta\left(z\mapsto(AB)^z\right)'(0)\right)}{\exp\left(\zeta\left(z\mapsto A^z\right)'(0)\right)\exp\left(\zeta\left(z\mapsto B^z\right)\right)'(0)}.$$

Choosing a multiplicative gauge G with $G' = G_0 G$, we obtain a different variation formula of the multiplicative anomaly; namely,

$$\begin{aligned}\partial_t \left(\zeta(A_tB_tG)' - \zeta(A_tG)' - \zeta(B_tG)'\right) =& \zeta(A_t'B_tG)' + \zeta(A_tB_t'G)' - \zeta(A_t'G)' - \zeta(B_t'G)' \\ =& \zeta(A_t'(B_t-1)G)' + \zeta((A_t-1)B_t'G)' \\ =& \zeta(A_t'(B_t-1)G') + \zeta((A_t-1)B_t'G') \\ =& \zeta(A_t'(B_t-1)G_0G) + \zeta((A_t-1)B_t'G_0G).\end{aligned}$$

■

Remark Note that the mechanism explored in this chapter also works whenever there is a representation $\int_{\mathbb{R}^N} \underbrace{\int_X e^{i\vartheta(x,x,\xi)}a(x,x,\xi)d\mathrm{vol}_X(x)}_{=:\alpha(\xi)} d\xi$ with poly-log-homogeneous α. In particular, we may consider algebras that do not have the form $\mathcal{A}_\Gamma$ where Γ intersects the co-normal bundle of the identity cleanly. Above, we used that $\langle k, \delta_{\mathrm{diag}}\rangle$ can be written as $\langle Pk, \delta_0\rangle$ for some pseudo-differential operator P, i.e. we used the clean intersection property to obtain the poly-log-homogeneous distribution form. However, for $\Re(z)$ sufficiently small, the gauged $k(z)$ is continuous, that is, $\langle k(z), \delta_{\mathrm{diag}}\rangle$ is well-defined and extends, thus, not needing the clean intersection property.

■

4. The heat trace, fractional, and shifted fractional Laplacians on flat tori

In this section, we will apply Theorem 3.1 to some examples which are well-known or can be easily checked through spectral considerations.

Example (the Heat Trace on the flat torus $\mathbb{R}^N/\Gamma$) Let $\Gamma \subseteq \mathbb{R}^N$ be a discrete group generated by a basis of $\mathbb{R}^N$, $|\Delta|$ the Dirichlet Laplacian on $\mathbb{R}^N$, δ the Dirichlet Laplacian on $\mathbb{R}^N/\Gamma$, and T the semi-group generated by $-\delta$ on $\mathbb{R}^N/\Gamma$. It is well-known that

$$\operatorname{tr} T(t) = \frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi t)^{\frac{N}{2}}} \sum_{\gamma\in\Gamma} \exp\left(-\frac{\|\gamma\|_{\ell_2(N)}^2}{4t}\right)$$

holds; cf. e.g. equation 3.2.3.28 in [17]. Furthermore, the kernel κ_δ of δ is given by the kernel $\kappa_{|\Delta|}$ via $\kappa_\delta(x,y) = \sum_{\gamma\in\Gamma} \kappa_{|\Delta|}(x, y\gamma)$; cf. e.g. section 3.2.2 in [17]. In other words,

$$\kappa_\delta(x,y) = \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^N} e^{i\langle x-y-\gamma,\xi\rangle}(2\pi)^{-N} \|\xi\|_{\ell_2(N)}^2 \, d\xi.$$

Hence, using functional calculus, we obtain

$$\kappa_{T(t)}(x,y) = \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^N} e^{i\langle x-y-\gamma,\xi\rangle}(2\pi)^{-N} e^{-t\|\xi\|_{\ell_2(N)}^2} d\xi.$$

Considering some gauge of $T(t)$ we obtain from the Laurent expansion (Theorem 3.1)

$$\begin{aligned}&\zeta(T(t))(0) \\ =& \int_{\mathbb{R}^N/\Gamma\times B_{\mathbb{R}^N}} \sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle}(2\pi)^{-N} e^{-t\|\xi\|_{\ell_2(N)}^2} \, d\mathrm{vol}_{\mathbb{R}^N/\Gamma\times B_{\mathbb{R}^N}}(x,\xi) \\ &+ \int_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})} \sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle}(2\pi)^{-N} \left(e^{-t\|\cdot\|_{\ell_2(N)}^2}\right)_0(\xi) \, d\mathrm{vol}_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})}(x,\xi)\end{aligned}$$

$$+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}(T(t))_\iota}{(N+d_\iota)^{l_\iota+1}}.$$

Since $\left(\xi\mapsto e^{-t\|\xi\|^2_{\ell_2(N)}}\right)\in\mathcal{S}(\mathbb{R}^N)$, we can choose $I=\varnothing$ and $\left(e^{-t\|\cdot\|^2_{\ell_2(N)}}\right)_0=e^{-t\|\cdot\|^2_{\ell_2(N)}}$ which yields

$$\begin{aligned}
&\zeta(T(t))(0)\\
=&\int_{\mathbb{R}^N/\Gamma\times B_{\mathbb{R}^N}}\sum_{\gamma\in\Gamma}e^{-i\langle\gamma,\xi\rangle}(2\pi)^{-N}e^{-t\|\xi\|^2_{\ell_2(N)}}\,d\mathrm{vol}_{\mathbb{R}^N/\Gamma\times B_{\mathbb{R}^N}}(x,\xi)\\
&+\int_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})}\sum_{\gamma\in\Gamma}e^{-i\langle\gamma,\xi\rangle}(2\pi)^{-N}e^{-t\|\xi\|^2_{\ell_2(N)}}\,d\mathrm{vol}_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N})}(x,\xi)\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(2\pi)^N}\int_{B_{\mathbb{R}^N}}\sum_{\gamma\in\Gamma}e^{-i\langle\gamma,\xi\rangle}e^{-t\|\xi\|^2_{\ell_2(N)}}\,d\mathrm{vol}_{B_{\mathbb{R}^N}}(\xi)\\
&+\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(2\pi)^N}\int_{\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N}}\sum_{\gamma\in\Gamma}e^{-i\langle\gamma,\xi\rangle}e^{-t\|\xi\|^2_{\ell_2(N)}}\,d\mathrm{vol}_{\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N}}(\xi)\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(2\pi)^N}\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^N}e^{-i\langle\gamma,\xi\rangle}e^{-t\|\xi\|^2_{\ell_2(N)}}\,d\xi\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi^2)^{\frac{N}{2}}}\sum_{\gamma\in\Gamma}\pi^{\frac{N}{2}}t^{-\frac{N}{2}}e^{-\frac{\|\gamma\|^2_{\ell_2(N)}}{4t}}\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi t)^{\frac{N}{2}}}\sum_{\gamma\in\Gamma}\exp\left(-\frac{\|\gamma\|^2_{\ell_2(N)}}{4t}\right),
\end{aligned}$$

i.e. precisely what we wanted to obtain.

■

Please note that the following example exceeds the applicability of the ζ-function Laurent expansion as it is for now. However, we will show in section 5 that the formulae still hold.

Example (fractional Laplacians on $\mathbb{R}/_{2\pi\mathbb{Z}}$) On $\mathbb{T}:=\mathbb{R}/_{2\pi\mathbb{Z}}$, let us consider the operator $H:=\sqrt{|\Delta|}$ where $|\Delta|$ denotes the (non-negative) Laplacian. It is well-known that the spectrum $\sigma(H)=\mathbb{N}_0$ is discrete and each non-zero eigenvalue has multiplicity 2. Furthermore, the symbol of H^z has the kernel

$$\kappa_{H^z}(x,y)=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{i(x-y-2\pi n)\xi}\frac{|\xi|^z}{2\pi}d\xi.$$

The singular part is given for $n=0$ and $\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{i(x-y-2\pi n)\xi}\frac{|\xi|^z}{2\pi}d\xi$ is regular.

Let $\alpha\in(-1,0)$. Since ζ is the spectral ζ-function, we obtain (μ_λ denoting the multiplicity of λ and $\Re(z)<-1$)

$$\zeta\left(s\mapsto H^sH^\alpha\right)(z)=\sum_{\lambda\in\sigma(H)\setminus\{0\}}\mu_\lambda\lambda^{z+\alpha}=2\sum_{n\in\mathbb{N}}n^{z+\alpha}=2\zeta_R(-z-\alpha)$$

where ζ_R denotes Riemann's ζ-function. In particular,

$$\zeta\left(s\mapsto H^sH^\alpha\right)(0)=2\zeta_R(-\alpha).$$

On the other hand, we have the Laurent expansion (Theorem 3.1)

$$\begin{aligned}\zeta\left(s\mapsto H^sH^\alpha\right)(z)=\sum_{k\in\mathbb{N}_0}\frac{1}{k!}\Bigg(&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}}e^{i\vartheta}\sigma\left((\ln H)^k H^\alpha\right)\ d\mathrm{vol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}}\\&+\int_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}})}e^{i\vartheta}\sigma\left((\ln H)^k H^\alpha\right)_0\ d\mathrm{vol}_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}})}\\&+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left((\ln H)^k H^\alpha\right)_\iota}{(1+d_\iota)^{l_\iota+1}}\Bigg)z^k,\end{aligned}$$

i.e.

$$\begin{aligned}\zeta\left(s\mapsto H^sH^\alpha\right)(0)=&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}}e^{i\vartheta}\sigma\left(H^\alpha\right)\ d\mathrm{vol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}}\\&+\int_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}})}e^{i\vartheta}\sigma\left(H^\alpha\right)_0\ d\mathrm{vol}_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}})}\\&+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(H^\alpha\right)_\iota}{(1+d_\iota)^{l_\iota+1}}.\end{aligned}$$

Plugging in our kernel yields

$$\begin{aligned}\zeta\left(s\mapsto H^sH^\alpha\right)(0)=&\sum_{n\in\mathbb{Z}}\int_0^{2\pi}\int_{-1}^1 e^{-2\pi in\xi}\frac{|\xi|^\alpha}{2\pi}\ d\xi\ dx\\&+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_0^{2\pi}\int_{\mathbb{R}_{\leq1}\cup\mathbb{R}_{\geq1}}e^{-2\pi in\xi}\frac{|\xi|^\alpha}{2\pi}\ d\xi\ dx\\&-\frac{1}{1+\alpha}\int_0^{2\pi}\int_{\partial B_{\mathbb{R}}}\frac{|\xi|^\alpha}{2\pi}\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)\ dx\\=&\int_{-1}^1|\xi|^\alpha\,d\xi+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\alpha\,d\xi\\&-\frac{1}{1+\alpha}\int_{\partial B_{\mathbb{R}}}|\xi|^\alpha\,d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi).\end{aligned}$$

Since $\alpha\in(-1,0)$ and $\mathrm{vol}_{\partial B_{\mathbb{R}}}$ is the sum of point measures $\delta_{-1}+\delta_1$, we obtain

$$\int_{-1}^1|\xi|^\alpha\,d\xi=2\int_0^1\xi^\alpha d\xi=\frac{2}{\alpha+1}=\frac{1}{1+\alpha}\int_{\partial B_{\mathbb{R}}}|\xi|^\alpha\,d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi),$$

i.e.

$$\zeta\left(s\mapsto H^sH^\alpha\right)(0)=\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\alpha\,d\xi.$$

Using that the Fourier transform of $\xi\mapsto|\xi|^\alpha$ is

$$\int_{\mathbb{R}}e^{-2\pi ix\xi}|\xi|^\alpha\,d\xi=\frac{2\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(\alpha+1)}{|2\pi x|^{\alpha+1}}$$

and Riemann's functional equation

$$\zeta_R(z)=2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta_R(1-z),$$

we obtain (in the sense of meromorphic extensions)

$$\zeta\left(s\mapsto H^sH^\alpha\right)(0)=\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\alpha\,d\xi=\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{2\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(\alpha+1)}{|2\pi n|^{\alpha+1}}$$

$$
\begin{aligned}
&=\frac{2\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}\cdot 2\sum_{n\in\mathbb{N}}\frac{1}{n^{\alpha+1}}\\
&=2\underbrace{2(2\pi)^{(-\alpha)-1}\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(1-(-\alpha))\zeta_R(1-(-\alpha))}_{=\zeta_R(-\alpha)}.
\end{aligned}
$$

■

Remark Using identification via meromorphic extension of

$$\zeta_R(z)=\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\sin\left(\frac{-z\pi}{2}\right)\Gamma(z+1)}{|2\pi n|^{z+1}}$$

and, therefore,

$$\forall z\in\mathbb{C}\setminus\{-1\}:\ \sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^z\,d\xi=2\zeta_R(-z)$$

as well as

$$\int_{-1}^{1}|\xi|^z\,d\xi=\frac{1}{1+z}\int_{\partial B_{\mathbb{R}}}|\xi|^z\,d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi),$$

we can extend the example above to all $\alpha\in\mathbb{C}\setminus\{-1\}$, i.e.

$$\zeta_R=\left(\alpha\mapsto\frac{1}{2}\zeta\left(s\mapsto H^sH^{-\alpha}\right)(0)\right).$$

■

Example (generalized ζ-determinant of $s\mapsto H^sH^\alpha$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$) In order to calculate $\det_\zeta\left(s\mapsto H^sH^\alpha\right)=\exp\left(\zeta\left(s\mapsto H^sH^\alpha\right)'(0)\right)$, it suffices to know the derivative $\zeta\left(s\mapsto H^sH^\alpha\right)'(0)$. From the spectral ζ-function we directly obtain

$$\zeta\left(s\mapsto H^sH^\alpha\right)'(0)=\partial\left(z\mapsto 2\zeta_R(-z)\right)(\alpha)=-2\zeta_R'(-\alpha).$$

On the other hand, we may invest (Theorem 3.1)

$$
\begin{aligned}
\zeta\left(s\mapsto H^sH^\alpha\right)(z)=\sum_{k\in\mathbb{N}_0}\frac{1}{k!}\Bigg(&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}}e^{i\vartheta}\sigma\left((\ln H)^kH^\alpha\right)\ d\mathrm{vol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}}\\
&+\int_{\Delta(\mathbb{T})\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}})}e^{i\vartheta}\sigma\left((\ln H)^kH^\alpha\right)_0\ d\mathrm{vol}_{\Delta(\mathbb{T})\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}})}\\
&+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left((\ln H)^kH^\alpha\right)_\iota}{(1+d_\iota)^{l_\iota+1}}\Bigg)z^k,
\end{aligned}
$$

again, to find

$$
\begin{aligned}
\zeta\left(s\mapsto H^sH^\alpha\right)'(0)=&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}}e^{i\vartheta}\sigma\left(\ln HH^\alpha\right)\ d\mathrm{vol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}}\\
&+\int_{\Delta(\mathbb{T})\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}})}e^{i\vartheta}\sigma\left(\ln HH^\alpha\right)_0\ d\mathrm{vol}_{\Delta(\mathbb{T})\times(\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}})}\\
&+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(\ln HH^\alpha\right)_\iota}{(1+d_\iota)^{l_\iota+1}}.
\end{aligned}
$$

Using the symbol $\frac{\ln|\xi|}{2\pi}$ of $\ln H$ on $\mathbb{R}$, yields that

$$\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{i(x-y-2\pi n)\xi}\frac{|\xi|^\alpha\ln|\xi|}{2\pi}\ d\xi$$

is the kernel of $\ln HH^{\alpha}$ on $\mathbb{T}$. Again, the singular part is given for $n=0$ yielding $\#I=1$, $d_\iota=\alpha$, and $l_\iota=1$, as well as

$$\begin{aligned}\zeta\left(s\mapsto H^sH^{\alpha}\right)'(0)=&\int_0^{2\pi}\int_{-1}^{1}\sum_{n\in\mathbb{Z}}e^{-2\pi in\xi}\frac{|\xi|^{\alpha}\ln|\xi|}{2\pi}\ d\xi\ dx\\&+\int_0^{2\pi}\int_{\mathbb{R}_{<-1}\cup\mathbb{R}_{>1}}\sum_{n\in\mathbb{Z}\setminus\{0\}}e^{-2\pi in\xi}\frac{|\xi|^{\alpha}\ln|\xi|}{2\pi}\ d\xi\ dx\\&+\frac{1}{(1+\alpha)^2}\int_0^{2\pi}\int_{\partial B_{\mathbb{R}}}\frac{|\xi|^{\alpha}\ln|\xi|}{2\pi}\ \mathrm{dvol}_{\partial B_{\mathbb{R}}}(\xi)\ dx\\=&\int_{-1}^{1}|\xi|^{\alpha}\ln|\xi|\ d\xi+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^{\alpha}\ln|\xi|\ d\xi+\frac{2}{(1+\alpha)^2}.\end{aligned}$$

Note that

$$\int_{-1}^{1}|\xi|^{\alpha}\ln|\xi|\ d\xi=2\int_0^1\xi^{\alpha}\ln\xi\ d\xi=-\frac{2}{(\alpha+1)^2}$$

holds for $\Re(\alpha)>-1$ and, hence, by meromorphic extension

$$\begin{aligned}\zeta\left(s\mapsto H^sH^{\alpha}\right)'(0)=&\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^{\alpha}\ln|\xi|\ d\xi\\=&\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}\partial\left(\beta\mapsto|\xi|^{\beta}\right)(\alpha)\ d\xi\\=&\partial\left(\beta\mapsto\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^{\beta}\ d\xi\right)(\alpha)\\=&\partial\left(\beta\mapsto2\zeta_R(-\beta)\right)(\alpha)\\=&-2\zeta_R'(-\alpha).\end{aligned}$$

∎

Similarly, we can take higher order derivatives.

Example ($\partial^k\zeta\left(s\mapsto H^sH^{\alpha}\right)(0)$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$) Regarding higher order derivatives the spectral ζ-function yields

$$\partial^k\zeta\left(s\mapsto H^sH^{\alpha}\right)(0)=\partial^k\left(z\mapsto2\zeta_R(-z)\right)(\alpha)=(-1)^k\cdot2\partial^k\zeta_R(-\alpha).$$

From

$$\begin{aligned}\zeta\left(s\mapsto H^sH^{\alpha}\right)(z)=\sum_{k\in\mathbb{N}_0}\frac{1}{k!}\Bigg(&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}}e^{i\vartheta}\sigma\left((\ln H)^kH^{\alpha}\right)\ \mathrm{dvol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}}\\&+\int_{\Delta(\mathbb{T})\times\left(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}}\right)}e^{i\vartheta}\sigma\left((\ln H)^kH^{\alpha}\right)_0\ \mathrm{dvol}_{\Delta(\mathbb{T})\times\left(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}}\right)}\\&+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\,\mathrm{res}\left(\left((\ln H)^kH^{\alpha}\right)_\iota\right)}{(1+d_\iota)^{l_\iota+1}}\Bigg)z^k\end{aligned}$$

(Theorem 3.1) we obtain

$$\begin{aligned}\partial^k\zeta\left(s\mapsto H^sH^{\alpha}\right)(0)=&\int_0^{2\pi}\int_{-1}^{1}\sum_{n\in\mathbb{Z}}e^{-2\pi in\xi}\frac{|\xi|^{\alpha}\left(\ln|\xi|\right)^k}{2\pi}\ d\xi\ dx\\&+\int_0^{2\pi}\int_{\mathbb{R}\setminus B_{\mathbb{R}}}\sum_{n\in\mathbb{Z}\setminus\{0\}}e^{-2\pi in\xi}\frac{|\xi|^{\alpha}\left(\ln|\xi|\right)^k}{2\pi}\ d\xi\ dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{(-1)^{k+1}k!}{(1+\alpha)^{k+1}}\int_0^{2\pi}\int_{\partial B_{\mathbb{R}}}\frac{|\xi|^\alpha}{2\pi}\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)\ dx\\
=&2\int_0^1 \xi^\alpha(\ln\xi)^k\,d\xi+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\alpha(\ln|\xi|)^k\,d\xi\\
&-\frac{2\cdot(-1)^k k!}{(1+\alpha)^{k+1}}\\
=&2\partial^k\left(\beta\mapsto\int_0^1\xi^\beta d\xi\right)(\alpha)-\frac{2\cdot(-1)^k k!}{(1+\alpha)^{k+1}}\\
&+\partial^k\left(\beta\mapsto\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\beta\,d\xi\right)(\alpha)\\
=&2\underbrace{\partial^k\left(\beta\mapsto(1+\beta)^{-1}\right)(\alpha)}_{=(-1)^k k!(1+\alpha)^{-(k+1)}}-\frac{2\cdot(-1)^k k!}{(1+\alpha)^{k+1}}+\partial^k\left(\beta\mapsto 2\zeta_R(-\beta)\right)(\alpha)\\
=&(-1)^k\cdot 2\partial^k\zeta_R(-\alpha).
\end{aligned}
$$

■

Finally, let us calculate the residue of $\zeta\left(s\mapsto H^sH^{-1}\right)$.

Example ($\mathrm{res}_0\,\zeta\left(s\mapsto H^sH^{-1}\right)$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$) $\zeta\left(s\mapsto H^sH^{-1}\right)(z)=2\zeta_R(1-z)$ shows that $\mathrm{res}_0\,\zeta\left(s\mapsto H^sH^{-1}\right)=-2\,\mathrm{res}_1\,\zeta_R=-2$. Also, using the Laurent expansion (Theorem 3.1) of $\zeta(A)$ for $A=\left(s\mapsto H^sH^{-1}\right)$, we obtain

$$
\mathrm{res}_0\,\zeta\left(s\mapsto H^sH^{-1}\right)=-\int_0^{2\pi}\int_{\partial B_{\mathbb{R}}}\frac{|\xi|^{-1}}{2\pi}\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}\ dx=-2.
$$

■

Furthermore, we can consider shifted fractional Laplacians which do not have singular amplitudes, that is, these are actually covered by the theory we have developed so far. They will also lead to the crucial observation that will help incorporate the case of singular amplitudes and, thus, justify the example of fractional Laplacians.

Example (shifted fractional Laplacians on $\mathbb{R}/_{2\pi\mathbb{Z}}$) Again, let $H:=\sqrt{|\Delta|}$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$, $h\in(0,1]$, and $G:=h+H$. Then,

$$
\zeta\left(s\mapsto G^{s+\alpha}\right)(z)=\sum_{n\in\mathbb{Z}}(h+|n|)^{z+\alpha}=2\sum_{n\in\mathbb{N}_0}(h+n)^{z+\alpha}-h^{z+\alpha}=2\zeta_H(-z-\alpha;h)-h^{z+\alpha}
$$

where $\zeta_H(z;h)$ denotes the Riemann-Hurwitz-ζ-function. In order to use our formalism above (Theorem 3.1), we will need to write $\xi\mapsto(h+|\xi|)^\alpha$ as a series of poly-homogeneous functions. Using

$$
(h+|\xi|)^\alpha=\sum_{k\in\mathbb{N}_0}\binom{\alpha}{k}|\xi|^{\alpha-k}h^k
$$

for $|\xi|\geq 1$ yields that the kernel of $G^{z+\alpha}$

$$
k_{G^{z+\alpha}}(x,y)=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{i(x-y-2\pi n)\xi}\frac{1}{2\pi}(h+|\xi|)^{z+\alpha}d\xi
$$

is, in fact, poly-log-homogeneous. For $\alpha=-1$, the critical term in zero is given by the $k=0$ term of $\sum_{k\in\mathbb{N}_0}\binom{\alpha}{k}|\xi|^{\alpha-k}h^k$, i.e.

$$
\mathrm{res}_0\,\zeta\left(s\mapsto G^{s-1}\right)=-\int_{\partial B_{\mathbb{R}}}|\xi|^{-1}\,d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)=-2.
$$

On the other hand,

$$\begin{aligned}\operatorname{res}_0 \zeta\left(s \mapsto G^{s-1}\right) &= \operatorname{res}_0\left(z \mapsto 2\zeta_H(-z+1;h) - h^{z+\alpha}\right) = 2\operatorname{res}_0\left(z \mapsto \zeta_H(-z+1;h)\right) \\ &= -2\operatorname{res}_0\left(z \mapsto \zeta_H(z-1;h)\right) = -2\operatorname{res}_1 \zeta_H(\cdot;h) = -2.\end{aligned}$$

For $\alpha \neq -1$ and $|\xi| \geq 1$,

$$(h+|\xi|)^\alpha = \sum_{k\in\mathbb{N}_0} \binom{\alpha}{k} h^k |\xi|^{\alpha-k}$$

implies $\alpha - k \in I_0$ if and only if $k = \alpha+1 \in \mathbb{N}_0$. However, since $\binom{\alpha}{\alpha+1} = 0$ for $\alpha \in \mathbb{N}_0$, we obtain $I_0 = \varnothing$ and

$$\begin{aligned}\zeta\left(s \mapsto G^{s+\alpha}\right)(0) &= \sum_{n\in\mathbb{Z}} \int_{-1}^1 e^{-2\pi i n\xi}(h+|\xi|)^\alpha d\xi \\ &\quad + \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}\setminus[-1,1]} e^{-2\pi i n\xi}(h+|\xi|)^\alpha d\xi \\ &\quad + \sum_{k\in\mathbb{N}_0} \frac{-1}{1+\alpha-k} \int_{\partial B_{\mathbb{R}}} \binom{\alpha}{k} h^k |\xi|^{\alpha-k} \,d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi) \\ &= \int_{-1}^1 (h+|\xi|)^\alpha d\xi - \sum_{k\in\mathbb{N}_0} \frac{2}{1+\alpha-k}\binom{\alpha}{k} h^k \\ &\quad + \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i n\xi}(h+|\xi|)^\alpha d\xi\end{aligned}$$

Observing

$$\begin{aligned}\int_{-1}^1 (h+|\xi|)^\alpha d\xi &= 2\int_0^1 (h+\xi)^\alpha d\xi = 2\int_h^{1+h} \xi^\alpha d\xi = \frac{2}{\alpha+1}\left((1+h)^{\alpha+1} - h^{\alpha+1}\right) \\ &= \frac{-2h^{\alpha+1}}{\alpha+1} + \frac{2}{\alpha+1}\sum_{k\in\mathbb{N}_0}\binom{\alpha+1}{k} h^k \\ &= \frac{-2h^{\alpha+1}}{\alpha+1} + 2\sum_{k\in\mathbb{N}_0} \frac{1}{\alpha-k+1}\binom{\alpha}{k} h^k\end{aligned}$$

leaves us with

$$\zeta\left(s \mapsto G^{s+\alpha}\right)(0) = \frac{-2h^{\alpha+1}}{\alpha+1} + \underbrace{\sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i n\xi}(h+|\xi|)^\alpha d\xi}_{\text{non-singular}}.$$

This is precisely what we expect since the principal part of $\zeta_H(z;h)$ near 1 is $\frac{h^{1-z}}{z-1}$ (cf. equation 3.1.1.10 in [17]), i.e.

$$\zeta\left(s \mapsto G^{s+\alpha}\right)(0) = 2\zeta_H(-z-\alpha;h) - h^{z+\alpha}$$

has principal part $2\frac{h^{1+\alpha}}{-\alpha-1}$.

Unfortunately, evaluating $\sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i n\xi}(h+|\xi|)^\alpha d\xi$ is a wee tricky. We will use that

$$\int_{\mathbb{R}} (h+|\xi|)^\alpha d\xi = 2\int_{\mathbb{R}_{\geq 0}} (h+\xi)^\alpha d\xi = 2\int_{\mathbb{R}_{\geq h}} \xi^\alpha d\xi = -\frac{2h^{\alpha+1}}{\alpha+1}$$

holds for $\Re\alpha < -1$ and, hence,

$$\zeta\left(s \mapsto G^{s+\alpha}\right)(0) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-2\pi i n\xi}(h+|\xi|)^\alpha d\xi$$

by meromorphic extension. Furthermore, we obtain

$$\begin{aligned}
\zeta\left(s\mapsto G^{s+\alpha}\right)(0) &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2\pi i n\xi}(h+|\xi|)^{\alpha}d\xi\\
&= \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}_{\ge 0}} e^{-2\pi i n\xi}(h+|\xi|)^{\alpha}d\xi + \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}_{<0}} e^{-2\pi i n\xi}(h+|\xi|)^{\alpha}d\xi\\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\int_{\mathbb{R}_{\ge h}} e^{-2\pi i n\xi}\xi^{\alpha}d\xi + \sum_{n\in\mathbb{Z}}\int_{-\infty}^{0-} e^{-2\pi i n\xi}(h-\xi)^{\alpha}d\xi\\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\int_{\mathbb{R}_{\ge h}} e^{-2\pi i n\xi}\xi^{\alpha}d\xi + \sum_{n\in\mathbb{Z}} -\int_{\infty}^{h+} e^{-2\pi i n(h-\xi)}\xi^{\alpha}d\xi\\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\int_{\mathbb{R}_{\ge h}} e^{-2\pi i n\xi}\xi^{\alpha}d\xi + \sum_{n\in\mathbb{Z}} e^{-2\pi i n h}\int_{\mathbb{R}_{>h}} e^{2\pi i n\xi}\xi^{\alpha}d\xi\\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\int_{\mathbb{R}_{\ge h}} e^{-2\pi i n\xi}\xi^{\alpha}d\xi + \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\int_{\mathbb{R}_{>h}} e^{-2\pi i n\xi}\xi^{\alpha}d\xi\\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\left(\int_{\mathbb{R}} e^{-2\pi i n\xi}1_{\mathbb{R}_{\ge h}}(\xi)\xi^{\alpha}d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi}1_{\mathbb{R}_{>h}}(\xi)\xi^{\alpha}d\xi\right).
\end{aligned}$$

For $\varepsilon\in(0,1)$ let

$$\varphi_\varepsilon(x) := \begin{cases} 0 & ,\ x\in\mathbb{R}_{\le h-\varepsilon}\\ \varepsilon^{-1}(x-h+\varepsilon) & ,\ x\in(h-\varepsilon,h)\\ 1 & ,\ x\in\mathbb{R}_{\ge h}\end{cases}$$

and

$$\psi_\varepsilon(x) := \begin{cases} 0 & ,\ x\in\mathbb{R}_{\le h}\\ \varepsilon^{-1}(x-h) & ,\ x\in(h,h+\varepsilon)\\ 1 & ,\ x\in\mathbb{R}_{\ge h+\varepsilon}\end{cases}.$$

Then

$$\begin{aligned}
\zeta\left(s\mapsto G^{s+\alpha}\right)(0) &= \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\left(\int_{\mathbb{R}} e^{-2\pi i n\xi}1_{\mathbb{R}_{\ge h}}(\xi)\xi^{\alpha}d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi}1_{\mathbb{R}_{>h}}(\xi)\xi^{\alpha}d\xi\right)\\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h}\lim_{\varepsilon\searrow 0}\left(\int_{\mathbb{R}} e^{-2\pi i n\xi}\varphi_\varepsilon(\xi)\xi^{\alpha}d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi}\psi_\varepsilon(\xi)\xi^{\alpha}d\xi\right)
\end{aligned}$$

can be evaluated using the Poisson summation formula on a lattice Λ (cf. Chapter VII.2 Theorem 2.4 in [19])

$$\sum_{\lambda\in\Lambda} f(x+\lambda) = \sum_{\lambda\in\Lambda}\mathcal{F}f(\lambda)e^{2\pi i\lambda x}$$

which yields (we can move $\lim_{\varepsilon\searrow 0}$ freely in and out of integrals and series due to meromorphic extension, dominated convergence, and since the series converges absolutely for $\Re(\alpha) < -1$)

$$\begin{aligned}
\zeta\left(s\mapsto G^{s+\alpha}\right)(0) &= \lim_{\varepsilon\searrow 0}\sum_{n\in\mathbb{Z}} e^{2\pi i n h}\left(\int_{\mathbb{R}} e^{-2\pi i n\xi}\varphi_\varepsilon(\xi)\xi^{\alpha}d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi}\psi_\varepsilon(\xi)\xi^{\alpha}d\xi\right)\\
&= \lim_{\varepsilon\searrow 0}\sum_{n\in\mathbb{Z}}\left(\varphi_\varepsilon(h+n)(h+n)^{\alpha} + \psi_\varepsilon(h+n)(h+n)^{\alpha}\right)\\
&= \lim_{\varepsilon\searrow 0}\left(\sum_{n\in\mathbb{N}_0}\varphi_\varepsilon(h+n)(h+n)^{\alpha} + \sum_{n\in\mathbb{N}}\psi_\varepsilon(h+n)(h+n)^{\alpha}\right)\\
&= \sum_{n\in\mathbb{N}_0}(h+n)^{\alpha} + \sum_{n\in\mathbb{N}}(h+n)^{\alpha}
\end{aligned}$$

$$=2\zeta_H(-\alpha;h)-h^\alpha.$$

Considering higher order derivatives, we obtain

$$\partial^m\zeta\left(s\mapsto G^sG^\alpha\right)(0)=2(-1)^m\partial^m\zeta_H(-\alpha;h)-h^\alpha(\ln h)^m$$

from the spectral ζ-function while the Laurent expansion yields

$$\begin{aligned}
&\partial^m\zeta\left(s\mapsto G^sG^\alpha\right)(0)\\
=&\sum_{n\in\mathbb{Z}}\int_{-1}^{1}e^{-2\pi in\xi}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}\setminus[-1,1]}e^{-2\pi in\xi}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&+\sum_{k\in\mathbb{N}_0}\sum_{j=0}^{m}\frac{(-1)^{j+1}j!\int_{\partial B_{\mathbb{R}}}\partial^{m-j}\left(\beta\mapsto\binom{\beta}{k}h^k|\xi|^{\beta-k}\right)(\alpha)\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)}{(\alpha-k+1)^{j+1}}\\
=&\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2\pi in\xi}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&-2\int_{\mathbb{R}_{\ge1}}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&+\partial^m\left(\beta\mapsto\sum_{k\in\mathbb{N}_0}\frac{-\int_{\partial B_{\mathbb{R}}}\binom{\beta}{k}h^k|\xi|^{\beta-k}\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)}{\beta-k+1}\right)(\alpha)\\
=&\partial^m\left(\beta\mapsto\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2\pi in\xi}(h+|\xi|)^\beta\ d\xi\right)(\alpha)\\
&-2\partial^m\left(\beta\mapsto\int_{\mathbb{R}_{\ge1}}(h+|\xi|)^\beta\ d\xi\right)(\alpha)+\partial^m\left(\beta\mapsto\sum_{k\in\mathbb{N}_0}\frac{-2\binom{\beta}{k}h^k}{\beta-k+1}\right)(\alpha)\\
=&\partial^m\left(\beta\mapsto2\zeta_H(-\beta;h)-h^\beta\right)(\alpha)\\
&-2\partial^m\left(\beta\mapsto-\frac{(1+h)^{\beta+1}}{\beta+1}\right)(\alpha)-2\partial^m\left(\beta\mapsto\frac{(1+h)^{\beta+1}}{\beta+1}\right)(\alpha)\\
=&2(-1)^m\partial^m\zeta_H(-\alpha;h)-h^\alpha(\ln h)^m.
\end{aligned}$$

■

5. Mollification of singular amplitudes

In this section we will address the fact that many applications consider amplitudes which are homogeneous on $\mathbb{R}^N\setminus\{0\}$. In particular for pseudo-differential operators, this does not add too many problems because we can use a cut-off function near zero and extend the symbol as a distribution to $\mathbb{R}^N$ (which is uniquely possible up to certain critical degrees of homogeneity which are related to the residues). Then, we are left with a Fourier transform of a compactly supported distribution, i.e. the corresponding kernel is continuous and we can take the trace. In the general Fourier Integral Operator case, the situation is more complicated. Hence, in this section, we will show that the Laurent expansion holds for such amplitudes, as well, and not just modulo trace-class operators. We will prove this result by showing that we can always find a sequence of "nice" families of operators such that their ζ-functions converge compactly.

In the previous section, our calculations of $\zeta\left(s \mapsto H^s H^\alpha\right)$ have been pushing the boundaries of our formula in the sense that the Laurent expansion of Fourier Integral Operators assumes integrability of all amplitudes $a(z)$ on $B_{\mathbb{R}^N}$. This is obviously not true for $a(z)(x,y,\xi) = |\xi|^{z+\alpha}$ (at least not for all $z \in \mathbb{C}$). Hence, we would have to consider the Laurent expansion in a more general version where we also allowed

$$z \mapsto \int_X \int_{B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} a(z)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)$$

to have a non-vanishing principal part.

However, we may use $\zeta\left(s \mapsto G^s G^\alpha\right)$ to justify the calculations as they are by taking the limit $h \searrow 0$ in $\zeta\left(s \mapsto G^s G^\alpha\right)$. In fact, it is possible to show

$$\lim_{h \searrow 0} \zeta\left(s \mapsto G^s G^\alpha\right) = \zeta\left(s \mapsto H^s H^\alpha\right) \qquad \text{compactly.}$$

Here, we regularized the kernel $a(z)(x,y,\xi) = |\xi|^z$ by adding an $h \in (0,1)$ yielding a perturbed amplitude $a_h(z)(x,y,\xi) = (h + |\xi|)^z$ which has no singularities. Showing that the limit $h \searrow 0$ exists, then, justifies our calculations. Using Vitali's theorem (cf. e.g. chapter 1 in [9]) we can largely generalize this approach.

Theorem 5.1 (Vitali)**.** *Let $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$, $f \in C^\infty(\Omega)^{\mathbb{N}}$ locally bounded, and let*

$$\{z \in \Omega;\ (f_n(z))_{n\in\mathbb{N}} \ \textit{converges}\}$$

have an accumulation point in Ω. Then, f is compactly convergent.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of gauged Fourier Integral Operators with C^∞-amplitudes and A a gauged Fourier Integral Operator whose amplitudes may contain singularities. Furthermore, let $A_n(z) \to A(z)$ for every z in the generalized sense (cf. Chapter IV in [11]). Let $d \in \mathbb{R}$ such that $\forall z \in \mathbb{C}:\ (\Re(z) < d \ \Rightarrow\ A(z)$ is of trace-class) and $\Omega := \mathbb{C}_{\Re(\cdot) < d-1}$. Then, for every $z \in \Omega$, $(A_n(z))_{n\in\mathbb{N}}$ is eventually a sequence of bounded operators and $A_n|_\Omega \to A|_\Omega$ converges pointwise in norm. Furthermore, let $(\lambda_k(z))_{k\in\mathbb{N}}$ be the sequence of eigenvalues of $A(z)$ counting multiplicities and $(\lambda_k(z) + h_k^n(z))_{k\in\mathbb{N}}$ be the sequence of eigenvalues of $A_n(z)$ counting multiplicities. Suppose that $h^n(z) := \sum_{k\in\mathbb{N}} |h_k^n(z)|$ exists and converges to zero for $z \in \Omega$. Then,

$$|\zeta(A_n)(z) - \zeta(A)(z)| = \left|\sum_{k\in\mathbb{N}} (\lambda_k(z) + h_k^n(z)) - \sum_{k\in\mathbb{N}} \lambda_k(z)\right| = \left|\sum_{k\in\mathbb{N}} h_k^n(z)\right| \le h^n(z) \to 0$$

for $z \in \Omega$ shows

$$\{z \in \Omega;\ (\zeta(A_n)(z))_{n\in\mathbb{N}} \ \text{converges}\} = \Omega.$$

Let $\tilde\Omega \subseteq \mathbb{C}$ be open and connected with $\Omega \subseteq \tilde\Omega$ such that all $\zeta(A_n)|_{\tilde\Omega}$ are holomorphic and $\left\{\zeta(A_n)|_{\tilde\Omega};\ n \in \mathbb{N}\right\}$ is locally bounded. Then,

$$\lim_{n\to\infty} \zeta(A_n)|_{\tilde\Omega} = \zeta(A)|_{\tilde\Omega}.$$

In particular, if h^n admits an analytic continuation to $\tilde\Omega$, then $\lim_{n\to\infty} \zeta(A_n)|_{\tilde\Omega} = \zeta(A)|_{\tilde\Omega}$.

Remark Note that $A_n(z) \to A(z)$ in the generalized sense implies that the $h_k^n(z)$ exist and for every k and z we have $\lim_{n\to\infty} h_k^n(z) \to 0$. However, in general, we will not have any uniform bound on them, let alone find an $h^n(z)$; cf. Section IV.3.5 in [11].

■

Definition 5.2. *Let A be an operator with purely discrete spectrum. For every $\lambda \in \sigma(A)$ let μ_λ be the multiplicity of λ. Then, we define the spectral ζ-function $\zeta_\sigma(A)$ to be the meromorphic extension of*

$$\zeta_\sigma(A)(s) := \sum_{\lambda \in \sigma(A) \smallsetminus \{0\}} \mu_\lambda \lambda^{-s}$$

and the spectral Θ-function $\Theta_\sigma(A)$

$$\forall t \in \mathbb{R}_{>0}: \ \Theta_\sigma(A)(t) := \sum_{\lambda \in \sigma(A)} \mu_\lambda \exp(-t\lambda)$$

if they exist.

Definition 5.3. *Let $T \in \mathbb{R}_{>0}$ and $\varphi \in C(\mathbb{R}_{>0})$. We define the upper Mellin transform as*

$$\mathcal{M}^T(\varphi)(s) := \int_{(0,T)} \varphi(t) t^{s-1} dt$$

and the lower Mellin transform

$$\mathcal{M}_T(\varphi)(s) := \int_{\mathbb{R}_{\geq T}} \varphi(t) t^{s-1} dt$$

(if the integrals exist). If both integrals exist and with non-empty intersection Ω of domains of holomorphy (that is, the maximal connected and open subset admitting an analytic continuation of the function), then we define the generalized Mellin transform of φ to be the meromorphic extension of

$$\mathcal{M}(\varphi) := \mathcal{M}^T(\varphi)|_\Omega + \mathcal{M}_T(\varphi)|_\Omega.$$

Example Let $\varphi(t) := t^\alpha$ for some $\alpha \in \mathbb{C}$. Then

$$\mathcal{M}^T(\varphi)(s) = \int_{(0,T)} t^{s+\alpha-1} dt = \frac{T^{s+\alpha}}{s+\alpha}$$

for $\Re(s) > \alpha$ extending to $\mathbb{C} \smallsetminus \{-\alpha\}$ and

$$\mathcal{M}_T(\varphi)(s) = \int_{\mathbb{R}_{\geq T}} t^{s+\alpha-1} dt = -\frac{T^{s+\alpha}}{s+\alpha}$$

for $\Re(s) < \alpha$ extending to $\mathbb{C} \smallsetminus \{-\alpha\}$. Hence, $\mathcal{M}(\varphi)$ exists with

$$\mathcal{M}(\varphi)(s) = \frac{T^{s+\alpha}}{s+\alpha} - \frac{T^{s+\alpha}}{s+\alpha} = 0$$

on $\mathbb{C} \smallsetminus \{-\alpha\}$, i.e. $\mathcal{M}(\varphi) = 0$.

■

Example Let $\lambda \in \mathbb{R}_{>0}$ and $s \in \mathbb{C}$ with $\Re(s) > 0$. Then

$$\int_{\mathbb{R}_{>0}} e^{-\lambda t} t^{s-1} dt = \int_{\mathbb{R}_{>0}} e^{-\tau} \tau^{s-1} \lambda^{-s} dt = \lambda^{-s} \Gamma(s)$$

shows that $\lambda \mapsto \int_{\mathbb{R}_{>0}} e^{-\lambda t} t^{s-1} dt$ extends analytically to $\mathbb{C} \smallsetminus \mathbb{R}_{\leq 0}$.

■

Example Let A be an operator with purely discrete spectrum. For every $\lambda \in \sigma(A)$ let μ_λ be the multiplicity of λ and $\Re(\lambda) \geq 0$. $\mathcal{M}(1) = 0$, then, implies

$$\begin{aligned}
\mathcal{M}\left(\Theta_\sigma(A)\right)(s) &= \sum_{\lambda \in \sigma(A)} \mu_\lambda \mathcal{M}\left(t \mapsto \exp(-t\lambda)\right)(s) \\
&= \sum_{\lambda \in \sigma(A) \smallsetminus \{0\}} \mu_\lambda \mathcal{M}\left(t \mapsto \exp(-t\lambda)\right)(s) \\
&= \sum_{\lambda \in \sigma(A) \smallsetminus \{0\}} \mu_\lambda \lambda^{-s} \Gamma(s)
\end{aligned}$$

$$=\zeta_\sigma(A)(s)\Gamma(s).$$

■

Lemma 5.4. $\lim_{h\searrow 0}\mathcal{M}\left(t\mapsto\exp(-th)\right)=\mathcal{M}(1)=0$ *compactly.*

Proof. For $\Re(s)>1$, we obtain

$$\begin{aligned}\frac{1}{\Gamma(s)}\mathcal{M}\left(t\mapsto\exp(-th)\right)(s)=&\frac{1}{\Gamma(s)}\int_{\mathbb{R}_{>0}}e^{-th}t^{s-1}dt\\ =&h^{-s}\\ =&\sum_{k\in\mathbb{N}_0}(k+h)^{-s}-\sum_{k\in\mathbb{N}_0}(k+1+h)^{-s}\\ =&\zeta_H(s;h)-\zeta_H(s;1+h).\end{aligned}$$

Hence,

$$\mathcal{M}\left(t\mapsto\exp(-th)\right)(s)=\Gamma(s)\zeta_H(s;h)-\Gamma(s)\zeta_H(s;1+h)$$

holds on $\mathbb{C}\setminus\mathbb{Z}_{\le 1}$. Furthermore, $\Gamma(s)\zeta_H(s;h)-\Gamma(s)\zeta_H(s;1+h)$ is locally bounded on $\mathbb{C}\setminus\mathbb{Z}_{\le 1}$ for $h\searrow 0$ which implies

$$\begin{aligned}\lim_{h\searrow 0}\mathcal{M}\left(t\mapsto\exp(-th)\right)(s)=&\lim_{h\searrow 0}\left(\Gamma(s)\zeta_H(s;h)-\Gamma(s)\zeta_H(s;1+h)\right)\\ =&\Gamma(s)\zeta_H(s;0)-\Gamma(s)\zeta_H(s;1)\\ =&\Gamma(s)\zeta_R(s)-\Gamma(s)\zeta_R(s)\\ =&0,\end{aligned}$$

i.e. $\lim_{h\searrow 0}\mathcal{M}\left(t\mapsto\exp(-th)\right)$ exists and vanishes on $\mathbb{C}\setminus\mathbb{Z}_{\le 1}$. Vitali's theorem, thence, proves the assertion.

□

Corollary 5.5. *Let A and A_h be operators with spectral ζ-functions. Let $\zeta_\sigma(A)$ be the meromorphic extension of $\sum_{k\in N}\lambda_k^{-s}$ for some $N\subseteq\mathbb{N}$ and $\zeta_\sigma(A_h)$ the meromorphic extension of $\sum_{j=1}^{n}\tilde h_j^{-s}+\sum_{k\in N}(\lambda_k+h_k)^{-s}$ where all $\tilde h_j\in\mathbb{R}_{>0}$. Suppose A_h converges to A in the generalized sense and the meromorphic extension f_h of $\sum_{k\in N}(\lambda_k+h_k)^{-s}$ is locally bounded and converges to $\zeta_\sigma(A)$ pointwise.*

Then, $\zeta_\sigma(A_h)$ converges to $\zeta_\sigma(A)$ compactly.

Proof. The assertion is a direct consequence of $\sum_{j=1}^{n}\tilde h_j^{-s}\to 0$ compactly (Lemma 5.4) and $f_h\to\zeta_\sigma(A)$ compactly (Vitali's theorem).

□

Proposition 5.6. *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution on $\mathbb{R}_{>0}\times M$ with I finite and α_0 regular. Then, $\zeta(\alpha)$ can be mollified.*

In particular,

$$\begin{aligned}\zeta(\alpha)(z)=&\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\ge 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times M}\\ &+\sum_{\iota\in I}\int_{(0,1)}r^{\dim M+d_\iota+z}(\ln r)^{l_\iota}dr\operatorname{res}\alpha_\iota(z)\end{aligned}$$

is the compact limit of

$$\begin{aligned}\zeta(\alpha_h)(z)=&\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\ge 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times M}\\ &+\sum_{\iota\in I}\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota}dr\operatorname{res}\alpha_\iota(z)\end{aligned}$$

for $h_\iota \in \mathbb{R}_{>0}$, $h_\iota \searrow 0$.

Proof. The part

$$\int_{\mathbb{R}_{>0}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$

creates no problems in the formalism used to obtain the Laurent expansion. Hence, we only need to consider

$$\begin{aligned}
&\sum_{\iota\in I}\int_{(0,1)} r^{\dim M+d_\iota+z}(\ln r)^{l_\iota} dr \operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\int_{(0,1)} \partial^{l_\iota}\left(s\mapsto r^{\dim M+d_\iota+s}\right)(z) dr \operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(s\mapsto \int_{(0,1)} r^{\dim M+d_\iota+s}dr\right)(z)\operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(\frac{1}{\dim M+d_\iota+s+1}\right)(z)\operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\frac{(-1)^{l_\iota} l_\iota!}{(\dim M+d_\iota+z+1)^{l_\iota+1}}\operatorname{res}\alpha_\iota(z).
\end{aligned}$$

Introducing $h_\iota \in \mathbb{R}_{>0}$ we obtain

$$\begin{aligned}
&\sum_{\iota\in I}\int_{(0,1)} (h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota} dr \operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\int_{(0,1)} \partial^{l_\iota}\left(s\mapsto (h_\iota+r)^{\dim M+d_\iota+s}\right)(z) dr \operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(s\mapsto \int_{(0,1)} (h_\iota+r)^{\dim M+d_\iota+s}dr\right)(z)\operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(s\mapsto \frac{(1+h_\iota)^{\dim M+d_\iota+s+1}-h_\iota^{\dim M+d_\iota+s+1}}{\dim M+d_\iota+s+1}\right)(z)\operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\operatorname{res}\alpha_\iota(z)\\
&-\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\operatorname{res}\alpha_\iota(z).
\end{aligned}$$

Since each of the $(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}$ is locally bounded for $h_\iota\to 0$ (taking derivatives in Lemma 5.4) and

$$(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\to\begin{cases}0 & j\neq l_\iota\\ 1 & j=l_\iota\end{cases}$$

for $h_\iota\to 0$, we obtain

$$\begin{aligned}
&\lim_{h_\iota\searrow 0}\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\frac{(-1)^{l_\iota} l_\iota!}{(\dim M+d_\iota+z+1)^{l_\iota+1}}\operatorname{res}\alpha_\iota(z)
\end{aligned}$$

compactly. Furthermore,

$$h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j} = h_\iota^{\dim M+d_\iota+z+1+j-l_\iota}(h_\iota\ln h_\iota)^{l_\iota-j}$$

being locally bounded for $h_\iota \to 0$ and converging to zero compactly shows

$$\begin{aligned}
&\zeta(\alpha_h)(z)\\
=&\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\
&+\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\operatorname{res}\alpha_\iota(z)\\
&-\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\operatorname{res}\alpha_\iota(z)\\
\to&\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\
&+\sum_{\iota\in I}\frac{(-1)^{l_\iota}l_\iota!}{(\dim M+d_\iota+z+1)^{l_\iota+1}}\operatorname{res}\alpha_\iota(z)\\
=&\zeta(\alpha)(z)
\end{aligned}$$

where the convergence is compact by Vitali's theorem.

□

Example (re-re-visiting $\zeta\left(s\mapsto H^sH^\alpha\right)$) Let $\Gamma\subseteq\mathbb{R}^N$ be a discrete group generated by a basis of $\mathbb{R}^N$, $|\Delta|$ the Dirichlet Laplacian on $\mathbb{R}^N$, δ the Dirichlet Laplacian on $\mathbb{R}^N/\Gamma$, and $H:=\sqrt{\delta}$. Then,

$$\zeta\left(s\mapsto H^s\right)(z)=\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^N}e^{-i\langle\gamma,\xi\rangle_{\ell_2(N)}}(2\pi)^{-N}\|\xi\|_{\ell_2(N)}^z\ d\xi$$

where

$$\sum_{\gamma\in\Gamma\setminus\{0\}}\int_{\mathbb{R}^N}e^{-i\langle\gamma,\xi\rangle_{\ell_2(N)}}(2\pi)^{-N}\|\xi\|_{\ell_2(N)}^z\ d\xi$$

is regular, i.e.

$$\alpha_0(z)(\xi)\hat{=}\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)\sum_{\gamma\in\Gamma\setminus\{0\}}e^{-i\langle\gamma,\xi\rangle_{\ell_2(N)}}(2\pi)^{-N}\|\xi\|_{\ell_2(N)}^z$$

and

$$\sum_{\iota\in I}\alpha_\iota(z)(\xi)=\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)(2\pi)^{-N}\|\xi\|_{\ell_2(N)}^z.$$

Hence, Proposition 5.6 is applicable.

■

In the following, we will use Abel's summation.

Lemma 5.7 (Abel's summation). *Let $a,b\in G^{\mathbb{N}}$ for some group G and $\forall n\in\mathbb{N}$: $B_n:=\sum_{k=1}^n b_k$. Then,*

$$\sum_{k=1}^n a_kb_k=a_{n+1}B_n+\sum_{k=1}^n(a_k-a_{k+1})B_k.$$

Proposition 5.8. *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution on $\mathbb{R}_{>0}\times M$ with $I\subseteq\mathbb{N}$, α_0 regular on $(0,1)\times M$,*

$$\alpha_\iota(z)(r,\xi)=r^{d_\iota+z}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\xi),$$

where $(\Re(d_\iota))_{\iota\in I}$ is bounded from above, each $\left(\frac{1}{\dim M+d_\iota+z+1}\right)_{\iota\in I}\in\ell_2(I)$, $(l_\iota)_{\iota\in I}\in\ell_\infty(I)$, $l:=\|(l_\iota)_{\iota\in I}\|_{\ell_\infty(I)}$, and each $\sum_{\iota\in I}\tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$. Then, $\zeta(\alpha)$ can be mollified.

In particular,

$$\zeta(\alpha)(z)=\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$
$$+\sum_{\iota\in I}\int_{(0,1)}r^{\dim M+d_\iota+z}(\ln r)^{l_\iota}dr\operatorname{res}\alpha_\iota(z)$$

is the compact limit of

$$\zeta(\alpha_h)(z)=\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$
$$+\sum_{\iota\in I}\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota}dr\operatorname{res}\alpha_\iota(z)$$

for $h:=(h_\iota)_{\iota\in I}\in\ell_\infty(I;\mathbb{R}_{>0})$ *and* $h\searrow 0$ *in* $\ell_\infty(I)$ *such that*

$$Z_\iota(z):=|\zeta_H(l-d_\iota-z;h_\iota)-\zeta_H(l-d_\iota-z;1+h_\iota)|$$

defines $(Z_\iota(z))_{\iota\in I}\in\ell_\infty(I)$ *which is bounded on an exhausting family of compacta as* $h\searrow 0$*.*

Proof. Proposition 5.6 yields the assertion for finite I. Hence, we may assume $I=\mathbb{N}$ without loss of generality. Furthermore, we only need to consider the part

$$\begin{aligned}A(h):=&\sum_{\iota\in I}\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota}dr\operatorname{res}\alpha_\iota(z)\\=&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\\&-\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j},\end{aligned}$$

i.e. show that it converges compactly to zero. Recall that $\sum_{\iota\in I}\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}$ converges absolutely and $|\dim M+d_\iota+z+1|\to\infty$ $(\iota\to\infty)$. Hence, we will assume, without loss of generality, $\forall\iota\in I:\ |\dim M+d_\iota+z+1|\geq 1$ (as there can only be finitely many with $|\dim M+d_\iota+z+1|<1$ which is handled by Proposition 5.6). Then, we observe (for $h_0:=\|h\|_{\ell_\infty(I)}<e-1$)

$$\begin{aligned}&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\right|\\\leq&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{j!|\operatorname{res}\alpha_\iota(z)|}{|\dim M+d_\iota+z+1|^{j+1}}\left|(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\right|\\\leq&l!\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{|\operatorname{res}\alpha_\iota(z)|}{|\dim M+d_\iota+z+1|}\left|(1+h_\iota)^{\dim M+d_\iota+z+1}\right|\underbrace{(\ln(1+h_0))^{l_\iota-j}}_{\leq 1\ (h_0<e-1)}\\\leq&l!\cdot l\sum_{\iota\in I}\frac{|\operatorname{res}\alpha_\iota(z)|}{|\dim M+d_\iota+z+1|}(1+h_0)^{\dim M+\Re(d_\iota+z)+1}\\\leq&l!\cdot l\underbrace{(1+h_0)^{\max\{\dim M+\Re(z)+1+\sup_{\iota\in I}\Re(d_\iota),0\}}}_{\to 1\ (h\searrow 0)}\sum_{\iota\in I}\frac{|\operatorname{res}\alpha_\iota(z)|}{|\dim M+d_\iota+z+1|}\end{aligned}$$

which is locally bounded by absolute convergence of $\sum_{\iota\in I}\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}$ and compact convergence of $(1+h_\iota)^{\dim M+d_\iota+z+1}$. Furthermore, we obtain (for $h_0\le e^{-1}$)

$$\begin{aligned}
&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\right|\\
\le&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{l!\left|\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}\left(h_\iota\ln h_\iota\right)^l\right|}{|\dim M+d_\iota+z+1|}h_\iota^{\dim M+1}\\
\le& l\cdot l!h_0^{\dim M+1}\sum_{\iota\in I}\frac{\left|\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}\left(h_\iota\ln h_\iota\right)^l\right|}{|\dim M+d_\iota+z+1|}.
\end{aligned}$$

Note that

$$|h_\iota\ln h_\iota|^l\to\begin{cases}1 & ,\ l=0\\ 0 & ,\ l\ne 0\end{cases}$$

for $h_\iota\to 0$, i.e. it suffices to show that

$$\sum_{\iota\in I}\frac{\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}}{\dim M+d_\iota+z+1}$$

converges absolutely. Since

$$\left|h_\iota^{d_\iota+z-l}\right|=|\zeta_H(l-d_\iota-z;h_\iota)-\zeta_H(l-d_\iota-z;1+h_\iota)|=Z_\iota(z)$$

holds (we can choose $(Z_\iota(z))_{\iota\in I}$ locally bounded because $z\mapsto\zeta_H(l-d_\iota-z;h_\iota)-\zeta_H(l-d_\iota-z;1+h_\iota)$ converges to zero compactly as $h_\iota\searrow 0)^8$, we observe

$$\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}}{\dim M+d_\iota+z+1}\right|\le\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|Z_\iota(z)$$

which is bounded by absolute convergence of $\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|$ and the assumed boundedness of $(Z_\iota(z))_{\iota\in I}$. Furthermore, local boundedness (with respect to z) follows from local boundedness of $\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|$ and Z_ι. Observing

$$\begin{aligned}
&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}\underbrace{(\ln(1+h_\iota))^{l_\iota-j}}_{\to\delta_{j,l_\iota}}\\
\to&\sum_{\iota\in I}\frac{(-1)^{l_\iota}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{l_\iota+1}}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\right|\\
=&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+2+j-l_\iota}(h_\iota\ln h_\iota)^{l_\iota-j}h_\iota\right|\\
\le& h\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+2+j-l_\iota}(h_\iota\ln h_\iota)^{l_\iota-j}\right|\\
\to& 0 \qquad\qquad (h\searrow 0)
\end{aligned}$$

---

[8]Since we have to construct a sequence $H\in\ell_\infty(I;\mathbb{R}_{>0})^{\mathbb{N}}$ where each element H_n is of the form h, it suffices to have uniform boundedness of $(Z_\iota)_{\iota\in I}$ on some compact set Ω_n for H_n and choose $(\Omega_n)_{n\in\mathbb{N}}$ to satisfy $\forall n\in\mathbb{N}\ \Omega_n\subseteq\Omega_{n+1}$ ad $\bigcup_{n\in\mathbb{N}}\Omega_n=\mathbb{C}\smallsetminus\{d_\iota;\ \iota\in I\}$.

for $\Re(\dim M + d_\iota + z + 2 - l) > 0$ shows

$$\begin{aligned}
A(h) &= \sum_{\iota\in I}\int_{(0,1)}(h_\iota + r)^{\dim M + d_\iota + z}(\ln(h_\iota + r))^{l_\iota}\,dr\,\operatorname{res}\alpha_\iota(z)\\
&= \sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}}(1+h_\iota)^{\dim M + d_\iota + z + 1}(\ln(1+h_\iota))^{l_\iota - j}\\
&\quad - \sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}}h_\iota^{\dim M + d_\iota + z + 1}(\ln h_\iota)^{l_\iota - j}\\
&\to \sum_{\iota\in I}\frac{(-1)^{l_\iota} l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{l_\iota + 1}}
\end{aligned}$$

compactly and, thus,

$$\zeta(\alpha_h) \to \zeta(\alpha)$$

compactly.

□

Theorem 5.9. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly*-log-*homogeneous distribution on $\mathbb{R}_{>0}\times M$ with $I \subseteq \mathbb{N}$, α_0 regular on $(0,1)\times M$,*

$$\alpha_\iota(z)(r,\xi) = r^{d_\iota + z}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\xi),$$

where $(\Re(d_\iota))_{\iota\in I}$ is bounded from above, each $\left(\frac{1}{\dim M + d_\iota + z + 1}\right)_{\iota\in I} \in \ell_2(I)$, and each $\sum_{\iota\in I}\tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$. Then, $\zeta(\alpha)$ can be mollified.

In particular,

$$\begin{aligned}
\zeta(\alpha)(z) &= \int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\
&\quad + \sum_{\iota\in I}\int_{(0,1)} r^{\dim M + d_\iota + z}(\ln r)^{l_\iota}\,dr\,\operatorname{res}\alpha_\iota(z)
\end{aligned}$$

is the compact limit of

$$\begin{aligned}
\zeta(\alpha_h)(z) &= \int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\
&\quad + \sum_{\iota\in I}\int_{(0,1)}(h_\iota + r)^{\dim M + d_\iota + z}(\ln(h_\iota + r))^{l_\iota}\,dr\,\operatorname{res}\alpha_\iota(z)
\end{aligned}$$

for $h := (h_\iota)_{\iota\in I} \in \ell_\infty(I;\mathbb{R}_{>0})$ and $h \searrow 0$ in $\ell_\infty(I)$ such that

$$Z_\iota(z) := l_\iota\sum_{j=0}^{l_\iota}|\zeta_H(l_\iota - j - d_\iota - z; h_\iota) - \zeta_H(l_\iota - j - d_\iota - z; 1 + h_\iota)|$$

is bounded on an exhausting family of compacta as $h \searrow 0$.

Proof. The proof works precisely as the proof of Proposition 5.8. The only difference is that we have to show local boundedness of

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}}(1+h_\iota)^{\dim M + d_\iota + z + 1}(\ln(1+h_\iota))^{l_\iota - j}$$

and

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}}h_\iota^{\dim M + d_\iota + z + 1}(\ln h_\iota)^{l_\iota - j}$$

since the estimates do not hold anymore. Since

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}$$

is a well-defined meromorphic function, it is locally bounded. Furthermore, $(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}$ can be chosen uniformly bounded on any half plane $\{z\in\mathbb{C};\ \Re(z)<r\}$ for any $r\in\mathbb{R}$, i.e. we can construct a sequence that is eventually uniformly convergent on any given compactum. Hence,

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}$$

is fine. Thus, choosing $|h_\iota\ln h_\iota|<1$ and $|\dim M+d_\iota+z+1|\ge1$ without loss of generality,

$$\begin{aligned}
&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\right|\\
&\le\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|l_\iota!\sum_{j=0}^{l_\iota}\left|h_\iota^{\dim M+d_\iota+z+1+j-l_\iota}\right|\left|(h_\iota\ln h_\iota)^{l_\iota-j}\right|\\
&\le\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|l_\iota!\,\|h\|_{\ell_\infty(I)}^{\dim M+1}\sum_{j=0}^{l_\iota}\left|h_\iota^{d_\iota+z+j-l_\iota}\right|\\
&\le\|h\|_{\ell_\infty(I)}^{\dim M+1}\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|Z_\iota(z)
\end{aligned}$$

completes the proof.

□

6. On structural singularities and the generalized Kontsevich-Vishik trace

In this section, we will discuss the integrals appearing in the Laurent coefficients. Most importantly, this will yield the generalized Kontsevich-Vishik density

$$\begin{aligned}
&\int_{B_{\mathbb{R}^N}(0,1)}e^{i\vartheta(x,x,\xi)}a(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)\\
&+\int_{\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N}}e^{i\vartheta(x,x,\xi)}a_0(0)(x,x,\xi)\ d\mathrm{vol}_{\mathbb{R}_{\ge1}\times\partial B_{\mathbb{R}^N}}(\xi)\ d\mathrm{vol}_X(x)\\
&+\sum_{\iota\in I\setminus I_0}\frac{-\int_{\partial B_{\mathbb{R}^N}}e^{i\vartheta(x,x,\xi)}a_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)}{N+d_\iota}\ d\mathrm{vol}_X(x),
\end{aligned}$$

as well as the fact that this density is globally defined in the $I_0=\varnothing$ case, that is in the absence of terms with critical degree of homogeneity. We will be able to calculate interesting examples by the end of section 7 leading up to (and including) Theorem 7.5.

Considering classical pseudo-differential operators it is common to start with the Kontsevich-Vishik trace which is constructed by removing those terms from the asymptotic expansion which have degree of homogeneity with real part greater than or equal to $-\dim X$ where X denotes the underlying manifold, i.e. if k is the kernel of the pseudo-differential operator, then the regularized kernel is given by

$$k_{KV}:=k-\sum_{j=0}^{N}k_{d-j}$$

where $d-j$ is the degree of homogeneity of the corresponding term in the expansion of the amplitude $a \sim \sum_{j\in\mathbb{N}_0} a_{d-j}$ and N sufficiently large. Then, $k_{KV} \in C(X \times X)$, i.e. $\int_X k_{KV}(x,x)d\mathrm{vol}_X(x)$ is well-defined. In other words, k_{KV} and α_0 play the same role and we would like to interpret $\zeta(\alpha_0)(0)$ as a generalized version of the Kontsevich-Vishik trace. The term $\sum_{j=0}^N \int_X k_{d-j}(x,x)d\mathrm{vol}_X(x)$ would, hence, be analogous to spinning off $\sum_{\iota\in I} \zeta(\alpha_\iota)(0)$. Unfortunately, we have to issue a couple of caveats.

(i) The observation above is fine if we are in local coordinates. However, when patching things together some of the terms in our Laurent expansion will not patch to global densities on X. This is no problem for Fourier Integral Operators, per se, as they are simply defined as a sum of local representations and in each of these representations the Laurent expansion holds. It will become a problem if we want to write down formulae in terms of kernels, though (especially if we require local terms to patch together defining densities globally).

(ii) Since $\mathcal{F}(a_{d-j}(x,y,\cdot))(z)$ is homogeneous of degree $-\dim X - d + j$ (where $\mathcal{F}$ denotes the Fourier transform), we obtain $\mathcal{F}(a_{d-j}(x,y,\cdot))(0) = 0$ for $d-j < -\dim X$, i.e. $k_{d-j}(x,x) = \lim_{y\to x} k_{d-j}(x,y) = \lim_{y\to x} \mathcal{F}(a_{d-j}(x,y,\cdot))(y-x) = \mathcal{F}(a_{d-j}(x,x,\cdot))(0)$. Thus, $k_{KV}(x,x)$ is independent of N.

However, this property does not extend to α_0 as we can easily construct a counter-example. Let $a(x,y,\xi)$ be homogeneous of degree $d < -n$ in the third argument and the phase function $\vartheta(x,y,\xi) = -\langle \Theta(x,y), \xi\rangle_{\ell_2(n)}$ such that $\Theta(x,x)$ has no zeros. Then,

$$k(x,y) = \int_{\mathbb{R}^n} e^{-i\langle\Theta(x,y),\xi\rangle_{\ell_2(n)}} a(x,y,\xi)d\xi = \mathcal{F}(a(x,y,\cdot))(\Theta(x,y))$$

shows that $k(x,x)$ is well-defined and continuous. Furthermore, since $\mathcal{F}(a(x,y,\cdot))$ is homogeneous, $k(x,x)$ vanishing implies $\mathcal{F}(a(x,y,\cdot)) = 0$ on $\{r\Theta(x,x);\ r \in \mathbb{R}_{>0}\}$.

On the other hand, for pseudo-differential operators the terms a_{d-j} with $d-j = -\dim X$ define a global density on the manifold giving rise to the residue trace. If this extends to poly-log-homogeneous distributions, then we obtain the residue trace globally from $\sum_{\iota\in I_0} \alpha_\iota$. Furthermore, this would imply that

$$\mathfrak{fp}_0\alpha = \alpha - \sum_{\iota\in I_0} \alpha_\iota$$

induces a global density, if α does and the contributions of the α_ι for $\iota \in I_0$ to the constant term Laurent coefficient vanish (in particular if $I_0 = \varnothing$), which allows us to interpret $\zeta(\mathfrak{fp}_0\alpha)(0)$ as the generalization of the Kontsevich-Vishik trace.

This, of course, needs to be interpreted in a gauged sense. $\zeta(\mathfrak{fp}_0\alpha)(0)$ corresponds to the kernel $k(x,y) - k_{d-j}(x,y)$ where $d-j = -\dim X$. Hence, all terms k_{d-j} with $j \in \mathbb{N}_{0,<d+\dim X}$ still appear in $\mathfrak{fp}_0\alpha$ but not in k_{KV}. Since $\zeta(\mathfrak{fp}_0\alpha)(0)$ is but constructed by gauging, we should do the same for k_{d-j}, i.e. consider k_{d-j+z} which is continuous for $\Re(z)$ sufficiently small and vanishes along the diagonal. Therefore,

$$\zeta(\mathfrak{fp}_0\alpha)(0) = \int_X k_{KV}(x,x)d\mathrm{vol}_X(x).$$

holds in the regularized sense; particularly so since Corollary 1.8 guarantees that $\zeta(\mathfrak{fp}_0\alpha)(0)$ is independent of the gauge. In other words, the objective is to show that

$$\begin{aligned}\sum_\chi \operatorname{res}\alpha^\chi(0) &= \sum_\chi \left\langle \int_{\partial B_{\mathbb{R}^N}} \tilde{\alpha}^\chi(0) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}, f \right\rangle \\ &= \sum_\chi \left\langle P \int_{\partial B_{\mathbb{R}^N}} \tilde{\alpha}^\chi(0) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}, \delta_0 \right\rangle \\ &= \sum_\chi \left\langle \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} \tilde{\alpha}^\chi(0)(x,y,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi), \delta_0 \right\rangle\end{aligned}$$

is globally well-defined ($\sum_\chi$ denotes a partition of unity and P is a suitable pseudo-differential operator) if the a^χ are log-homogeneous with degree of homogeneity $-N$.

At this point, we return to the fact that we can find a representation

$$\int_{\mathbb{R}^{2\dim X} \setminus B_{\mathbb{R}^{2\dim X}}} e^{i\langle (x,y),\xi\rangle_{\ell_2(2\dim X)}} \hat{a}(\xi)\ d\mathrm{vol}_{\mathbb{R}^{2\dim X} \setminus B_{\mathbb{R}^{2\dim X}}}(\xi)$$

of

$$\int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi)\ d\mathrm{vol}_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}}(\xi)$$

where $\hat{a}$ is poly-log-homogeneous with degree of homogeneity $-2\dim X$ and logarithmic order l if a has degree of homogeneity $-N$ and logarithmic order l. Thus, we want to show that the locally defined

$$\int_{\partial B_{\mathbb{R}^{2\dim X}}} e^{i\langle (x,y),\xi\rangle_{\ell_2(2\dim X)}} \tilde{a}^\chi(\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi)$$

patch together if a^χ is log-homogeneous with degree of homogeneity $-2\dim X$.

Let φ be a suitable test function, and

$$\int_{X^2} \int_{\mathbb{R}^{2\dim X}} e^{i\langle (x,y),\xi\rangle_{\ell_2(2\dim X)}} a(x,y,\xi)\varphi(x,y)\ d\xi\ d\mathrm{vol}_{X^2}(x,y)$$

and

$$\int_{X^2} \int_{\mathbb{R}^{2\dim X}} e^{i\vartheta(x,y,\xi)} a^\chi(x,y,\xi)\varphi(x,y)\ d\xi\ d\mathrm{vol}_{X^2}(x,y)$$

be two representations of $\langle u, f\rangle$ where ϑ is another linear phase function. Proposition 2.4.1 in [8] warrants the existence of a C^∞-map Θ taking values $\Theta(x,y) \in GL(\mathbb{R}^{2\dim X})$ such that

$$\vartheta(x,y,\xi) = \langle (x,y), \Theta(x,y)\xi\rangle_{\ell_2(2\dim X)}$$

holds. Hence,

$$\begin{aligned}&\int_{X^2} \int_{\mathbb{R}^{2\dim X}} e^{i\vartheta(x,y,\xi)} a^\chi(x,y,\xi)\varphi(x,y)\ d\xi\ d\mathrm{vol}_{X^2}(x,y) \\ =& \int_{X^2} \int_{\mathbb{R}^{2\dim X}} e^{i\langle x,\Theta(x)\xi\rangle_{\ell_2(2\dim X)}} a^\chi(x,\xi)\varphi(x)\ d\xi\ d\mathrm{vol}_{X^2}(x) \\ =& \int_{X^2} \int_{\mathbb{R}^{2\dim X}} e^{i\langle x,\xi\rangle_{\ell_2(2\dim X)}} a^\chi(x,\Theta(x)^{-1}\xi)\varphi(x) \left|\det\Theta(x)^{-1}\right|\ d\xi\ d\mathrm{vol}_{X^2}(x).\end{aligned}$$

In other words, the amplitude a transforms into $a^\chi(x,\Theta(x)^{-1}\xi)\left|\det\Theta(x)^{-1}\right|$ for some C^∞-function Θ taking values in $GL(\mathbb{R}^{2\dim X})$, more precisely

$$a(x,y,\xi) = a^\chi(\chi(x,y),\Theta(x,y)^{-1}\xi)\left|\det\Theta(x,y)^{-1}\right| \left|\det\chi'(x,y)\right|$$

for some diffeomorphism χ, and we need to show

$$\begin{aligned}\operatorname{res}\alpha(0) &= \int_{\partial B_{\mathbb{R}^{2\dim X}}} \hat{a}(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi)\\ &= \int_{\partial B_{\mathbb{R}^{2\dim X}}} \hat{a}^\chi(\Theta^{-1}\xi)\left|\det\Theta^{-1}\right| d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi)\\ &\overset{?}{=} \int_{\partial B_{\mathbb{R}^{2\dim X}}} \hat{a}^\chi(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi)\\ &= \operatorname{res}\alpha^\chi(0)\end{aligned}$$

where $\hat{a}$ and $\hat{a}^\chi$ are the restrictions to the polyhomogeneous part of α and α^χ, i.e. $\hat{a}(r\xi) = r^{d_\iota}\tilde{\alpha}(\xi)$.

Lemma 6.1. *Let $a \in C(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree d, $k \in \mathbb{N}_0$, $z \in \mathbb{C}$, and $T \in GL(\mathbb{R}^n)$. Then*

$$\begin{aligned}&\int_{\partial B_{\mathbb{R}^n}} a(T\xi)\left\|T\xi\right\|^z (\ln\|T\xi\|)^k d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)\\ =&\frac{(-1)^k}{|\det T|}\int_{\partial B_{\mathbb{R}^n}} a(\xi)\left\|T^{-1}\xi\right\|^{-n-d-z}\left(\ln\left\|T^{-1}\xi\right\|\right)^k d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi).\end{aligned}$$

This lemma (cf. e.g. equation (2.13) in [14] or Lemma 2.20 in [16] with minimal changes to the proof), and the fact that $\tilde{a}$ is the restriction of a homogeneous function with degree of homogeneity $-N$ if a is log-homogeneous with degree of homogeneity $-N$, yield (using $N = 2\dim X$, a suitable $U \subseteq_{\mathrm{open}} \mathbb{R}^N$, a diffeomorphism $\chi\colon U \to \chi[U]$, and a $\varphi \in C_c^\infty(\chi[U])$)

$$\begin{aligned}&\int_U\int_{\partial B_{\mathbb{R}^N}} \tilde{a}(x,\xi)\varphi(\chi(x)) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx\\ =&\int_U\int_{\partial B_{\mathbb{R}^N}} \tilde{a}^\chi(\chi(x),\Theta(x)^{-1}\xi)\left|\det\Theta(x)^{-1}\right||\det\chi'(x)|\varphi(\chi(x)) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx\\ =&\int_U\int_{\partial B_{\mathbb{R}^N}} \tilde{a}^\chi(\chi(x),\Theta(x)^{-1}\xi)\left|\det\Theta(x)^{-1}\right||\det\chi'(x)|\varphi(\chi(x)) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx\\ =&\int_U\left|\det\Theta(x)^{-1}\right|\int_{\partial B_{\mathbb{R}^N}} \tilde{a}^\chi(\chi(x),\Theta(x)^{-1}\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)|\det\chi'(x)|\varphi(\chi(x))dx\\ =&\int_U\int_{\partial B_{\mathbb{R}^N}} \tilde{a}^\chi(\chi(x),\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)|\det\chi'(x)|\varphi(\chi(x))dx\\ =&\int_{\chi[U]}\int_{\partial B_{\mathbb{R}^N}} \tilde{a}^\chi(x,\xi)\varphi(x) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx,\end{aligned}$$

i.e. the following theorem.

Theorem 6.2. $\operatorname{res}\langle u,f\rangle = \operatorname{res}\alpha(0) = \int_{\partial B_{\mathbb{R}^N}} \tilde{\alpha}(0) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ *is form-invariant under change of coordinates if $\alpha(0)$ has degree of homogeneity $-N$.*

In particular, $\sum_\chi\sum_{\iota\in I_0^\chi}\operatorname{res}\alpha_\iota^\chi(0)$ induces a global density and $\sum_\chi\zeta\left(\mathfrak{fp}_0\alpha^\chi\right)(0)$ induces a globally defined density provided $\sum_\chi\sum_{\iota\in I_0^\chi}\partial\operatorname{res}\alpha_\iota^\chi(0)$ vanishes.

Remark Note that this means that if a is polyhomogeneous and ι_0 is the index such that a_{ι_0} is homogeneous of degree $-N$, then

$$\sum_{\iota\in I_0}\int_X\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}a_\iota(x,x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) d\mathrm{vol}_X(x)$$

$$= \int_X \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} a_{\iota_0}(x,x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) d\mathrm{vol}_X(x).$$

This, of course, extends to higher order residues

$$\int_X \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} a_{\iota}(x,x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) d\mathrm{vol}_X(x).$$

with $\iota \in I_0$ and $l_\iota > 0$; this generalizes Corollary 4.8 in [14] on the residue traces for log-polyhomogeneous pseudo-differential operators.

■

Uniqueness of the residue trace, then, directly implies the following proposition.

Proposition 6.3. *Let $a \sim \sum_{j\in\mathbb{N}_0} a_{m-j}$ be the amplitude of a Fourier Integral Operator where $m \in \mathbb{Z}$ and a_{m-j} is homogeneous of degree $m-j$. If the residue trace is the (projectively) unique non-trivial continuous trace, then none of the $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a_{m-j}(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$ with $m-j \neq -N$ can define a global density, in general, unless they are trivial (i.e. vanish constantly).*

In particular, removing non-trivial terms from $\zeta(\mathfrak{fp}_0\alpha)$ will, in general, destroy global well-definedness of the induced density.

Now, we may ask when the residue vanishes. As a first result we obtain the well-known fact that the residue trace vanishes for odd-class operators on odd-dimensional manifolds.

Observation 6.4. *Let $\alpha(-\xi) = -\alpha(\xi)$. Then,* $\operatorname{res}\alpha = \int_{\partial B_{\mathbb{R}^N}} \alpha(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) = 0$.

Note that the property $\alpha(-\xi) = -\alpha(\xi)$ is invariant under change of linear phase functions with the same "N". Choosing non-linear phase functions or changing N might destroy this property. In fact, having phase functions with $\vartheta(-\xi) = -\vartheta(\xi)$ will yield

$$\operatorname{res}(a,\vartheta) = \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(\xi)} a(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) = -\left(\operatorname{res}(a,\vartheta)\right)^*,$$

i.e. $\Re\left(\operatorname{res}(a,\vartheta)\right) = 0$ but not necessarily $\Im\left(\operatorname{res}(a,\vartheta)\right) = 0$.

On the other hand, if $N = 1$, then

$$\int_{\partial B_{\mathbb{R}}} \alpha(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi) = \alpha(1) + \alpha(-1)$$

shows that $\operatorname{res}\alpha$ vanishes if and only if α is odd. Equivalently, we obtain

$$\int_{\partial B_{\mathbb{R}}} e^{i\vartheta(x,\xi)} a(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi) = e^{i\vartheta(x,1)} a(x,1) + e^{i\vartheta(x,-1)} a(x,-1).$$

Note, this implies there are two residue traces for $N = 1$; namely, $\alpha_{-1}(1)$ and $\alpha_{-1}(-1)$.

For $N > 1$, the de Rham co-homology of $\partial B_{\mathbb{R}^N}$ is given by

$$\forall k \in \mathbb{N}_0:\ H^k_{\mathrm{dR}}(\partial B_{\mathbb{R}^N}) \cong \begin{cases} \mathbb{R} & , k \in \{0, N-1\} \\ 0 & , k \in \mathbb{N} \setminus \{N-1\} \end{cases}.$$

In other words, there exists $\omega_0 \in \Omega^{N-1}(\partial B_{\mathbb{R}^N}, \mathbb{C})$ such that $\int_{\partial B_{\mathbb{R}^N}} \omega_0 = 1$ and

$$\forall \omega \in \Omega^{N-1}(\partial B_{\mathbb{R}^N}, \mathbb{C})\ \exists c \in \mathbb{C}\ \exists \tilde\omega \in \Omega^{N-2}(\partial B_{\mathbb{R}^N}, \mathbb{C}):\ \omega = c\omega_0 + d\tilde\omega.$$

Thus, we obtain the following statements.

(i) $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) = 0$ if and only if the differential form $e^{i\vartheta(x,\cdot)} a(x,\cdot) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ is exact.

(ii) $\Re\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right) = 0$ if and only if the differential form $\cos(\vartheta(x,\cdot))\, a(x,\cdot) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ is exact.

(iii) $\Im\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right) = 0$ if and only if the differential form $\sin(\vartheta(x,\cdot))\, a(x,\cdot) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ is exact.

Remark Since we are integrating $\dim M$-forms over a manifold M, we assume that all manifolds are orientable as we can only integrate pseudo-$\dim M$-forms if M is non-orientable. So far everything can be re-formulated for pseudo-forms and, thus, on non-orientable manifolds. From this point onwards, though, statements will need orientability; in particular with respect to uniqueness of residue traces and the commutator structure since

$$H_{\mathrm{dR}}^{\dim M}(M) \cong \begin{cases} \mathbb{R} & , M \text{ orientable, connected} \\ 0 & , M \text{ non-orientable, connected} \end{cases}.$$

■

Definition 6.5. *Let A be a poly-homogeneous Fourier Integral Operator on a compact manifold X and $\operatorname{res}_0 \zeta(A)$ be locally given by*

$$\int_X \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a(x,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ d\mathrm{vol}_X(x).$$

Then, we call the $(N-1+\dim X)$-form $\varrho(A)$ on $X \times \partial B_{\mathbb{R}^N}$ locally defined as

$$\varrho(A) := \exp\circ(i\vartheta) \cdot a\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}$$

the residue form of A (in other words, $*\varrho(A) = e^{i\vartheta} a$ where $*$ denotes the Hodge-$*$-operator).

Proposition 6.6. *Let $Y \subseteq X$ be a connected component. Then, $\int_{Y\times\partial B_{\mathbb{R}^N}} \varrho(A) = 0$ if and only if $\varrho(A)$ is exact on $Y \times \partial B_{\mathbb{R}^N}$.*

More precisely, let $X = Y_1 \uplus \ldots \uplus Y_k$ be composed of finitely many connected components ($\uplus$ denotes the disjoint union) and let $\varrho(A)|_{Y_j\times\partial B_{\mathbb{R}^N}} = c_j\omega_j + d\tilde{\omega}_j$ be the corresponding decompositions of $\varrho(A)$ with $\omega_j = \mathrm{vol}_{Y_j\times\partial B_{\mathbb{R}^N}}(Y_j \times \partial B_{\mathbb{R}^N})^{-1} d\mathrm{vol}_{Y_j\times\partial B_{\mathbb{R}^N}}$. Then,

$$\int_{X\times\partial B_{\mathbb{R}^N}} \varrho(A) = \sum_{j=1}^{k} c_j.$$

Using the Hodge-$*$-operator $*$, the co-derivative $d^* := (-1)^{N_X(N_X-1)+1} * d *$ with $N_X := N + \dim X - 1$, as well as

$$\begin{aligned} \varrho(A) = d\omega \ \Leftrightarrow\ e^{i\vartheta} a &= * d\omega \\ &= * d * (-1)^{N_X-1} * \omega \\ &= d^* (-1)^{N_X(N_X-1)+1} (-1)^{N_X-1} * \omega \\ &= d^* \left((-1)^{N_X^2} * \omega\right), \end{aligned}$$

and the divergence $\operatorname{div} F = *d*F^\flat = (-1)^{N_X(N_X-1)+1} d^* F^\flat$ with the musical isomorphism

$$\cdot^\flat : T\left(X \times \partial B_{\mathbb{R}^N}\right) \to T^*\left(X \times \partial B_{\mathbb{R}^N}\right);\ \sum_i F_i \partial_i \mapsto \sum_i F_i dx_i,$$

we can re-formulate Proposition 6.6.

Theorem 6.7. *Let X be connected. Then, the following are equivalent.*

(i) $\int_X \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a(x,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ d\mathrm{vol}_X(x) = 0$.

(ii) *There exists an $(N + \dim X - 2)$-form ω on $X \times \partial B_{\mathbb{R}^N}$ such that $d\omega = e^{i\vartheta} a\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}$ locally.*

(iii) *There exists a 1-form ω on $X \times \partial B_{\mathbb{R}^N}$ such that $d^*\omega = e^{i\vartheta} a$ locally.*

(iv) *There exists a vector field F on $X \times \partial B_{\mathbb{R}^N}$ such that $\operatorname{div} F = e^{i\vartheta} a$ locally.*

Remark These results hold if we replace $\partial B_{\mathbb{R}^N}$ by any other connected manifold M and consider the residue terms $\operatorname{res}\alpha = \int_M \hat{\alpha} d\mathrm{vol}_M$ for poly-log-homogeneous distributions. In particular, we obtain $\operatorname{res}\alpha = 0$ if and only if there exists a vector field F on M such that $\hat{\alpha} = \operatorname{div} F$.

■

Remark *Condition (iv) can be extended to* $X \times \left(\mathbb{R}^N \setminus \{0\}\right)$. Let $M := X \times \partial B_{\mathbb{R}^N}$, $(g_i)_i$ the local frame in which $e^{i\vartheta} a$ is given by α, and $(g^i)_i$ the dual frame. Let $\tilde{M} := \mathbb{R}_{>0} \times M \cong X \times \left(\mathbb{R}^N \setminus \{0\}\right)$ and the metric tensor is of the form

$$\tilde{g}(r,\xi) = \begin{pmatrix} 1 & 0 \\ 0 & r^{2\dim M} g(\xi) \end{pmatrix},$$

i.e. $d\mathrm{vol}_{\tilde{M}}(r,\xi) = \sqrt{\det \tilde{g}(r,\xi)} dr \wedge d\xi = r^{\dim M} \sqrt{\det g(\xi)} dr \wedge d\xi = r^{\dim M} dr \wedge d\mathrm{vol}_M(\xi)$. Let F be a vector field on M and $\tilde{F}$ be a vector field on $\tilde{M}$. Then,

$$\begin{aligned} \operatorname{div} F(\xi) &= \operatorname{tr} \operatorname{grad} F(\xi) = \operatorname{tr} \sum_{j=1}^{\dim M} \sum_{i=1}^{\dim M} \partial_j F_i(\xi) g^j(\xi) \otimes g^i(\xi) \\ &= \sum_{j=1}^{\dim M} \sum_{i=1}^{\dim M} \partial_j F_i(\xi) g^{ji}(\xi) \end{aligned}$$

and

$$\begin{aligned} \operatorname{div} \tilde{F}(r,\xi) &= \operatorname{tr} \sum_{j=0}^{\dim M} \sum_{i=0}^{\dim M} \partial_j \tilde{F}_i(r,\xi) \tilde{g}^j \otimes \tilde{g}^i \\ &= \partial_0 \tilde{F}_0(r,\xi) + r^{2\dim M} \sum_{j=1}^{\dim M} \sum_{i=1}^{\dim M} \partial_j \tilde{F}_i(r,\xi) g^{ji}(\xi). \end{aligned}$$

In other words, we obtain $\operatorname{div} \tilde{F}(1,\xi) = \operatorname{div} F(\xi)$ if $\partial_0 \tilde{F}_0(1,\xi) = 0$ and $\partial_j \tilde{F}_i(1,\xi) = \partial_j F_i(\xi)$. On the other hand, we want $\operatorname{div} F(\xi) = \tilde{\alpha}(\xi)$ and $\operatorname{div} \tilde{F}(r,\xi) = f(r)\tilde{\alpha}(\xi)$ with $f(1) = 1$. Choosing $\tilde{F}_0 = 0$ and $\tilde{F}_i(r,\xi) = f(r) F_i(\xi)$ implies $\operatorname{div} \tilde{F}(r,\xi) = f(r)\tilde{\alpha}(\xi)$ and $\operatorname{div} \tilde{F}(1,\xi) = \operatorname{div} F(\xi)$.

Thus, knowing (iv) we can construct a vector field $\tilde{F}$ such that $e^{i\vartheta} = \operatorname{div} \tilde{F}$ on $X \times \left(\mathbb{R}^N \setminus \{0\}\right)$ and $\tilde{F}$ satisfies the conditions above. Conversely, if $\tilde{F}$ has the described properties, then $\tilde{F}|_{X\times\partial B_{\mathbb{R}^N}}$ satisfies (iv).

■

At this point, using the framework of gauged poly-log-homogeneous distributions, we can follow the lines of Theorem 1.1 in [4] to obtain the following theorem (Theorem 1.2 in [4]) which we state here for completeness.

Theorem 6.8. *Let $\mathcal{A}_\Gamma$ be an algebra of classical Fourier Integral Operators associated with the canonical relation Γ such that the twisted relation Γ' ($A \in \mathcal{A}_\Gamma \Leftrightarrow k_A \in I(X^2, \Gamma')$) has clean and connected intersection with the co-normal bundle of diagonal in X^2. Then, the residue-trace of $A \in \mathcal{A}_\Gamma$ vanishes if and only if A is a smoothing operator plus a sum of commutators $[P_i, A_i]$ where the P_i are pseudo-differential operators and the $A_i \in \mathcal{A}_\Gamma$.*

Guillemin also proved the following (more general) version of Theorem 6.8 (cf. Proposition 4.11 in [5]).

Proposition 6.9. *Let Γ be connected. Then, the commutator of $\mathcal{A}_\Gamma$ is of co-dimension one in $\mathcal{A}_\Gamma$ modulo smoothing operators.*

Hence, $\operatorname{res}_0 \circ \zeta$ is either zero or the unique trace on $\mathcal{A}_\Gamma$ up to a constant factor. Regarding the trace of smoothing operators, Theorems A.1 and A.2 in [5] yield the commutator structure of smoothing operators (the following two definitions, the theorem, and the remark can all be found in the appendix of [5]).

Definition 6.10. *Let H be a separable Hilbert space and $e := (e_i)_{i\in\mathbb{N}}$ an orthonormal basis of H. An operator $A \in L(H)$ is called smoothing with respect to e if and only if*

$$\forall n \in \mathbb{N}\ \exists c \in \mathbb{R} :\ |\langle Ae_i, e_j\rangle_H| \le c(i+j)^{-n}.$$

Definition 6.11. *Let H be a separable Hilbert space, e an orthonormal basis, $\Omega \subseteq_{\text{open}} \mathbb{K}^n$ with $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and $A \in L(H)^\Omega$ such that each $A(s)$ is smoothing with respect to e. Then, A is said to be smooth/holomorphic if and only if all $s \mapsto \langle A(s)e_i, e_j\rangle_H$ are $C^\infty(\Omega)$.*

Theorem 6.12.
(i) *If A is smoothing with respect to e and $\operatorname{tr} A = 0$, then A can be written as a finite sum of commutators $[B_i, C_i]$ where the B_i and C_i are smoothing with respect to e.*
(ii) *If a family $A \in L(H)^\Omega$ of smoothing operators is smooth/holomorphic, then A can be written as a finite sum of commutators $s \mapsto [B_i(s), C_i]$ on every compact $K \subseteq \Omega$ where the $B_i(s)$ and C_i are smoothing, and the B_i are smooth/holomorphic.*

Remark
(i) Let X be a compact Riemannian manifold, $H = L_2(X)$, and e the family of eigenfunctions of the Laplacian on X. An operator $A \in L(L_2(X))$ is smoothing with respect to e if it is smoothing with respect to the Sobolev norms.
(ii) Let $H = L_2(\mathbb{R}^n)$ and e the family of Hermite functions. An operator $A \in L(H)$ is e-smoothing if it is smoothing with respect to the Schwartz semi-norms.

■

These theorems yield the following table assuming that the (unique) residue trace $\operatorname{res}_0 \circ\zeta$ is non-trivial and $\mathcal{A}_\Gamma = \langle\mathfrak{A}\rangle + \langle[\mathcal{A}_\Gamma, \mathcal{A}_\Gamma]\rangle + \{\text{smoothing operators}\}$ for some $\mathfrak{A} \in \mathcal{A}_\Gamma$ with $\operatorname{res}_0 \zeta(\mathfrak{A}) \ne 0$.

$I_0 \ne \varnothing$		$I_0 = \varnothing$	
$\operatorname{res}_0 \zeta(A) \ne 0$	$\operatorname{res}_0 \zeta(A) = 0$	$\zeta(A)(0) \ne 0$	$\zeta(A)(0) = 0$
$A = \alpha\mathfrak{A} + S + \sum_{i=1}^k C_i$ $C_i \in [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma]$ $\alpha = (\operatorname{res}_0 \zeta(\mathfrak{A}))^{-1} \operatorname{res}_0 \zeta(A)$ S smoothing	$A = S + \sum_{i=1}^k C_i$ $C_i \in [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma]$ S smoothing		$A = \sum_{i=1}^k C_i$ C_i commutators

Remark Note that the obstruction to the generalized Kontsevich-Vishik trace is given by the derivatives of the a_ι for $\iota \in I_0$. Using the example above Theorem 1.15, we obtain that these are residue traces themselves if the operator is poly-homogeneous. These residues are explicitly calculated for gauged families $A(z) = BQ^z$ in Proposition 3.4.

■

7. Stationary phase approximation

In this section we would like to get to know a little more about the singularity structure of

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi,$$

primarily to calculate the integrals

$$\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi).$$

We will skip many calculations in this chapter because they are very tedious and differ only slightly (if at all) from the calculations that can be found in any account on stationary phase approximation (e.g. chapter 7.7 in [7]).

We will prove the following theorem.

Theorem 7.3 *Let $k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi$ be the kernel of a Fourier Integral Operator with poly-*log*-homogeneous amplitude $a = a_0 + \sum_{\iota\in I} a_\iota$ and phase function satisfying $\partial_3^2 \left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right)$ whenever $\partial_3\vartheta(x,y,\xi) = 0$. Let $\tilde{I} := I \cup \{0\}$ and choose a decomposition $a = a^0 + \sum_{s=1}^S a^s$ such that there is no stationary point in the support of $a^0(x,y,\cdot)$ and exactly one stationary point $\hat{\xi}^s(x,y) \in \partial B_{\mathbb{R}^N}$ of $\vartheta(x,y,\cdot)$ in the support of each $a^s(x,y,\cdot)$.*

Let $\hat{\vartheta}^s(x,y) = \vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Theta^s(x,y) = \partial_{\partial B}^2 \vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\operatorname{sgn}\Theta^s(x,y)$ the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta^s(x,y)$, $\mathrm{grad}_{\partial B_{\mathbb{R}^N}} = \partial_{\partial B}$ and $\mathrm{div}_{\partial B_{\mathbb{R}^N}}$ are the gradient and divergence operators on the $(N-1)$-sphere $\partial B_{\mathbb{R}^N}$, and

$$\Delta_{\partial B,\Theta^s(x,y)} = \left\langle \Theta^s(x,y)^{-1}\partial_{\partial B}, \partial_{\partial B}\right\rangle = -\,\mathrm{div}_{\partial B_{\mathbb{R}^N}}\, \Theta^s(x,y)^{-1}\,\mathrm{grad}_{\partial B_{\mathbb{R}^N}}.$$

Furthermore, let

$$h_{j,\iota}^s(x,y) := \frac{(2\pi)^{\frac{N-1}{2}} \left|\det\Theta^s(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta^s(x,y)}}{j!(2i)^j} \Delta_{\partial B,\Theta^s(x,y)}^j a_\iota^s\left(x,y,\hat{\xi}^s(x,y)\right)$$

and

$$g_{j,\iota}^s(x,y) := \begin{cases} \partial^{l_\iota}\left(z \mapsto \Gamma(q+1+z)\, i^{q+1+z}\left(\hat{\vartheta}^s(x,y) + i0\right)^{-q-1-z}\right)(0) & , \ q \in \mathbb{C}\setminus(-\mathbb{N}_0) \\ \partial^{l_\iota}\left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i\ (-q)!} \int_{c+i\mathbb{R}} \frac{(-\sigma)^{-q}(c_{\ln}+\ln\sigma)}{\left(-i\hat{\vartheta}^s(x,y)+0-\sigma\right)^{z+1}} d\sigma\right)(0) & , \ q \in -\mathbb{N}_0 \end{cases}$$

with $q := d_\iota + \frac{N+1}{2} - j$, $c \in \mathbb{R}_{>0}$, and some constant $c_{\ln} \in \mathbb{C}$.

Then,

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a^0(x,y,\xi) d\xi + \sum_{\iota\in\tilde{I}} \sum_{s=1}^S \sum_{j\in\mathbb{N}_0} h_{j,\iota}^s(x,y) g_{j,\iota}^s(x,y)$$

holds in a neighborhood of the diagonal in X^2.

This will yield the following theorems.

Theorem 7.5 *Let A be a Fourier Integral Operator with kernel*

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi$$

whose phase function ϑ satisfies $\forall x \in X\ \forall \xi \in \mathbb{R}^N:\ \vartheta(x,x,\xi) = 0$, and whose amplitude has an asymptotic expansion $a \sim \sum_{\iota\in\mathbb{N}} a_\iota$ where each a_ι is log*-homogeneous with* *degree of homogeneity* d_ι *and logarithmic order* l_ι, *and* $\Re(d_\iota) \to -\infty$. *Let* $N_0 \in \mathbb{N}$ *such that* $\forall \iota \in \mathbb{N}_{>N_0}: \ \Re(d_\iota) < -N$ *and let*

$$k^{\mathrm{sing}}(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} \sum_{\iota=1}^{N_0} a_\iota(x,y,\xi) d\xi$$

denote the singular part of the kernel.

Then, the regularized kernel $k - k^{\mathrm{sing}}$ *is continuous along the diagonal and independent of the particular choice of* N_0 *(along the diagonal). Furthermore, the generalized Kontsevich-Vishik density is given by*

$$\left(k - k^{\mathrm{sing}}\right)(x,x) \mathrm{dvol}_X(x) = \int_{\mathbb{R}^N} a(x,x,\xi) - \sum_{\iota=1}^{N_0} a_\iota(x,x,\xi) d\xi \mathrm{dvol}_X(x).$$

THEOREM 7.7 *Let* A *be a Fourier Integral Operator with phase function* ϑ *satisfying* $\partial_3^2\left(\vartheta|_{X\times X\times \partial B_{\mathbb{R}^N}}\right)(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right)$ *whenever* $\partial_3\vartheta(x,y,\xi) = 0$ *(in a neighborhood of the diagonal) and* $\hat{\xi}^s$ $\left(s \in \mathbb{N}_{\le S}\right)$ *the stationary points. Furthermore, let*

$$\forall x \in X \ \ \forall s \in \mathbb{N}_{\le S}: \ \vartheta\left(x,x,\hat{\xi}^s(x,x)\right) \ne 0.$$

Then,

$$\left(\ X \ni x \mapsto k(x,x) \in \mathbb{C} \ \right) \in C(X)$$

and

$$\operatorname{tr} A = \int_X k(x,x) \mathrm{dvol}_X(x)$$

is well-defined, i.e. A *is a Hilbert-Schmidt operator. Furthermore,* ζ*-functions of such operators have no poles (since the trace integral always exists).*

For the remainder of the section, let a be log-homogeneous. Then, we obtain

$$\begin{aligned} k(x,y) &:= \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi \\ &= \int_{\mathbb{R}_{>0}} \int_{\partial B_{\mathbb{R}^N}} r^{N-1} e^{ir\vartheta(x,y,\eta)} a(x,y,r\eta) \mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta) dr \\ &= \int_{\mathbb{R}_{>0}} r^{N+d-1} (\ln r)^l \underbrace{\int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} a(x,y,\eta) \mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta)}_{=:I(x,y,r)} dr. \end{aligned}$$

Let (x,y) be off the critical manifold, i.e. $\forall \eta \in \partial B_{\mathbb{R}^N}: \ \partial_3\vartheta(x,y,\eta) \ne 0$. Then, we observe

$$\begin{aligned} \forall n \in \mathbb{N}: \ |I(x,y,r)| &= \frac{1}{r} \left| \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} \mathcal{D}a(x,y,\eta) \mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta) \right| \\ &= \frac{1}{r^n} \left| \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} \mathcal{D}^n a(x,y,\eta) \mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta) \right| \\ &\le \frac{1}{r^n} \left\| \mathcal{D}^n a \right\|_{L_\infty(X\times X\times \partial B_{\mathbb{R}^N})}, \end{aligned}$$

where

$$\mathcal{D}a(x,y,\eta) := \partial_3^* \frac{a(x,y,\eta)\partial_3\vartheta(x,y,\eta)}{\|\partial_3\vartheta(x,y,\eta)\|_{\ell_2(N)}^2},$$

which proves the well-known fact that k is C^∞ away from the critical manifold.

On the critical manifold, we will assume that

$$\partial_3^2 \left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right)$$

if $\partial_3\vartheta(x,y,\xi) = 0$ (note that this holds for pseudo-differential operators). Then, we are in a position to apply Morse' Lemma.

Lemma 7.1 (Morse' Lemma)**.** *Let $(x_0,y_0,\xi_0) \in X\times X\times\partial B_{\mathbb{R}^N}$ be stationary (in particular, $\partial_{\partial B}\vartheta(x_0,y_0,\xi_0) = 0$) and $\partial_{\partial B}^2\vartheta(x_0,y_0,\xi_0) \in GL\left(\mathbb{R}^{N-1}\right)$ where $\partial_{\partial B}$ denotes the spherical derivative, i.e. the derivative in $\partial B_{\mathbb{R}^N}$.*

Then, there are neighborhoods $U \subseteq_{\text{open}} X\times X$ of (x_0,y_0) and $V \subseteq_{\text{open}} \partial B_{\mathbb{R}^N}$ of ξ_0 and a function $\hat\xi \in C^\infty(U,V)$ such that

$$\forall (x,y,\xi) \in U\times V:\ \partial_{\partial B}\vartheta(x,y,\xi) = 0 \iff \xi = \hat\xi(x,y).$$

Furthermore, there is a function $\eta \in C^\infty\left(U\times V,\mathbb{R}^N\right)$ such that

$$\forall (x,y,\xi) \in U\times V:\ \eta(x,y,\xi) - \left(\xi - \hat\xi(x,y)\right) \in O\left(\left\|\xi - \hat\xi(x,y)\right\|_{\ell_2(N)}^2\right)$$

and

$$\partial_3\eta\left(x,y,\hat\xi(x,y)\right) = 1.$$

Corollary 7.2. *Let ϑ be as in Morse' Lemma (Lemma 7.1). Then, stationary points of $\vartheta(x,y,\cdot)$ are isolated in $\partial B_{\mathbb{R}^N}$. In particular, there are only finitely many.*

Proof. For given stationary (x,y,ξ) we can find a neighborhood $V \subseteq_{\text{open}} \partial B_{\mathbb{R}^N}$ such that $\xi = \hat\xi(x,y)$; thus, stationary points are locally unique. By compactness of $\partial B_{\mathbb{R}^N}$ they are isolated and at most finitely many. □

Hence, we may assume that

$$k(x,y) = \sum_{s=0}^{S} \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a^s(x,y,\xi)d\xi$$

where a^0 has no stationary points in its support and each of the a^s has exactly one branch $(x,y,\hat\xi^s(x,y))$ in its support. As we have already treated the a^0 case, we will assume, without loss of generality, that a is of the form of one of the a^s.

Let $\eta_{\partial B}$ be defined as the spherical part of η and

$$\Theta(x,y) := \partial_{\partial B}^2\vartheta\left(x,y,\hat\xi(x,y)\right).$$

Then,

$$\left\langle \partial_3^2\vartheta\left(x,y,\hat\xi(x,y)\right)\eta(x,y,\xi),\eta(x,y,\xi)\right\rangle_{\mathbb{R}^N} = \langle \Theta(x,y)\eta_{\partial B}(x,y,\xi),\eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}$$

and, defining $\hat\vartheta := \vartheta\left(x,y,\hat\xi(x,y)\right)$,

$$\begin{aligned} I(x,y,r) =& \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)} a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\ =& e^{ir\hat\vartheta} \int_{\partial B_{\mathbb{R}^N}} e^{i\frac{r}{2}\langle\Theta(x,y)\eta_{\partial B}(x,y,\xi),\eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}} a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi). \end{aligned}$$

Let $\sigma:\ \mathbb{R}^{N-1} \to \partial B_{\mathbb{R}^N}$ be a stereographic projection with pole $-\hat\xi(x,y)$ (which is assumed to be outside of $\operatorname{spt} a(x,y,\cdot)$),

$$\eta_\sigma(x,y,\xi) := \eta_{\partial B}(x,y,\sigma(\xi)),$$

and

$$a_\sigma(x,y,\xi) := a(x,y,\sigma(\xi))\sqrt{\det\left(\sigma'(\xi)^*\sigma'(\xi)\right)}.$$

Then,

$$\begin{aligned}I(x,y,r) =&e^{ir\hat{\vartheta}}\int_{\partial B_{\mathbb{R}^N}} e^{i\frac{r}{2}\langle\Theta(x,y)\eta_{\partial B}(x,y,\xi),\eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}} a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\\ =&e^{ir\hat{\vartheta}}\int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle\Theta(x,y)\eta_{\sigma}(x,y,\xi),\eta_{\sigma}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}} a_\sigma(x,y,\xi)d\xi\end{aligned}$$

and

$$\partial_3\eta_\sigma(x,y,\xi) = \partial_3\eta_{\partial B}(x,y,\sigma(\xi))\sigma'(\xi)$$

combined with the fact that $\partial_3\eta\left(x,y,\hat{\xi}(x,y)\right) = 1$ yields that $\eta_\sigma(x,y,\cdot)$ is invertible in a neighborhood of $\sigma^{-1}\left(\hat{\xi}(x,y)\right) = 0$ (we will also use $\eta_\sigma(x,y)(\cdot)$ for $\eta_\sigma(x,y,\cdot)$). Without loss of generality, let $a_\sigma(x,y,\cdot)$ have support in such a neighborhood and

$$\tilde{a}(x,y,\xi) \coloneqq a_\sigma(x,y,\eta_\sigma(x,y)^{-1}(\xi))\sqrt{\det\left(\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)^*\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)\right)}.$$

This yields

$$\begin{aligned}I(x,y,r) =&e^{ir\hat{\vartheta}}\int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle\Theta(x,y)\eta_{\sigma}(x,y,\xi),\eta_{\sigma}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}} a_\sigma(x,y,\xi)d\xi\\ =&e^{ir\hat{\vartheta}}\int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle\Theta(x,y)\xi,\xi\rangle_{\mathbb{R}^{N-1}}} \tilde{a}(x,y,\xi)d\xi.\end{aligned}$$

Using

$$\begin{aligned}&\mathcal{F}\left(z\mapsto e^{i\frac{1}{2}\langle r\Theta(x,y)\xi,\xi\rangle_{\mathbb{R}^{N-1}}}\right)(\xi)\\ =&\left|\det(r\Theta(x,y))\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}(r\Theta(x,y))} e^{-i\frac{1}{2}\left\langle(r\Theta(x,y))^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}}\\ =&\frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}(\Theta(x,y))} e^{-i\frac{1}{2}\left\langle(r\Theta(x,y))^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}}\end{aligned}$$

where $\operatorname{sgn}(\Theta(x,y))$ is the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta(x,y)$ (cf. Lemma 1.2.3 in [3] and noting that Duistermaat uses "$\mathcal{F} = \int_{\mathbb{R}^N}$" whereas we are using "$\mathcal{F} = (2\pi)^{-\frac{N}{2}}\int_{\mathbb{R}^N}$"), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} e^{i\frac{1}{2}\langle r\Theta(x,y)\xi,\xi\rangle_{\mathbb{R}^{N-1}}}\tilde{a}(x,y,\xi)d\xi\\ =&\left(\frac{2\pi}{r}\right)^{\frac{N-1}{2}}\left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta(x,y)}\sum_{j\in\mathbb{N}_0}\frac{(-i)^j r^{-j}}{j!2^j}\left\langle\Theta(x,y)^{-1}\partial_3,\partial_3\right\rangle_{\mathbb{R}^{N-1}}^j\tilde{a}(x,y,0).\end{aligned}$$

Hence, defining

$$h_j(x,y) \coloneqq \frac{(2\pi)^{\frac{N-1}{2}}\left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta(x,y)}}{j!(2i)^j}\left\langle\Theta(x,y)^{-1}\partial_3,\partial_3\right\rangle_{\mathbb{R}^{N-1}}^j\tilde{a}(x,y,0)$$

we obtain

$$\begin{aligned}k(x,y) =&\sum_{s=0}^S\int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l\int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}a^s(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ dr\\ =&\int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l\int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}a^0(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ dr\\ &+\sum_{s=1}^S\sum_{j\in\mathbb{N}_0}h_j^s(x,y)\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j}(\ln r)^l e^{ir\hat{\vartheta}^s(x,y)}\ dr.\end{aligned}$$

For $l = 0$ we may invest the well-known fact

$$\forall q\in\mathbb{C}_{\Re(\cdot)>-1}\ \forall s\in\mathbb{C}_{\Re(\cdot)>0}:\ \int_{\mathbb{R}_{>0}} t^q e^{-st}dt = \Gamma(q+1)s^{-q-1}$$

about the Laplace transform to obtain

$$\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j} e^{ir\hat{\vartheta}^s(x,y)} dr = \Gamma\left(d + \frac{N+1}{2} - j\right) i^{d+\frac{N+1}{2}-j} \left(\hat{\vartheta}^s(x,y) + i0\right)^{-d-\frac{N+1}{2}+j}$$

if $\Re\left(d + \frac{N+1}{2} - j\right) > 0$ where $f(t+i0) := \lim_{\varepsilon \searrow 0} f(t+i\varepsilon)$. By meromorphic extension, we obtain

$$\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j} e^{ir\hat{\vartheta}^s(x,y)} dr = \Gamma\left(d + \frac{N+1}{2} - j\right) i^{d+\frac{N+1}{2}-j} \left(\hat{\vartheta}^s(x,y) + i0\right)^{-d-\frac{N+1}{2}+j}$$

whenever $d + \frac{N+1}{2} - j \in \mathbb{C} \setminus (-\mathbb{N}_0)$ and, for $l \in \mathbb{N}_0$,

$$\begin{aligned}\int_{\mathbb{R}_{>0}} r^q (\ln r)^l e^{ir\hat{\vartheta}^s(x,y)} dr =& \partial^l \left(z \mapsto \int_{\mathbb{R}_{>0}} r^{q+z} e^{ir\hat{\vartheta}^s(x,y)} dr\right)(0) \\ =& \partial^l \left(z \mapsto \Gamma(q+1+z)\, i^{q+1+z} \left(\hat{\vartheta}^s(x,y) + i0\right)^{-q-1-z}\right)(0).\end{aligned}$$

If $d + \frac{N+1}{2} - j \in -\mathbb{N}_0$, i.e. $d + \frac{N-1}{2} - j \in -\mathbb{N}$, then we can use the following property

$$\int_{\mathbb{R}_{>0}} \int_0^t f(\tau) d\tau e^{-st} dt = \frac{1}{s} \int_{\mathbb{R}_{>0}} f(t) e^{-st} dt$$

to obtain

$$\forall q, s \in \mathbb{C}_{\Re(\cdot)>0} : \int_{\mathbb{R}_{>0}} t^q e^{-st} dt = \int_{\mathbb{R}_{>0}} \int_0^t q\tau^{q-1} d\tau e^{-st} dt = \frac{q}{s} \int_{\mathbb{R}_{>0}} t^{q-1} e^{-st} dt$$

and, hence,

$$\int_{\mathbb{R}_{>0}} t^q e^{-st} dt = \frac{s}{q+1} \int_{\mathbb{R}_{>0}} t^{q+1} e^{-st} dt = \frac{s^n}{\prod_{p=1}^n (q+p)} \int_{\mathbb{R}_{>0}} t^{q+n} e^{-st} dt$$

by meromorphic extension. Thus, for $q \in -\mathbb{N}$ and $n = -q-1$, we have

$$\int_{\mathbb{R}_{>0}} t^q e^{-st} dt = \frac{(-s)^{-q-1}}{(-q-1)!} \int_{\mathbb{R}_{>0}} t^{-1} e^{-st} dt$$

reducing the problem to finding $\int_{\mathbb{R}_{>0}} t^{-1} e^{-st} dt$. Consider the Borel measure

$$\mu_{q,s} : \mathcal{B}(\mathbb{R}_{>0}) \to \mathbb{C};\ A \mapsto \int_A t^{q-1} e^{-st} dt$$

on $\mathbb{R}_{>0}$ for $q, s \in \mathbb{C}_{\Re(\cdot)>0}$. Then,

$$\partial\left(\sigma \mapsto \int_{\mathbb{R}_{>0}} f(t) e^{-\sigma t} dt\right)(s) = -\int_{\mathbb{R}_{>0}} t f(t) e^{-st} dt$$

implies

$$\partial\left(\sigma \mapsto \mu_{q,\sigma}\right)(s) = -\mu_{q+1,s}$$

and, hence,

$$\partial\left(\sigma \mapsto \mu_{q,\sigma}\right)(s)\left(\mathbb{R}_{>0}\right) = -\mu_{q+1,s}\left(\mathbb{R}_{>0}\right) = -\frac{\Gamma(q+2)}{s^{q+2}} \to -\frac{1}{s} \quad (q \to -1).$$

In other words, $\int_{\mathbb{R}_{>0}} t^{-1} e^{-st} dt$ is logarithmic (up to a constant) and $\int_{\mathbb{R}_{>0}} t^q e^{-st} dt$ for $q \in -\mathbb{N}$ is log-homogeneous; namely,

$$\int_{\mathbb{R}_{>0}} r^q e^{ir\hat{\vartheta}^s(x,y)}\, dr = -\frac{\left(i\hat{\vartheta}^s(x,y) - 0\right)^{-q-1}}{(-q-1)!} \left(c_{\ln} + \ln\left(-i\hat{\vartheta}^s(x,y) + 0\right)\right)$$

with some constant $c_{\ln}$. Finally, we can add the $\ln r$ terms for $q \in -\mathbb{N}$ by investing the the multiplication property of the Laplace transform

$$\mathcal{L}(fg)(s) = \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \mathcal{L}(f)(\sigma)\mathcal{L}(g)(s-\sigma)d\sigma$$

where $c \in \mathbb{R}$ such that $c + i\mathbb{R}$ is a subset of the region of convergence for $\mathcal{L}(f) = \left(s \mapsto \int_{\mathbb{R}_{>0}} f(t)e^{-st}dt\right)$. Thence, for $c \in \mathbb{R}_{>0}$, $q \in -\mathbb{N}$, and $l \in \mathbb{N}_0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_{>0}} r^q (\ln r)^l e^{-tr} dr\Big|_{t=-i\hat{\vartheta}^s(x,y)+0} \\
&= \partial^l \left(z \mapsto \int_{\mathbb{R}_{>0}} r^q r^z e^{-tr} dr\right)(0)\Big|_{t=-i\hat{\vartheta}^s(x,y)+0} \\
&= \partial^l \left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i(-q-1)!}\int_{c+i\mathbb{R}} (-\sigma)^{-q-1}(c_{\ln} + \ln\sigma)(t-\sigma)^{-z-1} d\sigma\right)(0)\Big|_{t=-i\hat{\vartheta}^s(x,y)+0}.
\end{aligned}$$

Thus, we have proven the Theorem

Theorem 7.3. *Let $k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi$ be the kernel of a Fourier Integral Operator with poly-log-homogeneous amplitude $a = a_0 + \sum_{\iota\in I} a_\iota$. Let $\tilde{I} := I \cup \{0\}$ and choose a decomposition $a = a^0 + \sum_{s=1}^S a^s$ such that there is no stationary point in the support of $a^0(x,y,\cdot)$ and exactly one stationary point $\hat{\xi}^s(x,y) \in \partial B_{\mathbb{R}^N}$ of $\vartheta(x,y,\cdot)$ in the support of each $a^s(x,y,\cdot)$.*

Let $\hat{\vartheta}^s(x,y) = \vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Theta^s(x,y) = \partial_{\partial B}^2\vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\operatorname{sgn}\Theta^s(x,y)$ the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta^s(x,y)$, and $\Delta_{\partial B,\Theta^s(x,y)} = \left\langle \Theta^s(x,y)^{-1}\partial_{\partial B}, \partial_{\partial B}\right\rangle = -\operatorname{div}_{\partial B_{\mathbb{R}^N}} \Theta^s(x,y)^{-1}\operatorname{grad}_{\partial B_{\mathbb{R}^N}}$. Furthermore, let

$$h_{j,\iota}^s(x,y) := \frac{(2\pi)^{\frac{N-1}{2}}\left|\det\Theta^s(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta^s(x,y)}}{j!(2i)^j}\Delta_{\partial B,\Theta^s}^j a_\iota^s\left(x,y,\hat{\xi}^s(x,y)\right)$$

and

$$g_{j,\iota}^s(x,y) := \begin{cases} \partial^{l_\iota}\left(z \mapsto \Gamma(q+1+z)\, i^{q+1+z}\left(\hat{\vartheta}^s(x,y)+i0\right)^{-q-1-z}\right)(0) & , \ q \in \mathbb{C}\setminus(-\mathbb{N}_0) \\ \partial^{l_\iota}\left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i\ (-q)!}\int_{c+i\mathbb{R}} \frac{(-\sigma)^{-q}(c_{\ln}+\ln\sigma)}{\left(-i\hat{\vartheta}^s(x,y)+0-\sigma\right)^{z+1}} d\sigma\right)(0) & , \ q \in -\mathbb{N}_0 \end{cases}$$

with $q := d_\iota + \frac{N+1}{2} - j$, $c \in \mathbb{R}_{>0}$, and some constant $c_{\ln} \in \mathbb{C}$.

Then,

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a^0(x,y,\xi)d\xi + \sum_{\iota\in\tilde{I}}\sum_{s=1}^S\sum_{j\in\mathbb{N}_0} h_{j,\iota}^s(x,y)g_{j,\iota}^s(x,y)$$

holds in a neighborhood of the diagonal in X^2.

Remark Suppose $\partial_{\partial B}^2\vartheta$ is not invertible at some stationary point but we can split the third variable in a pair (ξ,ζ) such that $\partial_4^2\vartheta(x_0,y_0,\xi_0,\zeta_0)$ is invertible at the stationary point. Then, we can find open neighborhoods U of ξ_0 and V of ζ_0 as well as a function $\hat{\zeta}$ such that $\partial_4\vartheta(x,y,\xi,\zeta) = 0$ if and only if $\zeta = \hat{\zeta}(\xi)$. In particular, since $U \times V$ is open in the compact set $\partial B_{\mathbb{R}^N}$, we can use a partition of unity to reduce $I(x,y,r)$ into a sum of integrals of the form

$$\int_U\int_V e^{ir\vartheta(x,y,\xi,\zeta)}a(x,y,\xi,\zeta)\mathrm{dvol}_V(\zeta)\mathrm{dvol}_U(\xi).$$

Using stationary phase with respect to ζ, then, yields

$$\begin{aligned}&\int_U\int_V e^{ir\vartheta(x,y,\xi,\zeta)}a(x,y,\xi,\zeta)d\mathrm{vol}_V(\zeta)d\mathrm{vol}_U(\xi)\\ =&\int_U e^{ir\vartheta(x,y,\xi,\hat\zeta(\xi))}\int_V e^{ir\langle\partial_4^2\vartheta(x,y,\xi,\hat\zeta(\xi))\eta(\zeta),\eta(\zeta)\rangle_{\mathbb{R}^n}}a(x,y,\xi,\zeta)d\mathrm{vol}_V(\zeta)d\mathrm{vol}_U(\xi)\end{aligned}$$

which, again, yields an expansion of the form above but where the coefficients need to be integrated once more.

■

Example For a pseudo-differential operator, we have

$$\vartheta(x,y,\xi)=(x-y)^T\sigma(\xi).$$

Choosing coordinates such that $(x-y)=-\|x-y\|_{\ell_2(N)}e_N$ and letting e_N be the pole of the stereographic projection, we obtain

$$\sigma(\xi)=\begin{pmatrix}\frac{2\xi}{1+\|\xi\|_{\ell_2(N-1)}}\\ \frac{\|\xi\|_{\ell_2(N-1)}-1}{\|\xi\|_{\ell_2(N-1)}+1}\end{pmatrix}$$

and

$$\tilde\vartheta(\xi):=\frac{\vartheta(x,y,\xi)}{\|x-y\|_{\ell_2(N)}}=\frac{1-\|\xi\|_{\ell_2(N-1)}}{1+\|\xi\|_{\ell_2(N-1)}}.$$

From $\Theta(x,y):=\partial^2_{\partial B}\vartheta\left(x,y,\hat\xi(x,y)\right)$ and $\hat\xi(x,y)=\frac{x-y}{\|x-y\|_{\ell_2(N)}}=\sigma(0)$ in these coordinates, we obtain

$$\Theta(x,y)=\|x-y\|_{\ell_2(N)}\tilde\vartheta''(0)=-4\|x-y\|_{\ell_2(N)}.$$

Hence, using $z:=x-y$,

$$h_j(x,y)=\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}\|z\|_{\ell_2(N)}^{-\frac{N-1}{2}-j}e^{-\frac{i\pi}{4}(N-1)}}{j!(-8i)^j}\Delta^j_{\partial B}a\left(x,y,\frac{z}{\|z\|_{\ell_2(N)}}\right).$$

Let

$$\tilde h_j(x,y):=\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}e^{-\frac{i\pi}{4}(N-1)}}{j!(-8i)^j}\Delta^j_{\partial B}a\left(x,y,\frac{z}{\|z\|_{\ell_2(N)}}\right).$$

Then,

$$h_j(x,y)=\tilde h_j(x,y)\|z\|_{\ell_2(N)}^{-\frac{N-1}{2}-j}$$

and

$$\begin{aligned}&\sum_{j\in\mathbb{N}_0}h_j(x,y)\int_{\mathbb{R}_{>0}}r^{d+\frac{N-1}{2}-j}(\ln r)^le^{ir\hat\vartheta(x,y)}\,dr\\ =&\sum_{j\in\mathbb{N}_0}\tilde h_j(x,y)\|z\|_{\ell_2(N)}^{-\frac{N-1}{2}-j}\int_{\mathbb{R}_{>0}}r^{d+\frac{N-1}{2}-j}(\ln r)^le^{ir\|z\|_{\ell_2(N)}}\,dr\end{aligned}$$

In particular, for $l=0$ and $d+\frac{N-1}{2}-j\in\mathbb{C}\setminus(-\mathbb{N})$,

$$\begin{aligned}&\sum_{j\in\mathbb{N}_0}h_j(x,y)\int_{\mathbb{R}_{>0}}r^{d+\frac{N-1}{2}-j}e^{ir\hat\vartheta(x,y)}\,dr\\ =&\sum_{j\in\mathbb{N}_0}\tilde h_j(x,y)\Gamma\left(d+\frac{N+1}{2}-j\right)(-i)^{-d-\frac{N+1}{2}+j}\left(\|z\|_{\ell_2(N)}+i0\right)^{-d-N}\end{aligned}$$

yields the following proposition since, for $k=\delta_{\mathrm{diag}}$, we have $\vartheta(x,y,\xi)=\langle x-y,\xi\rangle$ and $a(x,y,\xi)=\frac{1}{2\pi}$, i.e. $d=0$ and

$$\tilde{h}_j(x,y):=\begin{cases}\frac{1}{2\pi}\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}e^{-\frac{i\pi}{4}(N-1)} & ,\ j=0\\ 0 & ,\ j\in\mathbb{N}\end{cases}.$$

Proposition 7.4.

$$\begin{aligned}\delta_{\mathrm{diag}}(x,y)=&\frac{1}{2\pi}\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}e^{-\frac{i\pi}{4}(N-1)}\Gamma\left(\frac{N+1}{2}\right)(-i)^{-\frac{N+1}{2}}\left(\|x-y\|_{\ell_2(N)}+i0\right)^{-N}\\&+\frac{1}{2\pi}\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}e^{-\frac{i\pi}{4}(N-1)}\Gamma\left(\frac{N+1}{2}\right)(-i)^{-\frac{N+1}{2}}\left(-\|x-y\|_{\ell_2(N)}+i0\right)^{-N}.\end{aligned}$$

In particular, for $N=1$, we obtain

$$\delta_{\mathrm{diag}}(x,y)=\frac{i}{2\pi}\left(\left(\|x-y\|_{\ell_2(N)}+i0\right)^{-1}-\left(\|x-y\|_{\ell_2(N)}-i0\right)^{-1}\right).$$

This is precisely what we expect; cf. end of section 4.4.3.1 in [17].

■

Remark Note that in the $N=1$ case everything collapses as there are no spherical derivatives. We will simply obtain

$$k_d(x,y)=\int_{\mathbb{R}_{>0}}r^d e^{ir\vartheta(x,y,1)}a_d(x,y,1)dr+\int_{\mathbb{R}_{>0}}r^d e^{ir\vartheta(x,y,-1)}a_d(x,y,-1)dr$$

and

$$\begin{aligned}&\int_{\mathbb{R}_{>0}}r^d e^{ir\vartheta(x,y,\pm1)}a_d(x,y,\pm1)dr\\&=\begin{cases}c_d a_d(x,y,\pm1)\left(\vartheta(x,y,\pm1)+i0\right)^{-d-1} & ,\ d\notin-\mathbb{N}\\ a_d(x,y,\pm1)\frac{(i\vartheta(x,y,\pm1)-0)^{-d-1}}{(-d-1)!}\left(c_d+\ln\left(-i\vartheta(x,y,\pm1)+0\right)\right) & ,\ d\in-\mathbb{N}\end{cases}\end{aligned}$$

with some constants c_d. Hence, for

$$k(x,y)\sim\sum_{j\in\mathbb{N}_0}\int_{\mathbb{R}}e^{i\vartheta(x,y,\xi)}a_{d-j}(x,y,\xi)d\xi$$

with $d\in\mathbb{Z}$ and a_{d-j} homogeneous of degree $d-j$, the coefficient of the logarithmic terms are

$$\sum_{j\in\mathbb{N}_{\geq d+1}}a_{d-j}(x,y,\pm1)\frac{(i\vartheta(x,y,\pm1)-0)^{j-d-1}}{(j-d-1)!}.$$

In particular, in the critical case where $\vartheta(x,y,\pm1)=0$ (in fact, we are only interested in $\vartheta(x,x,\pm1)$) we are reduced to the fact (cf. formulae (3) and (4) in [1]) that the densities of the residue traces at x (that is, $a_{-1}(x,x,\pm1)$) coincide with the coefficients of the logarithmic terms (that is, $\ln\left(-i\vartheta(x,x,\pm1)+0\right)$) in the singularity structure of k.

Furthermore, we can calculate the generalized Kontsevich-Vishik trace for $a=a_0+\sum_{\iota\in I}a_\iota$ if $\forall\iota\in I:\ d_\iota\in\mathbb{R}\setminus\{-1\}\ \wedge\ l_\iota=0$. Then, the kernel k satisfies (note $\vartheta(x,x,r)=0$ by assumption)

$$k(x,x)=\int_{\mathbb{R}_{>0}}a_0(x,x,r)dr+\sum_{\iota\in I}\int_{\mathbb{R}_{>0}}a_\iota(x,x,r)dr.$$

Since $1_{\mathbb{R}_{>0}}a_\iota(x,x,\cdot)$ is homogeneous of degree d_ι, we obtain that $\int_{\mathbb{R}_{>0}} a_\iota(x,x,r)dr$ vanishes for $d_\iota < -1$ since the Fourier transform $\mathcal{F}(1_{\mathbb{R}_{>0}}a_\iota(x,x,\cdot))$ over $\mathbb{R}$ is a homogeneous distribution of degree $-1-d_\iota$. For $d_\iota > -1$, we obtain

$$\int_{\mathbb{R}_{>0}} e^{i\vartheta(x,y,r)}a_\iota(x,x,r)dr = c_\iota a_\iota(x,y,1)\left(\vartheta(x,y,1)+i0\right)^{-d_\iota-1}$$

which is precisely the other singular contribution (that is the $f(x,y)(\varphi+0)^{-N}$ term in equation (3) of [1]) to the kernel singularity. In other words, the difference of $k(x,y)$ and its singular part $k^{\mathrm{sing}}(x,y)$ satisfies

$$\left(k-k^{\mathrm{sing}}\right)(x,x) = \int_{\mathbb{R}_{>0}} a_0(x,x,r)dr.$$

In order to use Theorem 3.1, we will have to show that the regularized singular terms vanish. This follows directly from the Laurent expansion with mollification. For $d_\iota > -1$, we have the two terms

$$\sum_{n\in\mathbb{N}_0}\frac{\int_X\int_0^1 e^{i\vartheta(x,x,\xi)}\partial^n a_\iota(0)(x,x,\xi)d\xi d\mathrm{vol}_X(x)}{n!}z^n$$
$$+\sum_{n\in\mathbb{N}_0}\sum_{j=0}^{n}\frac{(-1)^{j+1}j!\int_X e^{i\vartheta(x,x,1)}\partial^n a_\iota(0)(x,x,1)d\mathrm{vol}_X(x)}{n!(1+d_\iota)^{j+1}}z^n$$

to evaluate at $z=0$, i.e.

$$\begin{aligned}
&\lim_{h\searrow 0}\int_X\int_0^1 (h+r)^{d_\iota}a_\iota(x,x,1)drd\mathrm{vol}_X(x)\\
=&\int_X a_\iota(x,x,1)\lim_{h\searrow 0}\int_h^{1+h} r^{d_\iota}drd\mathrm{vol}_X(x)\\
=&\int_X a_\iota(x,x,1)\lim_{h\searrow 0}\frac{(1+h)^{d_\iota+1}-h^{d_\iota+1}}{d_\iota+1}d\mathrm{vol}_X(x)\\
=&\int_X \frac{a_\iota(x,x,1)}{d_\iota+1}d\mathrm{vol}_X(x)
\end{aligned}$$

and

$$\frac{-\int_X a_\iota(x,x,1)d\mathrm{vol}_X(x)}{1+d_\iota}.$$

Hence, the generalized Kontsevich-Vishik trace reduces to the pseudo-differential form. Let $a\sim\sum_{j\in\mathbb{N}_0}a_{d-j}$ and N be sufficiently large, then

$$\mathrm{tr}_{KV}\,A = \int_X\int_{\mathbb{R}_{>0}} a(x,x,r)-\sum_{j=0}^{N}a_{d-j}(x,x,r)\ dr\ d\mathrm{vol}_X(x)$$

which is independent of N.

■

In fact, we can generalize the case above.

Theorem 7.5. *Let A be a Fourier Integral Operator with kernel*

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi$$

whose phase function ϑ satisfies $\forall x\in X\ \forall\xi\in\mathbb{R}^N:\ \vartheta(x,x,\xi)=0$ and whose amplitude has an asymptotic expansion $a\sim\sum_{\iota\in\mathbb{N}}a_\iota$ where each a_ι is log-homogeneous *with* *degree of homogeneity d_ι and logarithmic order l_ι, and $\Re(d_\iota)\to-\infty$. Let $N_0\in\mathbb{N}$ such that $\forall\iota\in\mathbb{N}_{>N_0}:\ \Re(d_\iota)<-N$ and let*

$$k^{\mathrm{sing}}(x,y)=\int_{\mathbb{R}^N}e^{i\vartheta(x,y,\xi)}\sum_{\iota=1}^{N_0}a_\iota(x,y,\xi)d\xi$$

denote the singular part of the kernel.

Then, the regularized kernel $k-k^{\mathrm{sing}}$ is continuous along the diagonal and independent of the particular choice of N_0 (along the diagonal). Furthermore, the generalized Kontsevich-Vishik density[9] *is given by*

$$\left(k-k^{\mathrm{sing}}\right)(x,x)d\mathrm{vol}_X(x)=\int_{\mathbb{R}^N}a(x,x,\xi)-\sum_{\iota=1}^{N_0}a_\iota(x,x,\xi)d\xi d\mathrm{vol}_X(x).$$

Proof. Note that $k-k^{\mathrm{sing}}$ is regular because it has an amplitude in the Hörmander class $S^m(X\times X\times\mathbb{R}^N)$ for some $m\in\mathbb{R}_{<-N}$. Hence, it suffices to show that the ζ-regularized singular contributions of a_ι vanish for $d_\iota\neq-N$. Let $\iota\in\mathbb{N}$ such that $d_\iota\neq-N$. Then, we need to show that

$$\int_X\int_{B_{\mathbb{R}^N}(0,1)}a_\iota(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)+\frac{(-1)^{l_\iota+1}l_\iota!\int_{X\times\partial B_{\mathbb{R}^N}}\tilde a_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}}$$

vanishes. Mollifying

$$\begin{aligned}\int_{B_{\mathbb{R}^N}(0,1)}a_\iota(0)(x,x,\xi)d\xi&=\int_0^1\int_{\partial B_{\mathbb{R}^N}}r^{N-1}a_\iota(0)(x,x,r\nu)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\nu)dr\\&=\int_0^1\int_{\partial B_{\mathbb{R}^N}}r^{N+d_\iota-1}(\ln r)^{l_\iota}\tilde a_\iota(0)(x,x,\nu)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\nu)dr\end{aligned}$$

yields (note that $f_n\to f$ compactly implies $f_n'\to f'$ compactly for holomorphic functions)

$$\begin{aligned}\lim_{h\searrow0}\int_0^1(h+r)^{N+d_\iota-1}(\ln(h+r))^{l_\iota}dr&=\lim_{h\searrow0}\int_h^{1+h}r^{N+d_\iota-1}(\ln r)^{l_\iota}dr\\&=\lim_{h\searrow0}\int_h^{1+h}\partial^{l_\iota}\left(z\mapsto r^{N+d_\iota-1+z}\right)(0)dr\\&=\lim_{h\searrow0}\partial^{l_\iota}\left(z\mapsto\frac{(1+h)^{N+d_\iota+z}-h^{N+d_\iota+z}}{N+d_\iota+z}\right)(0)\\&=\partial^{l_\iota}\left(z\mapsto(N+d_\iota+z)^{-1}\right)(0)\\&=\left(z\mapsto\frac{(-1)^{l_\iota}l_\iota!}{(N+d_\iota+z)^{l_\iota+1}}\right)(0),\end{aligned}$$

i.e.

$$\int_X\int_{B_{\mathbb{R}^N}(0,1)}a_\iota(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)+\frac{(-1)^{l_\iota+1}l_\iota!\int_{X\times\partial B_{\mathbb{R}^N}}\tilde a_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}}$$

[9] Mind that this density is only locally defined. It only patches together (modulo pathologies) if we assume the kernel patched together in the first place and the derivatives of terms of critical dimension $d_\iota=-N$ regularize to zero, i.e. if $\zeta(\mathfrak{fp}_0A)(0)$ is tracial and independent of gauge.

$$
\begin{aligned}
&=\frac{(-1)^{l_\iota} l_\iota! \int_{X\times\partial B_{\mathbb{R}^N}} \tilde{a}_\iota(0)(x,x,\xi)\ \mathrm{dvol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}} \\
&\quad+\frac{(-1)^{l_\iota+1} l_\iota! \int_{X\times\partial B_{\mathbb{R}^N}} \tilde{a}_\iota(0)(x,x,\xi)\ \mathrm{dvol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}} \\
&=0.
\end{aligned}
$$

□

Remark Instead of using mollification directly, we could have used the generalized Mellin transform which yields

$$\int_{\mathbb{R}_{>0}} r^\alpha dr = \mathcal{M}\left(r \mapsto r^\alpha\right)(1) = 0$$

where $\int_{\mathbb{R}_{>0}} r^\alpha dr$ is understood in the regularized sense. However, this does not apply to the critical case $d_\iota = -N$ because the coefficients in the Laurent expansion are integrals over $\tilde{a}_\iota(0)$ on $B_{\mathbb{R}^N}$ and over $\partial^{l_\iota+1}\tilde{a}_\iota(0)$ outside $B_{\mathbb{R}^N}$. Hence, we cannot re-write those integrals such that the generalized Mellin transform appears as a factor and the critical terms will not vanish, in general.

■

At this point, we can return to Proposition 3.4 where we had the formula

$$\mathfrak{fp}_0\zeta(z\mapsto BQ^z) = \int_X \mathrm{tr}_x\left(\mathfrak{fp}_0 B\right) - \frac{1}{q}\mathrm{res}\left(B\ln Q\right)_x \mathrm{dvol}_X(x) - \mathrm{tr}\left(B1_{\{0\}}(Q)\right)$$

with B and Q poly-homogeneous, Q admitting holomorphic functional calculus and the logarithm, and with finite dimensional kernel (e.g. an elliptic classical pseudo-differential operator on a closed manifold with spectral cut), and q is the order of Q. In [16] (equation (2.14)) it was shown that

$$\mathfrak{fp}_0\zeta(z\mapsto BQ^z) = -\frac{1}{q}\mathrm{res}\left(B\ln Q\right) - \mathrm{tr}\left(B1_{\{0\}}(Q)\right)$$

holds if $(x\mapsto \mathrm{tr}_x(\mathfrak{fp}_0 B)) = 0$ (e.g. if B is a differential operator) and Sylvie Paycha conjectured that this formula should hold more generally. (Note that we are using a different notation as we might want to assume a global point of view rather than just considering everything a sum of local patches without patching properties. Under these stronger conditions, we cannot simply write $\int_X \mathrm{tr}_x(\mathfrak{fp}_0 B) - \frac{1}{q}\mathrm{res}\left(B\ln Q\right)_x \mathrm{dvol}_X(x) = \mathrm{tr}_{KV}(B) - \frac{1}{q}\mathrm{res}\left(B\ln Q\right)$ since they are not separately globally defined densities.) The following corollary shows an equivalent characterization of Paycha's conjecture for Fourier Integral Operators as in Theorem 7.5 (in particular for pseudo-differential operators) in terms of the regular part of B.

Corollary 7.6. *Let Q be as above and B be a Fourier Integral Operator whose phase function ϑ satisfies $\forall x\in X\ \forall\xi\in\mathbb{R}^N:\ \vartheta(x,x,\xi)=0$ and whose amplitude has an asymptotic expansion $b\sim\sum_{\iota\in\mathbb{N}} b_\iota$ where each b_ι is homogeneous (on $\mathbb{R}^N\setminus\{0\}$) with degree of homogeneity d_ι and $\Re(d_\iota)\to-\infty$. Furthermore, let $I\subseteq\mathbb{N}$ be such that the amplitude b decomposes into the form $b_0+\sum_{\iota\in I} b_\iota$ where b_0 is integrable in $\mathbb{R}^N$ (i.e. of Hörmander class $S^m(X\times X\times\mathbb{R}^N)$ with $m<-N$), and let B_0 the part of B corresponding to b_0. Then,*

$$
\begin{aligned}
\mathfrak{fp}_0\zeta(z\mapsto BQ^z) &= \int_X \mathrm{tr}_x\left(\mathfrak{fp}_0 B\right) - \frac{1}{q}\mathrm{res}\left(B\ln Q\right)_x \mathrm{dvol}_X(x) - \mathrm{tr}\left(B1_{\{0\}}(Q)\right) \\
&= \int_X \mathrm{tr}_x\left(B_0\right) - \frac{1}{q}\mathrm{res}\left(B\ln Q\right)_x \mathrm{dvol}_X(x) - \mathrm{tr}\left(B1_{\{0\}}(Q)\right).
\end{aligned}
$$

In particular, the following are equivalent.

(i) *Paycha's conjecture:* $\mathfrak{fp}_0\zeta(z\mapsto BQ^z) = -\frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)$.

(ii) $x\mapsto\int_{\mathbb{R}^N} b_0(x,x,\xi)d\xi d\mathrm{vol}_X(x)$ *is a globally defined density on* X *and*

$$\operatorname{tr}(B_0) = \int_X\int_{\mathbb{R}^N} b_0(x,x,\xi)d\xi d\mathrm{vol}_X(x) = 0.$$

Remark If we remove the question of global patching and simply consider sums of local representations, then we obtain

$$\begin{aligned}\mathfrak{fp}_0\zeta(z\mapsto BQ^z) &= \operatorname{tr}_{KV}(B) - \frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)\\ &= \operatorname{tr}(B_0) - \frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)\end{aligned}$$

by default. In particular,

(i) Paycha's conjecture: $\mathfrak{fp}_0\zeta(z\mapsto BQ^z) = -\frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)$.

and

(ii') $\operatorname{tr}(B_0) = \int_X\int_{\mathbb{R}^N} b_0(x,x,\xi)d\xi d\mathrm{vol}_X(x) = 0$.

are equivalent.

■

Finally, we will consider an example of linear phase functions which will be generalized to find algebras of Fourier Integral Operators which are Hilbert-Schmidt with continuous kernels.

Example Let $\vartheta(x,y,\xi) := \langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N}$ and $\Theta(x_0,y_0)\neq 0$. Then,

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N}} a(x,y,\xi)d\xi = \mathcal{F}\left(a(x,y,\cdot)\right)\left(-\Theta(x,y)\right)$$

is continuous in a sufficiently small neighborhood of (x_0,y_0) for homogeneous a because $\mathcal{F}\left(a(x,y,\cdot)\right)$ is homogeneous and $\Theta(x,y)$ non-zero. Hence, if Θ does not vanish on the diagonal, then $X\ni x\mapsto k(x,x)\in\mathbb{C}$ is continuous and, by compactness of X, $\int_X k(x,x)d\mathrm{vol}_X(x)$ well-defined.

The stationary phase approximation above generalizes this observation ($\hat{\xi}(x,y) = \pm\frac{\Theta(x,y)}{\|\Theta(x,y)\|_{\ell_2(N)}}$, i.e. $\hat{\vartheta}^s(x,y) = (-1)^s\|\Theta(x,y)\|_{\ell_2(N)}$ with $s\in\{0,1\}$).

Theorem 7.7. *Let* A *be a Fourier Integral Operator with phase function* ϑ *satisfying* $\partial_3^2\left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi)\in GL\left(\mathbb{R}^{N-1}\right)$ *whenever* $\partial_3\vartheta(x,y,\xi)=0$ *(in a neighborhood of the diagonal) and* $\left\{\hat{\xi}^s;\ s\in\mathbb{N}_{\leq n}\right\}$ *the set of stationary points. Furthermore, let*

$$\forall x\in X\ \forall s\in\mathbb{N}_{\leq n}:\ \vartheta\left(x,x,\hat{\xi}^s(x,x)\right)\neq 0.$$

Then,

$$\left(\ X\ni x\mapsto k(x,x)\in\mathbb{C}\ \right)\in C(X)$$

and

$$\operatorname{tr} A = \int_X k(x,x)d\mathrm{vol}_X(x)$$

is well-defined, i.e. A *is a Hilbert-Schmidt operator. Furthermore,* ζ*-functions of such operators have no poles (since the trace integral always exists).*

An example for such operators occurs on quotient manifolds. Let Γ be a co-compact discrete group on M acting continuously[10] and freely[11] on $M/_\Gamma$, $\tilde{k}$ a $\Gamma\times\Gamma$-invariant[12] Schwartz kernel on M, and k its projection to $M/_\Gamma$. Then, $k(x,y)=\sum_{\gamma\in\Gamma}\tilde{k}(x,\gamma y)$. Suppose $\tilde{k}$ is pseudo-differential, i.e. $\tilde{k}(x,y)=\int_{\mathbb{R}^N}e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}}a(x,y,\xi)d\xi$. Then,

$$k(x,y)=\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^N}e^{i\langle x-\gamma y,\xi\rangle_{\mathbb{R}^N}}a(x,\gamma y,\xi)d\xi.$$

Hence, for $\gamma=\mathrm{id}$ we have a pseudo-differential part and for $\gamma\neq\mathrm{id}$ the phase function $\vartheta_\gamma(x,y,\xi)=\langle x-\gamma y,\xi\rangle_{\mathbb{R}^N}$ has stationary points $\pm\frac{x-\gamma y}{\|x-\gamma y\|_{\ell_2(N)}}$, that is, $\vartheta_\gamma\left(x,y,\hat{\xi}^s(x,y)\right)=(-1)^s\|x-\gamma y\|_{\ell_2(N)}$ does not vanish in a neighborhood of the diagonal.

■

Remark Note that we may use the stationary phase approximation results to get insights into the Laurent coefficients of the ζ-function without having to consider all these Laplace transforms because those coefficients are of the form $c\cdot I(x,y,1)$ with some constant $c\in\mathbb{C}$, i.e. we do not need the radial integration and obtain an asymptotic expansion

$$\begin{aligned}&\int_{\partial B_{\mathbb{R}^N}}e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\\ =&\int_{\partial B_{\mathbb{R}^N}}e^{i\vartheta(x,y,\xi)}a^0(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)+\sum_{s=1}^{S}\sum_{j\in\mathbb{N}_0}e^{i\vartheta\left(x,y,\hat{\xi}(x,y)\right)}h_j^s(x,y)\\ =&\int_{\partial B_{\mathbb{R}^N}}e^{i\vartheta(x,y,\xi)}a^0(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\\ &+\sum_{s=1}^{S}e^{i\hat{\vartheta}^s(x,y)}\sum_{j\in\mathbb{N}_0}\frac{(2\pi)^{\frac{N-1}{2}}\left|\det\Theta^s(x,y)\right|^{-\frac{1}{2}}e^{\frac{i\pi}{4}\operatorname{sgn}\Theta^s(x,y)}}{j!(2i)^j}\Delta^j_{\partial B,\Theta^s}a^s\left(x,y,\hat{\xi}^s(x,y)\right)\end{aligned}$$

with $\hat{\vartheta}^s(x,y)=\vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Theta^s(x,y)=\partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Delta_{\partial B,\Theta^s(x,y)}=\left\langle\Theta^s(x,y)^{-1}\partial_{\partial B},\partial_{\partial B}\right\rangle=-\operatorname{div}_{\partial B_{\mathbb{R}^N}}\Theta^s(x,y)^{-1}\operatorname{grad}_{\partial B_{\mathbb{R}^N}}$, and $\hat{\xi}^s(x,y)$ is the unique stationary point of $\vartheta(x,y,\cdot)$ in $\partial B_{\mathbb{R}^N}\cap\operatorname{spt}a^s(x,y,\cdot)$ while a^0 has no such point in its support.

■

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[10] $\Gamma\times M/_\Gamma\ni(\gamma,x)\mapsto\gamma x\in M/_\Gamma$ is continuous

[11] $\forall\gamma\in\Gamma:\ (\exists x\in M/_\Gamma:\gamma x=x)\ \Rightarrow\ \gamma=\mathrm{id}$

[12] $\forall\gamma\in\Gamma\ \forall x,y\in M:\ \tilde{k}(x,y)=\tilde{k}(\gamma x,\gamma y)$

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Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom
E-mail address: tobias.hartung@kcl.ac.uk