Well-posedness and causality of the non-Newtonian Navier-Stokes equations on 3-dimensional Riemannian $C^{1,1}$-manifolds with respect to strong, local-in-time solutions

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Preface

The Navier-Stokes equations have been studied in a variety of cases for their importance in physics and engineering. Yet, it seems, especially non-Newtonian fluids create a lot of problems for the sheer diversity of viscous properties a fluid may have, though the Newtonian case, too, still holds a tight grasp on many interesting questions. Not even well-posedness of strong solutions global in time is known; in fact, this is a Millennium problem of the Clay Mathematics Institute ([**5**]). Furthermore, causality has never been addressed to my knowledge. In my Dipl. Math. thesis I considered a unified approach for large classes of viscosities in the case of C^∞-manifolds without boundary. In these notes, we will expand these findings to include many results interesting for applications, that is, we will consider 3-dimensional $C^{1,1}$-manifolds with or without boundary - the largest class of conceivable Riemannian manifolds. If the manifold has a boundary, then we will consider Dirichlet and Neumann boundary conditions. Fluids are assumed to be incompressible and isotropic.

This approach is largely influenced by Rainer Picard who developed a unified Hilbert space approach for well-posedness and causality of (linear) partial differential equations ([**15**]). It seems that this unified approach works perfectly for linear partial differential equations encountered in mathematical physics and, hence, he has studied many models as examples; the Stokes equation was one of them. In fact, he observed that it is possible to generalize the viscosity term which I will use as well. The other highly interesting question is, how little regularity of the manifold can we ask for and still obtain local well-posedness of strong solutions. Aside from a Rellich-Kondrachov type condition, $C^{1,1}$-manifolds are as low in regularity as we are able to reduce the problem without having to argue with very special assumptions on the manifold. This is a rather interesting topic in itself and far beyond the scope of these notes. However, it is interesting to keep in mind that this is precisely the lower end of regularity most problems in mathematical physics can support because most problems in mathematical physics contain an operator (here, the Stokes operator) which is a relative of the "mother operator" (cf., [**14**])

$$A = \begin{pmatrix} 0 & -\nabla^* \\ \nabla & 0 \end{pmatrix}$$

with a suitable domain in the L_2 space generated by of the set of Lipschitz continuous covariant tensors. Here, ∇ denotes the co-variant derivative $d\otimes$. In other words, the $C^{1,1}$ condition is necessary for the domain of A to be sufficiently rich and, hence, a minimal condition for the problem to be meaningful.

These notes are structured in two parts. In part 1 we will discuss the (functional) analytic background and in part 2 we will use the physical textbook formulation of Navier-Stokes for incompressible fluids "re-modeling" them to find an abstract non-linear Cauchy problem which we are going to solve afterwards.

I will start with chapter 1 which seems rather random but this is a rather intriguing interpretation of the projection theorem and yields many powerful applications. Ever since Rainer Picard has introduced this to me, I have used it quite extensively and often subtly hidden. Hence, I included this chapter for the reader to see some subtle applications of the projection theorem in proving some important theorems. These methods will be used everywhere.

The content of chapter 2 has also been taught to me by Rainer Picard and it will be used throughout the notes as the spaces we are working on are tensor products and the operators are mostly of the form $1 \otimes A$ or $A \otimes 1$ even though we will only write A in both cases due to the theory explored in chapter 2. More extensive representations of the topic can be found in chapter 1 of [**15**] and the appendix A of [**20**].

The L_p-spaces used in these notes will be introduced in chapter 3. This is properly standard Lebesgue theory and nothing special; however, for the sake of notation and completeness and since it is not very common to see these L_p spaces, I have added this chapter. Furthermore, as we are on a $C^{1,1}$-manifold, it is not at all obvious why Sobolev spaces of higher order should exist. This is subject of the last section of chapter 3 though readers interested in a more detailed account should refer to chapter 2 of [**15**].

Chapter 4 contains the analytic implicit function theorem. Since the usual approach to the theorem is rather abstract, I chose to adapt a prove that was shown to me by Jürgen Voigt. This proof first proves the implicit function theorem and regularity up to C^∞ in a constructive way (which is important since it makes a major difference if we are able to construct solutions of the Navier-Stokes equations or not) such that any second year mathematics student should be able to understand it, if you explain them a few facts about Banach spaces and linear operators. Other than that it is a direct generalization of the finite dimensional theorem. In order to obtain analyticity, we then have to pull out the big guns. The proof shown here is an adaptation of the one shown in [**3**].

Finally, chapter 5 concludes the analytical background part with some facts about Fredholm operators. These will come in handy as the linearized Navier-Stokes operator is a Fredholm operator and they will allow a major shortcut in proving that the linearized Navier-Stokes operator is an isomorphism (needed for the implicit function theorem).

Part 2 starts with chapter 6 on modeling Navier-Stokes. Here, we will start from the physical equations of fluid dynamics and "re-model" them into the partial differential equation we are going to solve after identifying the spaces to work in. At this point, the Rellich-Kondrachov condition becomes vital as the equation would be ill-stated otherwise. However, I will not go into detail of the physical implications of the changed viscosity term and, thus, non-Newtonian fluids since this would fill at least a book (cf., e.g., [**1**]).

Chapter 7 will is a rather short one though important as I think the content should be part of anyone's vocabulary working with non-linear partial differential equations. Chapter 7 contains the framework of the proof, that is, how to construct solutions assuming all theorems are applicable. This will leave us with two holes to fill. First we will have to show that the linearized Navier-Stokes operator is an isomorphism. This will be addressed in chapter 8 by showing that it is an injective Fredholm operator of index zero. This approach is also a standard approach and has previously been applied to Navier-Stokes successfully in multiple special cases (cf., e.g., [**2**]).

As a corollary we will, furthermore, obtain injectivity of the Navier-Stokes operator which will be used in the causality proof in chapter 9. This is the second gap to fill in order to make the construction of chapter 7 work. Here, I had to generalize the concept of causality (cf., [**15**]) to non-linear relations which is not as straight forward as it appears. It turns out there are two slightly different notions of causality here - weak and strong causality - both having physical meaning. In fact, strong causality is what you want in a classical deterministic theory (such as the Navier-Stokes system) whereas weak causality is the most we can hope for in a quantum system with non-vanishing vacuum fluctuations. For the proof to work, weak causality would suffice but for physics to work strong causality is needed and, as it turns out, physics is fine; we can prove strong causality of the Navier-Stokes equations. Finally, we can state the well-posedness and causality theorem for Navier-Stokes of non-Newtonian fluids for strong solutions local in time.

At last, I would like to thank Ralph Chill and Rainer Picard for uncountable discussions while supervising my Dipl. Math. thesis, thus, making these notes possible. Furthermore, I thank Ralph Chill, Rainer Picard, and Jürgen Voigt for introducing me to most of theory used in these notes. Finally, I want to thank my parents for their support and patience.

Contents

Part 1

Analytic Background

CHAPTER 1

Some remarks on the Projection Theorem

We will begin by having a closer look at the projection theorem and some interesting applications as this is used throughout these notes without further mentioning. Let H_0 and H_1 be Hilbert spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $A \subseteq H_0 \oplus H_1$ a closed linear relation. By $-A$ we denote the operational minus

$$-A := \{(u, -v) \in H_0 \oplus H_1;\ (u, v) \in A\}.$$

Then the adjoint relation A^* is defined by

$$\begin{aligned}
A^* :=& -(A^{-1})^{\perp} \\
=& -\Big(\{(u,v) \in H_0 \oplus H_1;\ (u,v) \in A\}^{-1}\Big)^{\perp} \\
=& -\{(v,u) \in H_1 \oplus H_0;\ (u,v) \in A\}^{\perp} \\
=& -\{(y,x) \in H_1 \oplus H_0;\ \forall (u,v) \in A:\ \langle (y,x),(v,u)\rangle_{H_1 \oplus H_0} = 0\} \\
=& -\{(y,x) \in H_1 \oplus H_0;\ \forall (u,v) \in A:\ \langle y,v\rangle_{H_1} + \langle x,u\rangle_{H_0} = 0\} \\
=& \{(y,-x) \in H_1 \oplus H_0;\ \forall (u,v) \in A:\ \langle y,v\rangle_{H_1} + \langle x,u\rangle_{H_0} = 0\} \\
=& \{(y,x) \in H_1 \oplus H_0;\ \forall (u,v) \in A:\ \langle y,v\rangle_{H_1} + \langle -x,u\rangle_{H_0} = 0\} \\
=& \{(y,x) \in H_1 \oplus H_0;\ \forall (u,v) \in A:\ \langle x,u\rangle_{H_0} = \langle y,v\rangle_{H_1}\}.
\end{aligned}$$

The last line shows that this is the definition we want, as well as,

$$A^* = -(A^{-1})^{\perp} = (-A^{-1})^{\perp} = ((-A)^{-1})^{\perp} = -(A^{\perp})^{-1} = (-A^{\perp})^{-1} = ((-A)^{\perp})^{-1},$$

i.e., $-$, ${}^{\perp}$, and ${}^{-1}$ commute. Note that $({}^*, {}^*)$ is a Galois connection on the set of linear relations in $H_0 \oplus H_1$ with the inclusion as partial ordering. For $U \subseteq H_0$ and $V \subseteq H_1$ we will use the notation

$$\begin{aligned}
[V]A :=& \{(u,v) \in A;\ v \in V\} \text{ the pre-set of } V \text{ with respect to } A \\
A[U] :=& \{(u,v) \in A;\ u \in U\} \text{ the post-set of } U \text{ with respect to } A.
\end{aligned}$$

Note that if A was a function one would call them pre-image and image.

Theorem 1.1 (Projection Theorem). *Let H_0 and H_1 be Hilbert spaces and $A \subseteq H_0 \oplus H_1$ a closed linear relation. Then we obtain the following orthogonal decompositions.*

$$\begin{aligned}
H_0 =& [\{0\}]A \oplus \overline{A^*[H_1]} \\
H_1 =& [\{0\}]A^* \oplus \overline{A[H_0]}
\end{aligned}$$

Proof.

$$\begin{aligned}
y \in \overline{A[H_0]}^{\perp} &\Leftrightarrow y \perp A[H_0] \\
&\Leftrightarrow \forall (u,v) \in A:\ \langle y,v\rangle_{H_1} = 0 \\
&\Leftrightarrow \forall (u,v) \in A:\ \langle y,v\rangle_{H_1} + \langle 0,u\rangle_{H_0} = 0 \\
&\Leftrightarrow \forall (u,v) \in A:\ \langle (0,y),(u,v)\rangle_{H_0 \oplus H_1} = 0 \\
&\Leftrightarrow (0,y) \in A^{\perp}
\end{aligned}$$

$$\Leftrightarrow (y,0) \in (A^{\perp})^{-1}$$
$$\Leftrightarrow (y,0) \in -(A^{\perp})^{-1}$$
$$\Leftrightarrow y \in [\{0\}]A^{*}.$$

The other identity follows from dualization.

□

Remark Note that the usual version of the projection theorem reduces to proving that an orthoprojection exists and that it is self-adjoint.

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COROLLARY 1.2. *Let A and A^* be closed linear operators and $A[H_0]$ closed. Then*

$$Au = f$$

admits a solution u if and only if $f \perp [\{0\}]A^*$. Furthermore, if u_0 is a solution then the set of solutions is given by $u_0 + [\{0\}]A$.

COROLLARY 1.3 (Fredholm Alternative). *Let A be a compact operator in H_0 and $\lambda \in \mathbb{C}$. Considering*

$$(*) \qquad (\lambda - A)u = f$$

in H_0 yields the following cases.

Either *$(*)$ admits a unique solution u for every $f \in H_0$*

or *$(*)$ admits a solution u if and only if $f \perp [\{0\}](\lambda^* - A^*)$. In this case, every element of $u + [\{0\}](\lambda - A)$ solves $(*)$ and solutions are unique in $([\{0\}](\lambda - A))^{\perp}$.*

Remark The corollary above trivializes the Fredholm alternative to "Either there is a solution or not." However, the Fredholm alternative stated in this form shows that (for any compact operator A) a non-zero $\lambda \in \mathbb{C}$ is either in the resolvent set or the point spectrum of A.

■

COROLLARY 1.4. *Let $f : H_0 \to \mathbb{C}$ be a continuous linear functional. Then $f = 0$ or* $\operatorname{codim}[\{0\}]f = 1$.

PROOF. Let f be non-zero. Then f^* is non-zero, i.e., $\dim f^*[\mathbb{C}] = 1$, and $H_0 = [\{0\}]f \oplus \overline{f^*[\mathbb{C}]}$ proves the assertion.

□

COROLLARY 1.5 (Riesz' Representation Theorem). *Let $f : H_0 \to \mathbb{C}$ be a continuous linear functional. Then, there exists $x \in H_0$ such that*

$$\forall y \in H_0 : \ f(y) = \langle y, x \rangle_{H_0}.$$

PROOF. If $f = 0$ then $x = 0$ ✓

If $f \neq 0$ then choose $x_0 \in ([\{0\}]f)^{\perp}$ with $\|x\|_{H_0} = 1$ and define $x := f(x_0)^* x_0$. Then (x) is a basis of $([\{0\}]f)^{\perp}$, i.e.,

$$\begin{aligned}\forall y \in H_0 : \ \langle y, x\rangle_{H_0} &= \langle y, f(x_0)^* x_0\rangle_{H_0} \\ &= f(x_0)\langle y, x_0\rangle_{H_0} \\ &= f\left(\langle y, x_0\rangle_{H_0} x_0\right) \\ &= f(\langle y, x_0\rangle_{H_0} x_0 + \underbrace{y - \langle y, x_0\rangle_{H_0} x_0}_{\in [\{0\}]f}) \\ &= f(y).\end{aligned}$$

Let $\tilde{x} \in H_0$ be such that $\forall y \in H_0 :\ f(y) = \langle y, x\rangle_{H_0}$ holds, as well. Then

$$\forall y \in H_0 :\ 0 = \langle y, x\rangle_{H_0} - \langle y, \tilde{x}\rangle_{H_0} = \langle y, x - \tilde{x}\rangle_{H_0}$$

holds and we conclude $x = \tilde{x}$, that is, x is unique.

□

The following example shows that we may also use the projection theorem to solve PDE.

Example Let $\Omega \subseteq \mathbb{R}^n$ open and non-empty, $C_c^\infty(\Omega, \mathbb{K})$ the set of $C^\infty(\mathbb{R}^n, \mathbb{K})$ functions with compact support in Ω, $\operatorname{grad}_c :\ C_c^\infty(\Omega, \mathbb{K}) \to C_c^\infty(\Omega, \mathbb{K}^n)$ the usual gradient, and $\operatorname{div}_c :\ C_c^\infty(\Omega, \mathbb{K}^n) \to C_c^\infty(\Omega, \mathbb{K})$ the usual divergence. Then it is easy that grad_c and $-\operatorname{div}_c$ are formally adjoint (partial integration) in $L_2(\Omega, \mathbb{K}) \oplus L_2(\Omega, \mathbb{K}^n)$, that is, $\operatorname{grad}_c \subseteq -\operatorname{div}_c^*$. Note that $A \subseteq B^*$ implies

$$\bar{B} = B^{**} \subseteq A^*$$

which shows that both operators are closable if A^* and B^* are operators (that is, A and B are densely defined). Hence,

$$\operatorname{grad}_0 := \overline{\operatorname{grad}_c},\ \operatorname{div}_0 := \overline{\operatorname{div}_c},\ \operatorname{grad} := -\operatorname{div}_c^*,\ \operatorname{div} := -\operatorname{grad}_c^*$$

exist and they are all densely defined closed linear operators.

For $A \in \{\operatorname{grad}_0, \operatorname{grad}, \operatorname{div}_0, \operatorname{div}\}$ we define $H(A)$ to be the Hilbert space $D(A)$ endowed with the graph norm $\|x\|_{H(A)}^2 = \|x\|_{L_2}^2 + \|Ax\|_{L_2}^2$. The projection theorem for inclusion $H(\operatorname{grad}_0) \hookrightarrow H(\operatorname{grad})$ now shows

$$H(\operatorname{grad}) = H(\operatorname{grad}_0) \oplus H(\operatorname{grad}_0)^\perp.$$

Let $f \in H(\operatorname{grad}_0)^\perp \cap D(\operatorname{div}\operatorname{grad})$. Then

$$\begin{aligned}\forall x \in H(\operatorname{grad}_0) :\ 0 =& \langle f, x\rangle_{H(\operatorname{grad})}\\ =& \langle f, x\rangle_{L_2(\Omega,\mathbb{R})} + \langle \operatorname{grad} f, \operatorname{grad} x\rangle_{L_2(\Omega,\mathbb{R}^n)}\\ =& \langle f, x\rangle_{L_2(\Omega,\mathbb{R})} + \langle \operatorname{grad} f, \operatorname{grad}_0 x\rangle_{L_2(\Omega,\mathbb{R}^n)}\\ =& \langle f, x\rangle_{L_2(\Omega,\mathbb{R})} + \langle -\operatorname{div}\operatorname{grad} f, x\rangle_{L_2(\Omega,\mathbb{R})}\\ =& \langle (1 - \operatorname{div}\operatorname{grad}) f, x\rangle_{L_2(\Omega,\mathbb{R})}\end{aligned}$$

implies $H(\operatorname{grad}_0)^\perp = \overline{[\{0\}](1 - \operatorname{div}\operatorname{grad})} = [\{0\}](1 - \operatorname{div}\operatorname{grad})$.

We may now use this to solve the inhomogeneous Dirichlet problem

$$\varphi - \operatorname{div}\operatorname{grad}\varphi = 0,\ \varphi - f \in H(\operatorname{grad}_0),\ f \in H(\operatorname{grad}).$$

Since $H(\operatorname{grad}) = H(\operatorname{grad}_0) \oplus [\{0\}](1 - \operatorname{div}\operatorname{grad})$, there are unique $f_0 \in H(\operatorname{grad}_0)$ and $f_1 \in [\{0\}](1 - \operatorname{div}\operatorname{grad})$ such that $f = f_0 + f_1$ and we obtain

$$(1 - \operatorname{div}\operatorname{grad}) f_1 = 0,\ f_1 - f = -f_0 \in H(\operatorname{grad}_0),\ f \in H(\operatorname{grad}),$$

i.e., $\varphi = f_1$ solves the inhomogeneous Dirichlet problem by projection.

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CHAPTER 2

Tensor products of Hilbert spaces

Let $n \in \mathbb{N}$ and $(H_k)_{k\in\mathbb{N}_{\le n}}$ be a family of real[1] Hilbert spaces. For $x \in \bigtimes_{k=1}^n H_k$ we define $x_1 \otimes \ldots \otimes x_n \in (\bigtimes_{k=1}^n H_k)^*$ to be the linear functional that suffices

$$\forall u \in \bigtimes_{k=1}^{n} H_k :\ (x_1 \otimes \ldots \otimes x_n)(u) = \langle x_1, u_1\rangle_{H_1} \cdot \ldots \cdot \langle x_n, u_n\rangle_{H_n}.$$

Let

$$W_\otimes := \operatorname{lin}\left\{x_1 \otimes \ldots \otimes x_n;\ x \in \bigtimes_{k=1}^{n} H_k\right\}$$

be equipped with the bilinear continuation of

$$\langle x_1 \otimes \ldots \otimes x_n, u_1 \otimes \ldots \otimes u_n\rangle_{H_1\otimes\ldots\otimes H_n} := \langle x_1, u_1\rangle_{H_1} \cdot \ldots \cdot \langle x_n, u_n\rangle_{H_n}.$$

(i) *Symmetry*

$$\begin{aligned}
&\left\langle \sum_i \alpha_i x_{i,1} \otimes \ldots \otimes x_{i,n}, \sum_j \beta_j y_{j,1} \otimes \ldots \otimes y_{j,n}\right\rangle_{H_1\otimes\ldots\otimes H_n} \\
&= \sum_i \sum_j \alpha_i \beta_j \langle x_{i,1}, y_{j,1}\rangle_{H_1} \ldots \langle x_{i,n}, y_{j,n}\rangle_{H_n} \\
&= \sum_i \sum_j \alpha_i \beta_j \langle y_{j,1}, x_{i,1}\rangle_{H_1} \ldots \langle y_{j,n}, x_{i,n}\rangle_{H_n} \\
&= \left\langle \sum_j \beta_j y_{j,1} \otimes \ldots \otimes y_{j,n}, \sum_i \alpha_i x_{i,1} \otimes \ldots \otimes x_{i,n}\right\rangle_{H_1\otimes\ldots\otimes H_n}
\end{aligned}$$

(ii) *Non-negativity*

Since the Gramian matrices $G_k := (\langle x_{i,k}, x_{j,k}\rangle_{H_k})_{i,j\in\mathbb{N}_{\le m}}$ are positive semi-definite, the matrices $(A^{(k)}_{ij})_{i,j\in\mathbb{N}_{\le m}} := \sqrt{G_k}$ are positive semi-definite as well. Thus,

$$\begin{aligned}
&\left\langle \sum_{i=1}^m \alpha_i x_{i,1} \otimes \ldots \otimes x_{i,n}, \sum_{j=1}^m \alpha_j x_{j,1} \otimes \ldots \otimes x_{j,n}\right\rangle_{H_1\otimes\ldots\otimes H_n} \\
&= \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j \langle x_{i,1}, x_{j,1}\rangle_{H_1} \cdot \ldots \cdot \langle x_{i,n}, x_{j,n}\rangle_{H_n} \\
&= \sum_{i=1}^m \sum_{j=1}^m \sum_{s_1} \ldots \sum_{s_n} \alpha_i \alpha_j A^{(1)}_{is_1} A^{(1)}_{s_1 j} \ldots A^{(n)}_{is_n} A^{(n)}_{s_n j} \\
&= \sum_{s_1} \ldots \sum_{s_n} \left(\sum_{i=1}^m A^{(1)}_{is_1} \ldots A^{(n)}_{is_n} \alpha_i\right)\left(\sum_{j=1}^m A^{(1)}_{s_1 j} \ldots A^{(n)}_{s_n j} \alpha_j\right)
\end{aligned}$$

[1] It works complex, as well, using the obvious adaptations to obtain sesqui-linearity.

$$= \left\langle \left(\sum_{i=1}^{m} A_{is_1}^{(1)} \dots A_{is_n}^{(n)} \alpha_i \right)_{(s_1,\dots,s_n)}, \left(\sum_{i=1}^{m} A_{is_1}^{(1)} \dots A_{is_n}^{(n)} \alpha_i \right)_{(s_1,\dots,s_n)} \right\rangle_{\ell_2(\mathbb{N}_{\le m}^n)} \geq 0$$

holds.

Hence, $(W_\otimes, \langle \cdot, \cdot \rangle_{H_1 \otimes \dots \otimes H_n})$ is a semi-scalar product space and called the algebraic tensor product of $(H_k)_{k \in \mathbb{N}_{\le n}}$. We will also denote algebraic tensor products as $H_1 \overset{a}{\otimes} \dots \overset{a}{\otimes} H_n$ or $\overset{a}{\bigotimes}_{k \in \mathbb{N}_{\le n}} H_k$.

Definition 2.1. *The completion*

$$H_1 \otimes \dots \otimes H_n := \bigotimes_{k=1}^{n} H_k := \overline{W_\otimes}^{\|\cdot\|_{H_1 \otimes \dots \otimes H_n}}$$

is called tensor product of $(H_k)_{k \in \mathbb{N}_{\le n}}$ *where* $\|\cdot\|_{H_1 \otimes \dots \otimes H_k}$ *denotes the semi-norm induced by* $\langle \cdot, \cdot \rangle_{H_1 \otimes \dots \otimes H_n}$.

The empty tensor product $\bigotimes_\varnothing$ *(sometimes denoted as* $\bigotimes_\varnothing H$ *with some arbitrary Hilbert space* H*) is defined as* $\bigotimes_\varnothing := \mathbb{R}$.

Remark
(i) Due to the completion process, elements $x, y \in H_1 \otimes \dots \otimes H_n$ with $\|x - y\|_{H_1 \otimes \dots \otimes H_n} = 0$ are identified. $H_1 \otimes \dots \otimes H_n$ is a Hilbert space, thus.
(ii) The choice $\bigotimes_\varnothing := \mathbb{R}$ is senseful because both $\bigotimes_\varnothing$ and $\mathbb{R}$ act as neutral elements

$$\bigotimes_{\varnothing} \otimes \bigotimes_{i \in I} H_i = \bigotimes_{i \in I} H_i \cong \mathbb{R} \otimes \bigotimes_{i \in I} H_i.$$

(iii) The tensor products introduced here are not tensor products in the algebraic sense as, in general, they fail to have the universal property[2]; cf., [**6**].

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Example Let H be a Hilbert space and $\Omega \subseteq \mathbb{R}$ measurable. The space $L_2(\Omega; H)$ is the completion of

$$\operatorname{lin}\{t \mapsto 1_I(t)x;\ x \in H,\ I \subseteq \Omega \text{ measurable and with finite measure}\}$$

with respect to the scalar product $(f, g) \mapsto \int_\Omega \langle f(t), g(t) \rangle_H dt$. For $I \subseteq \Omega$ measurable and $x \in H$ we define

$$1_I \otimes x := (t \mapsto 1_I(t)x).$$

Obviously

$$\begin{aligned} \langle 1_I \otimes x, 1_J \otimes y \rangle_{L_2(\Omega) \otimes H} &= \int_\Omega 1_I(t) 1_J(t) dt \langle x, y \rangle_H \\ &= \int_\Omega \langle 1_I(t)x, 1_J(t)y \rangle_H dt \\ &= \langle 1_I x, 1_J y \rangle_{L_2(\Omega; H)} \end{aligned}$$

holds. Thus, the closure of the linear continuation of $(t \mapsto 1_I(t)x) \mapsto 1_I \otimes x$ defines a unitary map $U\colon L_2(\Omega; H) \to L_2(\Omega) \otimes H$.

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[2]For two infinite dimensional Hilbert spaces H_1 and H_2, there is no Hilbert space H and bounded bi-linear map $j\colon H_1 \times H_2 \to H$ such that for every Hilbert space $\tilde{H}$ and bounded bi-linear map $\tilde{j}\colon H_1 \times H_2 \to \tilde{H}$ there is a bounded linear operator $L\colon H \to \tilde{H}$ satisfying $\tilde{j} = L \circ j$.

THEOREM 2.2 (Structure of Tensor Products). *Let H_1 and H_2 be separable, infinite dimensional Hilbert spaces. Then, there exists $T \in L(H_1 \otimes H_2, L(H_2, H_1))$ satisfying*

$$\forall h_1 \in H_1 \ \forall h_2, h_2' \in H_2 : \ T(h_1 \otimes h_2)h_2' = \langle h_2, h_2' \rangle_{H_2} h_1.$$

The operator T maps $H_1 \otimes H_2$ unitarily to $HS(H_2, H_1)$, the set of Hilbert-Schmidt-operators between H_2 and H_1.

Furthermore, let $a \in H_1 \otimes H_2$. Then, there exists $\lambda \in \ell_2(\mathbb{N})$, an orthonormal basis $(\eta_i)_{i \in \mathbb{N}}$ of H_1, and and orthonormal basis $(\chi_i)_{i \in \mathbb{N}}$ of H_2 such that $\|a\|_{H_1 \otimes H_2} = \|\lambda\|_{\ell_2(\mathbb{N})}$ and

$$a = \sum_{n \in \mathbb{N}} \lambda_n \eta_n \otimes \chi_n$$

hold.

PROOF. Let $(\varphi_i)_{i \in \mathbb{N}}$ be an orthonormal basis of H_1, $(\psi_i)_{i \in \mathbb{N}}$ an orthonormal basis of H_2, and

$$\tilde{a} := \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{a}_{ij} \varphi_i \otimes \varphi_j.$$

Then, $(\tilde{a}_{ij})_{(i,j) \in \mathbb{N}^2} \in \ell_2(\mathbb{N}^2)$ with $\tilde{a}_{ij} := 0$ for $i > m$ or $j > n$, and we obtain $\|\tilde{a}\|_{H_1 \otimes H_2} = \left\|(\tilde{a}_{ij})_{(i,j) \in \mathbb{N}^2}\right\|_{\ell_2(\mathbb{N}^2)}$ by the Pythagorean theorem. Hence, we may decompose any $a \in H_1 \otimes H_2$ as $a = \sum_{i,j \in \mathbb{N}} a_{ij} \varphi_i \otimes \varphi_j$ with $\|a\|_{H_1 \otimes H_2} = \left\|(a_{ij})_{(i,j) \in \mathbb{N}^2}\right\|_{\ell_2(\mathbb{N}^2)}$.

Let $a \in H_1 \otimes H_2$ satisfy $a = \sum_{i,j \in \mathbb{N}} a_{ij} \varphi_i \otimes \varphi_j$ and $h_2 \in H_2$. Then, we define

$$T(a)h_2 := \sum_{i,j \in \mathbb{N}} a_{ij} \langle \psi_j, h_2 \rangle_{H_2} \varphi_i$$

and observe for $h_1 = \sum_{i \in \mathbb{N}} \alpha_i \varphi_i \in H_1$, $h_2 = \sum_{j \in \mathbb{N}} \beta_j \psi_j \in H_2$, and $h_2' \in H_2$

$$T(h_1 \otimes h_2)h_2' = \sum_{i,j \in \mathbb{N}} \alpha_i \beta_j \langle \psi_j, h_2' \rangle_{H_2} \varphi_i = \left\langle \sum_{j \in \mathbb{N}} \beta_j \psi_j, h_2' \right\rangle \sum_{i \in \mathbb{N}} \alpha_i \varphi_i = \langle h_2, h_2' \rangle_{H_2} h_1,$$

as well as,

$$\begin{aligned}
\|T(a)h_2\|_{H_1}^2 &= \sum_{i \in \mathbb{N}} \left| \sum_{j \in \mathbb{N}} a_{ij} \langle \psi_j, h_2 \rangle_{H_2} \right|^2 \\
&= \sum_{i \in \mathbb{N}} \left| \left\langle \sum_{j \in \mathbb{N}} a_{ij} \psi_j, h_2 \right\rangle_{H_2} \right|^2 \\
&\le \sum_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} a_{ij} \psi_j \right\|_{H_2}^2 \|h_2\|_{H_2}^2 \\
&\le \sum_{i \in \mathbb{N}} \left(\sum_{j \in \mathbb{N}} |a_{ij}| \right)^2 \|h_2\|_{H_2}^2 \\
&\le \left(\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |a_{ij}| \right)^2 \|h_2\|_{H_2}^2 \\
&= \left\|(a_{ij})_{(i,j) \in \mathbb{N}^2}\right\|_{\ell_1(\mathbb{N}^2)}^2 \|h_2\|_{H_2}^2 .
\end{aligned}$$

Thus, T extends to a bounded operator on $H_1 \otimes H_2$ and the Hilbert-Schmidt norm $\|T(a)\|_{HS}$ of $T(a)$ satisfies

$$\|T(a)\|_{HS}^2 = \sum_{k \in \mathbb{N}} \|T(a)\psi_k\|_{H_1}^2) = \sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{N}} \left| \sum_{j \in \mathbb{N}} a_{ij} \langle \psi_j, \psi_k \rangle_{H_2} \right|^2 = \sum_{i,k \in \mathbb{N}} |a_{ik}|^2 = \|a\|_{H_1 \otimes H_2}^2 ,$$

i.e., $T: H_1 \otimes H_2 \to HS(H_2, H_1)$ is an isometry.

Let $S \in HS(H_2, H_1)$. Then, $S^*S \in L(H_2)$ is compact[3], self-adjoint, and non-negative. Thus, the spectral theorem yields the existence of $N \subseteq \mathbb{N}$, an orthonormal basis $(\chi_n)_{n\in N}$ of $([\{0\}]S^*S)^\perp$ ($[\{0\}]S^*S$ is the kernel of S^*S), and $\lambda \in \mathbb{R}_{>0}^N$ such that for every $h_2 \in H_2$

$$S^*Sh_2 = \sum_{n\in N} \lambda_n^2 \langle h_2, \chi_n\rangle_{H_2} \chi_n$$

holds. Let $\eta_n := \lambda_n^{-1} S\chi_n$ for $n \in N$. Then,

$$\langle \eta_n, \eta_m\rangle_{H_1} = \lambda_n^{-1}\lambda_m^{-1}\langle \chi_n, S^*S\chi_m\rangle_{H_2} = \lambda_n^{-1}\lambda_m \langle \chi_n, \chi_m\rangle_{H_2} = \delta_{mn}$$

shows that $(\eta_n)_{n\in N}$ is an orthonormal set. Defining $a_k := \sum_{n\in N_{<k}} \lambda_n \eta_n \otimes \chi_n$ for $k \in N$, we observe for $n \in N$

$$T(a_k)\chi_n = \sum_{j\in N_{<k}} \lambda_j \langle \chi_j, \chi_n\rangle_{H_2} \eta_j = \begin{cases} \lambda_n \eta_n & , n < k \\ 0 & , n \geq k \end{cases} = \begin{cases} S\chi_n & , n < k \\ 0 & , n \geq k \end{cases}.$$

Hence, $\left\| T(a_k)|_{([\{0\}]S)^\perp} \right\|_{HS(H_2,H_1)} \leq \left\| S|_{([\{0\}]S)^\perp} \right\|_{HS(H_2,H_1)}$. Since T is an isometry, this shows that $a := \sum_{n\in N} \lambda_n \eta_n \otimes \chi_n$ converges and $T(a)|_{([\{0\}]S)^\perp} = S|_{([\{0\}]S)^\perp}$ holds. However, by definition, we have $[\{0\}]T(a) = [\{0\}]S$, i.e., $T(a) = S$, showing surjectivity of T. Since isometries are injective, we directly obtain bijectivity, too. Furthermore, setting $\lambda_n := 0$ for $n \in \mathbb{N} \setminus N$, we obtain

$$\sum_{n\in\mathbb{N}} |\lambda_n|^2 = \left\langle \sum_{n\in N} \lambda_n \eta_n \otimes \chi_n, \sum_{m\in N} \lambda_m \eta_m \otimes \chi_m \right\rangle_{H_1\otimes H_2} = \|a\|_{H_1\otimes H_2}^2$$

which shows $\lambda \in \ell_2(\mathbb{N})$ and $\|\lambda\|_{\ell_2(\mathbb{N})} = \|a\|_{H_1\otimes H_2}$; thus, completing the proof. □

Let $(H_{0,k})_{k\in\mathbb{N}_{\leq n}}$ and $(H_{1,k})_{k\in\mathbb{N}_{\leq n}}$ be families of real Hilbert spaces and for each $k \in \mathbb{N}_{\leq n}$ let $A_k \subseteq H_{0,k} \oplus H_{1,k}$ be a linear operator[4]. We define

$$A_1 \dot{\otimes} \ldots \dot{\otimes} A_n : \bigotimes_{k\in\mathbb{N}_{\leq n}} H_{0,k} \to \bigotimes_{k\in\mathbb{N}_{\leq n}} H_{1,k}$$

as linear continuation of $x_1 \otimes \ldots \otimes x_n \mapsto (A_1x_1) \otimes \ldots \otimes (A_nx_n)$ with

$$D(A_1 \dot{\otimes} \ldots \dot{\otimes} A_n) := \bigotimes_{k\in\mathbb{N}_{\leq n}}^{a} D(A_k).$$

We will also use the abbreviation $\dot{\bigotimes}_{k\in\mathbb{N}_{\leq n}} A_k$ for $A_1 \dot{\otimes} \ldots \dot{\otimes} A_n$.

---

[3]Every Hilbert-Schmidt operator is compact.

[4]We do not distinguish between an operator (or, more generally, a function) and its graph as a function $f: X \to Y$ is, by definition, a right-unique, that is, single-valued, relation which is usually considered the graph of the function. Furthermore, we do not assume a function to be left-total since closed unbounded operators in Banach spaces may at most be densely defined.

Also note that we induce a topology on $X \oplus Y$ if X and Y are Banach spaces. This topology can be defined using the norm $\|(x,y)\|_{X\oplus Y} := \|x\|_X + \|y\|_Y$ or $\|(x,y)\|_{X\oplus Y} = \max\{\|x\|_X, \|y\|_Y\}$. In the Hilbert space case, it is common to choose $\|(x,y)\|_{X\oplus Y} := \sqrt{\|x\|_X^2 + \|y\|_Y^2}$ since then $X \oplus Y$ is a Hilbert space assuming X and Y are.

Since $A_1 \dot{\otimes} \ldots \dot{\otimes} A_n$ is a linear continuation, it is a linear subspace of

$$\bigotimes_{k \in \mathbb{N}_{\leq n}} H_{0,k} \oplus \bigotimes_{k \in \mathbb{N}_{\leq n}} H_{1,k},$$

thus, we only need to make sure that for all $(0, w) \in A_1 \dot{\otimes} \ldots \dot{\otimes} A_n$

$$w = 0$$

holds for $A_1 \dot{\otimes} \ldots \dot{\otimes} A_n$ to be an operator. We may express w by

$$w = \sum_i \alpha_i (A_1 x_{i,1}) \otimes \ldots \otimes (A_n x_{i,n})$$

with

$$0 = \sum_i \alpha_i x_{i,1} \otimes \ldots \otimes x_{i,n}$$

and, thus, we observe

$$\begin{aligned} \forall k \in \mathbb{N}_{\leq n} \ \forall u_k \in H_{0,k}: \ 0 &= \sum_i \alpha_i \langle x_{i,1}, u_1 \rangle_{H_{0,1}} \cdot \ldots \cdot \langle x_{i,n}, u_n \rangle_{H_{0,n}} \\ &= \left\langle \sum_i \alpha_i \langle x_{i,2}, u_2 \rangle_{H_{0,2}} \cdot \ldots \cdot \langle x_{i,n}, u_n \rangle_{H_{0,n}} x_{i,1}, u_1 \right\rangle_{H_{0,1}}. \end{aligned}$$

Without loss of generality we may assume that the $x_{i,k}$ are linearly independent in $H_{0,k}$ yielding

$$0 = \alpha_i \langle x_{i,2}, u_2 \rangle_{H_{0,2}} \cdot \ldots \cdot \langle x_{i,n}, u_n \rangle_{H_{0,n}}$$

for every i. Since none of the $x_{i,k}$ is zero

$$\forall i: \ \alpha_i = 0$$

needs to hold. Hence, $w = 0$. $A_1 \dot{\otimes} \ldots \dot{\otimes} A_n$ is an operator, thus. If it is closable, we will denote the closure by

$$A_1 \otimes \ldots \otimes A_n.$$

Remark In fact, if all A_k are closed operators, then they are Hilbert spaces with respect to the graph norm and the tensor product of the operators is isometrically isomorphic to the tensor product of Hilbert spaces. In particular, $A_1 \dot{\otimes} \ldots \dot{\otimes} A_n \cong A_1 \overset{a}{\otimes} \ldots \overset{a}{\otimes} A_n$

■

LEMMA 2.3. *Let H_0 and H_1 be Hilbert spaces, $S_0 \subseteq H_0$ be total, i.e.,* $\operatorname{lin} S_0$ *is dense in H_0, and $S_1 \subseteq H_1$ total. Then, $S_0 \overset{a}{\otimes} S_1$ is dense in $H_0 \otimes H_1$.*

PROOF. Let $x \in H_0$ and $y \in H_1$. Then, there are sequences $(x_n)_{n \in \mathbb{N}} \in (\operatorname{lin} S_0)^{\mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}} \in (\operatorname{lin} S_1)^{\mathbb{N}}$ with $x_n \to x$ in H_0 and $y_n \to y$ in H_1.

Let $n \in \mathbb{N}$. Then, there are $k, m \in \mathbb{N}$, $s_1^0, \ldots, s_k^0 \in S_0$, $s_1^1, \ldots, s_m^1 \in S_1$, and $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m \in \mathbb{R}$ such that $x_n = \sum_{j=1}^k \alpha_j s_j^0$ and $y_n = \sum_{j=1}^m \beta_j s_j^1$. Hence,

$$x_n \otimes y_n = \left(\sum_{i=1}^k \alpha_i s_i^0\right) \otimes \left(\sum_{j=1}^m \beta_j s_j^1\right) = \sum_{i=1}^k \sum_{j=1}^m \alpha_i \beta_j s_i^0 \otimes s_j^1 \in S_0 \overset{a}{\otimes} S_1.$$

Furthermore, we obtain

$$\begin{aligned} \|x_n \otimes y_n - x \otimes y\|_{H_0 \otimes H_1} &\leq \|x_n \otimes y_n - x \otimes y_n\|_{H_0 \otimes H_1} + \|x \otimes y_n - x \otimes y\|_{H_0 \otimes H_1} \\ &= \|(x_n - x) \otimes y_n\|_{H_0 \otimes H_1} + \|x \otimes (y_n - y)\|_{H_0 \otimes H_1} \\ &= \underbrace{\|x_n - x\|_{H_0}}_{\to 0} \underbrace{\|y_n\|_{H_1}}_{\text{bounded}} + \|x\|_{H_0} \underbrace{\|y_n - y\|_{H_1}}_{\to 0} \\ &\to 0. \end{aligned}$$

Hence, all simple tensors can be approximated by elements of $S_0 \overset{a}{\otimes} S_1$.

Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in H_0$, $y_1, \ldots, y_n \in H_1$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, and $\varepsilon \in \mathbb{R}_{>0}$. Since simple tensors can be approximated by elements of $S_0 \overset{a}{\otimes} S_1$, there are elements $u_i \in S_0 \overset{a}{\otimes} S_1$ such that

$$\|x_i \otimes y_i - u_i\|_{H_0 \otimes H_1} < \left(\sum_{j=1}^n |\alpha_j|\right)^{-1} \varepsilon$$

holds for every $i \in \mathbb{N}_{\leq n}$. Hence,

$$\left\|\sum_{i=1}^n \alpha_i x_i \otimes y_i - \sum_{i=1}^n \alpha_i u_i\right\|_{H_0 \otimes H_1} \leq \sum_{i=1}^n |\alpha_i| \, \|x_i \otimes y_i - u_i\|_{H_0 \otimes H_1} < \varepsilon$$

shows density of $S_0 \overset{a}{\otimes} S_1$ in $H_0 \overset{a}{\otimes} H_1$ and, thus, the assertion as well.

□

Corollary 2.4. *Let H_0 and H_1 be Hilbert spaces, and $O_0 \subseteq H_0$ and $O_1 \subseteq H_1$ two complete orthonormal sets. Then,*

$$[O_0] \otimes [O_1] := \{u \otimes v;\ u \in O_0 \ \wedge\ v \in O_1\}$$

is a complete orthonormal set in $H_0 \otimes H_1$.

Proof. We already know that $O_0 \overset{a}{\otimes} O_1$ is dense in $H_0 \otimes H_1$, i.e., $[O_0] \otimes [O_1]$ is total. Hence, it suffices to show that $[O_0] \otimes [O_1]$ is orthonormal. Let $u \otimes v, x \otimes y \in [O_0] \otimes [O_1]$. Then,

$$\langle u \otimes v, x \otimes y \rangle_{H_0 \otimes H_1} = \langle u, x \rangle_{H_0} \langle v, y \rangle_{H_1} = \begin{cases} 1 & , \ u = x \ \wedge \ v = y \\ 0 & , \ u \neq x \ \vee \ v \neq y \end{cases}$$

shows the assertion.

□

Proposition 2.5. *Let H_{00}, H_{01}, H_{10}, and H_{11} be Hilbert spaces, and $A \subseteq H_{00} \oplus H_{01}$ and $B \subseteq H_{10} \oplus H_{11}$ densely defined closable linear operators. Then, $A \dot{\otimes} B$ is closable and*

$$A \otimes B = \overline{A \dot{\otimes} B} \subseteq (A^* \dot{\otimes} B^*)^*$$

holds.

PROOF. Let $\xi = \sum_{i=1}^n \alpha_i x_i \otimes y_i \in D(A^*)\overset{a}{\otimes}D(B^*)$ and $\eta = \sum_{j=1}^m \beta_j u_j \otimes v_j \in D(A)\overset{a}{\otimes}D(B)$. Then, we observe

$$\begin{aligned}\langle A\dot{\otimes}B\eta, \xi\rangle_{H_{01}\otimes H_{11}} &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i\beta_j \langle Au_i, x_j\rangle_{H_{01}} \langle Bv_i, y_j\rangle_{H_{11}} \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i\beta_j \langle u_i, A^* x_j\rangle_{H_{00}} \langle v_i, B^* y_j\rangle_{H_{10}} \\ &= \langle \eta, A^*\dot{\otimes}B^*\xi\rangle_{H_{00}\otimes H_{10}},\end{aligned}$$

that is, $A^*\dot{\otimes}B^* \subseteq (A\dot{\otimes}B)^*$, which implies

$$A\dot{\otimes}B \subseteq \bar{A}\dot{\otimes}\bar{B} = A^{**}\dot{\otimes}B^{**} \subseteq (A^*\dot{\otimes}B^*)^*.$$

Since A and B are closable, A^* and B^* are densely defined, and Lemma 2.3 yields that $A^*\dot{\otimes}B^*$ is densely defined, i.e., $(A^*\dot{\otimes}B^*)^*$ is a closed linear operator.

□

Example Let H_1, H_2 be Hilbert spaces, $A \subseteq H_2 \oplus H_2$ be a closed and densely defined linear operator. The operator $\mathcal{A}$ defined by the $H_1 \otimes H_2$-closure of $x \otimes y \mapsto x \otimes Ay$ can be expressed by

$$\mathcal{A} = 1 \otimes A$$

■

PROPOSITION 2.6. *Let H_{00}, H_{01}, H_{10}, and H_{11} be Hilbert spaces, and $A \subseteq H_{00} \oplus H_{01}$ and $B \subseteq H_{10} \oplus H_{11}$ densely defined closable linear operators. Then,*

$$A \otimes B = \bar{A} \otimes \bar{B}.$$

PROOF. Clearly,

$$A\dot{\otimes}B \subseteq \bar{A}\dot{\otimes}\bar{B} \subseteq \bar{A} \otimes \bar{B}$$

holds and, hence, $A \otimes B \subseteq \bar{A} \otimes \bar{B}$. Let $x = \sum_{i=1}^n \alpha_i \xi_i \otimes \eta_i \in D(\bar{A})\overset{a}{\otimes}D(\bar{B}) = D(\bar{A}\dot{\otimes}\bar{B})$, $x \neq 0$, and $\varepsilon \in \mathbb{R}_{>0}$. Then, we can find $x_i \in D(A)$ and $y_i \in D(B)$ such that for every $i \in \mathbb{N}_{\leq n}$

$$\begin{aligned}\|x_i - \xi_i\|_{H_{00}} &< \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j| \, \|\eta_j\|_{H_{10}}}, \\ \|Ax_i - \bar{A}\xi_i\|_{H_{01}} &< \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j| \, \|\bar{B}\eta_j\|_{H_{11}}}, \\ \|y_i - \eta_i\|_{H_{10}} &< \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j| \, \|x_j\|_{H_{00}}},\end{aligned}$$

and

$$\|By_i - \bar{B}\eta_i\|_{H_{11}} < \frac{\varepsilon}{2\sum_{j=1}^n |\alpha_j| \, \|Ax_j\|_{H_{01}}}$$

hold. Setting $y := \sum_{j=1}^n \alpha_j x_j \otimes y_j \in D(A\dot{\otimes}B)$ yields

$$\begin{aligned}\|y - x\|_{H_{00}\otimes H_{10}} &= \left\| \sum_{i=1}^n \alpha_i (x_i \otimes y_i - \xi_i \otimes \eta_i) \right\|_{H_{00}\otimes H_{10}} \\ &= \left\| \sum_{i=1}^n \alpha_i (x_i \otimes (y_i - \eta_i) + (x_i - \xi_i) \otimes \eta_i) \right\|_{H_{00}\otimes H_{10}} \\ &\leq \sum_{i=1}^n |\alpha_i| \, \|x_i\|_{H_{00}} \, \|y_i - \eta_i\|_{H_{10}} + \sum_{i=1}^n |\alpha_i| \, \|x_i - \xi_i\|_{H_{00}} \, \|eta_i\|_{H_{10}}\end{aligned}$$

$$<\varepsilon$$

and

$$\begin{aligned}\left\|(A\dot{\otimes}B)y-(\bar{A}\dot{\otimes}\bar{B})x\right\|_{H_{01}\otimes H_{11}} &= \left\|\sum_{i=1}^{n}\alpha_i(Ax_i\otimes By_i-\bar{A}\xi_i\otimes\bar{B}\eta_i)\right\|_{H_{00}\otimes H_{10}} \\ &\leq \sum_{i=1}^{n}|\alpha_i|\left\|Ax_i\right\|_{H_{01}}\left\|By_i-\bar{B}\eta_i\right\|_{H_{11}} \\ &\quad+\sum_{i=1}^{n}|\alpha_i|\left\|Ax_i-\bar{A}\xi_i\right\|_{H_{00}}\left\|\bar{B}\eta_i\right\|_{H_{10}} \\ &<\varepsilon\end{aligned}$$

Thus, $x\in D(A\otimes B)$ and $A\otimes Bx=\bar{A}\dot{\otimes}\bar{B}x$, i.e., $\bar{A}\dot{\otimes}\bar{B}\subseteq A\otimes B$; thus,

$$\bar{A}\otimes\bar{B}\subseteq A\otimes B.$$

□

PROPOSITION 2.7. *Let H_0, H_1, and H_2 be Hilbert spaces. Then,*

$$(H_0\otimes H_1)\otimes H_2=H_0\otimes(H_1\otimes H_2)=H_0\otimes H_1\otimes H_2$$

in the sense of unitary equivalence.

PROOF. For $\varphi\in H_0$, $\psi\in H_1$, and $\chi\in H_2$, we set

$$U((\varphi\otimes\psi)\otimes\chi):=\varphi\otimes(\psi\otimes\chi)$$

and extend this mapping to $\left(H_0\overset{a}{\otimes}H_1\right)\overset{a}{\otimes}H_2$ by

$$\begin{aligned}U:\ \left(H_0\overset{a}{\otimes}H_1\right)\overset{a}{\otimes}H_2 &\to H_0\otimes(H_1\otimes H_2)\\ \sum_{j=1}^{m}\beta_j\left(\sum_{i=1}^{n_j}\alpha_i^jx_i^j\otimes y_i^j\right)\otimes z_j &\mapsto \sum_{j=1}^{m}\beta_j\sum_{i=1}^{n_j}\alpha_i^jx_i^j\otimes(y_i^j\otimes z_j).\end{aligned}$$

First, we will prove that this extension is still right-unique. Let $\varphi\in\left(H_0\overset{a}{\otimes}H_1\right)\overset{a}{\otimes}H_2$ with

$$\varphi=\sum_{j=1}^{m}\beta_j\left(\sum_{i=1}^{n_j}\alpha_i^jx_i^j\otimes y_i^j\right)\otimes z_j=\sum_{j=1}^{p}\delta_j\left(\sum_{i=1}^{k_j}\gamma_i^ju_i^j\otimes v_i^j\right)\otimes w_j.$$

Then, we observe for all $a\in H_0$, $b\in H_1$, and $c\in H_2$,

$$\begin{aligned}U\left(\sum_{j=1}^{m}\beta_j\left(\sum_{i=1}^{n_j}\alpha_i^jx_i^j\otimes y_i^j\right)\otimes z_j\right)(a,b\otimes c) &= \left(\sum_{j=1}^{m}\beta_j\sum_{i=1}^{n_j}\alpha_i^jx_i^j\otimes(y_i^j\otimes z_j)\right)(a,b\otimes c)\\ &=\sum_{j=1}^{m}\beta_j\sum_{i=1}^{n_j}\alpha_i^j\langle x_i^j,a\rangle_{H_0}\langle y_i^j\otimes z_j,b\otimes c\rangle_{H_1\otimes H_2}\\ &=\sum_{j=1}^{m}\beta_j\sum_{i=1}^{n_j}\alpha_i^j\langle x_i^j,a\rangle_{H_0}\langle y_i^j,b\rangle_{H_1}\langle z_j,c\rangle_{H_2}\\ &=\sum_{j=1}^{m}\beta_j\sum_{i=1}^{n_j}\alpha_i^j\langle x_i^j\otimes y_i^j,a\otimes b\rangle_{H_0\otimes H_1}\langle z_j,c\rangle_{H_2}\\ &=\sum_{j=1}^{m}\beta_j\left\langle\sum_{i=1}^{n_j}\alpha_i^jx_i^j\otimes y_i^j,a\otimes b\right\rangle_{H_0\otimes H_1}\langle z_j,c\rangle_{H_2}\\ &=\varphi(a\otimes b,c).\end{aligned}$$

The same calculation also shows

$$U\left(\sum_{j=1}^{p}\delta_j\left(\sum_{i=1}^{k_j}\gamma_i^j u_i^j\otimes v_i^j\right)\otimes w_j\right)(a,b\otimes c)=\varphi(a\otimes b,c).$$

Since these are continuous bi-linear functionals and $H_0\overset{a}{\otimes}\left(H_1\overset{a}{\otimes}H_2\right)$ is dense in $H_0\otimes(H_1\otimes H_2)$, we conclude right-uniqueness of U.

Furthermore, U is linear since, for $\kappa\in\mathbb{R}$ and $\varphi,\psi\in H_0\overset{a}{\otimes}\left(H_1\overset{a}{\otimes}H_2\right)$ with

$$\varphi=\sum_{j=1}^{m}\beta_j\left(\sum_{i=1}^{n_j}\alpha_i^j x_i^j\otimes y_i^j\right)\otimes z_j$$

and

$$\psi=\sum_{j=1}^{p}\delta_j\left(\sum_{i=1}^{k_j}\gamma_i^j u_i^j\otimes v_i^j\right)\otimes w_j,$$

we obtain

$$\kappa\varphi+\psi=\sum_{j=1}^{m+p}\zeta_j\left(\sum_{i=1}^{l_j}\eta_i^j\vartheta_i^j\otimes\lambda_i^j\right)\otimes\nu_j$$

with

$$\zeta_j=\begin{cases}\kappa\beta_j & ,\ j\in\mathbb{N}_{\le m}\\ \delta_{j-m} & ,\ j\in[m+1,m+p]\cap\mathbb{N}\end{cases},$$

$$l_j=\begin{cases}n_j & ,\ j\in\mathbb{N}_{\le m}\\ k_{j-m} & ,\ j\in[m+1,m+p]\cap\mathbb{N}\end{cases},$$

$$\eta_i^j=\begin{cases}\alpha_i^j & ,\ j\in\mathbb{N}_{\le m}\\ \gamma_i^{j-m} & ,\ j\in[m+1,m+p]\cap\mathbb{N}\end{cases},$$

$$\vartheta_i^j=\begin{cases}x_i^j & ,\ j\in\mathbb{N}_{\le m}\\ u_i^{j-m} & ,\ j\in[m+1,m+p]\cap\mathbb{N}\end{cases},$$

$$\lambda_i^j=\begin{cases}y_i^j & ,\ j\in\mathbb{N}_{\le m}\\ v_i^{j-m} & ,\ j\in[m+1,m+p]\cap\mathbb{N}\end{cases},$$

and

$$\nu_j=\begin{cases}z_j & ,\ j\in\mathbb{N}_{\le m}\\ w_{j-m} & ,\ j\in[m+1,m+p]\cap\mathbb{N}\end{cases}.$$

Hence,

$$\begin{aligned}U(\kappa\varphi+\psi)&=\sum_{j=1}^{m+p}\zeta_j\sum_{i=1}^{l_j}\eta_i^j\vartheta_i^j\otimes\left(\lambda_i^j\otimes\nu_j\right)\\&=\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{l_j}\eta_i^j\vartheta_i^j\otimes\left(\lambda_i^j\otimes\nu_j\right)+\sum_{j=m+1}^{m+p}\zeta_j\sum_{i=1}^{l_j}\eta_i^j\vartheta_i^j\otimes\left(\lambda_i^j\otimes\nu_j\right)\\&=\kappa\sum_{j=1}^{m}\beta_j\sum_{i=1}^{n_j}\alpha_i^j x_i^j\otimes\left(y_i^j\otimes z_j\right)+\sum_{j=1}^{p}\delta_j\sum_{i=1}^{k_j}\gamma_i^j u_i^j\otimes\left(v_i^j\otimes w_j\right)\\&=\kappa U(\varphi)+U(\psi).\end{aligned}$$

Now, we can show that U is an isometry. This follows from

$$\|U\varphi\|^2_{H_0\otimes(H_1\otimes H_2)}=\sum_{j,k=1}^{m}\beta_j\beta_k\sum_{i=1}^{n_j}\sum_{l=1}^{n_k}\alpha_i^j\alpha_l^k\langle x_i^j\otimes(y_i^j\otimes z_j),x_l^k\otimes(y_l^k\otimes z_k)\rangle_{H_0\otimes(H_1\otimes H_2)}$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{m} \beta_j \beta_k \sum_{i=1}^{n_j} \sum_{l=1}^{n_k} \alpha_i^j \alpha_l^k \langle x_i^j, x_l^k \rangle_{H_0} \langle y_i^j, y_l^k \rangle_{H_1} \langle z_j, z_k \rangle_{H_2} \\
&= \sum_{j,k=1}^{m} \beta_j \beta_k \left\langle \left(\sum_{i=1}^{n_j} \alpha_i^j x_i^j \otimes y_i^j \right) \otimes z_j, \left(\sum_{l=1}^{n_k} \alpha_l^k x_l^k \otimes y_l^k \right) \otimes z_k \right\rangle_{(H_0 \otimes H_1) \otimes H_2} \\
&= \|\varphi\|^2_{(H_0 \otimes H_1) \otimes H_2} .
\end{aligned}
$$

Finally, if we show that U has dense range, then we can extend U to a unitary operator. Since $H_0 \overset{a}{\otimes} \left(H_1 \overset{a}{\otimes} H_2 \right)$ is dense, it suffices to show that every

$$
\psi =: \sum_{j=1}^{m} \beta_j x_j \otimes \left(\sum_{i=1}^{n_j} \alpha_i^j y_i^j \otimes z_i^j \right) \in H_0 \overset{a}{\otimes} \left(H_1 \overset{a}{\otimes} H_2 \right)
$$

is an image of U. Let

$$
\varphi := \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_j} \alpha_i^j (x_i \otimes y_i^j) \otimes z_i^j \in \left(H_0 \overset{a}{\otimes} H_1 \right) \overset{a}{\otimes} H_2 .
$$

Then, linearity of U implies

$$
\begin{aligned}
U\varphi &= \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_j} \alpha_i^j U \left((x_i \otimes y_i^j) \otimes z_i^j \right) \\
&= \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n_j} \alpha_i^j x_i \otimes (y_i^j \otimes z_i^j) \\
&= \psi .
\end{aligned}
$$

The other assertion, $H_0 \otimes (H_1 \otimes H_2) = H_0 \otimes H_1 \otimes H_2$, follows similarly.

□

COROLLARY 2.8. *For $i \in \{0,1,2\}$ and $j \in \{0,1\}$, let H_{ij} be a Hilbert space and $A_0 \subseteq H_{00} \oplus H_{01}$, $A_1 \subseteq H_{10} \oplus H_{11}$, and $A_2 \subseteq H_{20} \oplus H_{21}$ densely defined closable linear operators. Then,*

$$
(A_0 \otimes A_1) \otimes A_2 = A_0 \otimes (A_1 \otimes A_2) = A_0 \otimes A_1 \otimes A_2 .
$$

PROPOSITION 2.9. *For $i, j \in \{0,1\}$, let H_{ij} be a Hilbert space, $A_0 \in L(H_{00}, H_{01})$, and $A_1 \in L(H_{10}, H_{11})$. Then, $A_0 \otimes A_1 \in L(H_{00} \otimes H_{10}, H_{01} \otimes H_{11})$ with*

$$
\|A_0 \otimes A_1\|_{L(H_{00} \otimes H_{10}, H_{01} \otimes H_{11})} = \|A_0\|_{L(H_{00}, H_{01})} \|A_1\|_{L(H_{10}, H_{11})} .
$$

PROOF. Let S_{ij} be a complete orthonormal set in H_{ij} and$x = \sum_{j=1}^{l} \kappa_j \varphi_j \otimes \psi_j \in H_{00} \overset{a}{\otimes} H_{10}$. Then, we can find sequences $\alpha^j, \beta^j \in \mathbb{R}^{\mathbb{N}}$, $\zeta \in S_{00}^{\mathbb{N}}$, and $\xi \in S_{10}^{\mathbb{N}}$ such that for every $j \in \mathbb{N}$

$$
\begin{aligned}
\kappa_j \varphi_j &= \sum_{n \in \mathbb{N}} \alpha_n^j \zeta_n , \\
\psi_j &= \sum_{n \in \mathbb{N}} \beta_n^j \xi_n ,
\end{aligned}
$$

and, hence,

$$
x = \sum_{j=1}^{l} (\kappa_j \varphi_j) \otimes \psi_j = \sum_{m,n \in \mathbb{N}} \underbrace{\sum_{j=1}^{l} \alpha_n^j \beta_m^j}_{=:\gamma_{nm}} \zeta_n \otimes \xi_m
$$

hold. Since A_0 and A_1 are continuous, we observe

$$
A_0 \otimes A_1 x = \sum_{j=1}^{l} A_0 \kappa_j \varphi_j \otimes A_1 \psi_j
$$

$$= \sum_{m,n\in\mathbb{N}} \sum_{j=1}^{n} \alpha_n^j \beta_m^j A_0\zeta_n \otimes A_1\xi_m$$
$$= \sum_{m,n\in\mathbb{N}} \gamma_{nm} A_0\zeta_n \otimes A_1\xi_m.$$

Let $y = \sum_{i=1}^{k} \lambda_i\sigma_i \otimes \tau_i \in H_{01} \overset{a}{\otimes} H_{11}$. Then, we can find sequences $\nu^j, \varrho^j \in \mathbb{R}^{\mathbb{N}}$, $\eta \in S_{01}^{\mathbb{N}}$, and $\vartheta \in S_{11}^{\mathbb{N}}$ such that for every $j \in \mathbb{N}$

$$\lambda_j\sigma_j = \sum_{n\in\mathbb{N}} \nu_n^j \eta_n,$$
$$\tau_j = \sum_{n\in\mathbb{N}} \varrho_n^j \vartheta_n,$$

and, hence,

$$y = \sum_{j=1}^{l} (\lambda_j\sigma_j) \otimes \tau_j = \sum_{m,n\in\mathbb{N}} \underbrace{\sum_{j=1}^{l} \nu_n^j \varrho_m^j}_{=:\delta_{nm}} \eta_n \otimes \vartheta_m$$

hold. We, thus, observe

$$\langle A_0 \otimes A_1 x, y\rangle_{H_{01}\otimes H_{11}} = \sum_{m,n,s,t\in\mathbb{N}} \gamma_{nm}\delta_{st} \langle A_0\zeta_n, \eta_s\rangle_{H_{01}} \langle A_1\xi_m, \vartheta_t\rangle_{H_{11}}$$
$$= \sum_{m,n,s,t\in\mathbb{N}} \gamma_{nm}\delta_{st} \langle A_0\zeta_n, \eta_s\rangle_{H_{01}} \langle \xi_m, A_1^*\vartheta_t\rangle_{H_{11}}$$
$$= \sum_{m,s\in\mathbb{N}} \left\langle A_0 \sum_{n\in\mathbb{N}} \gamma_{nm}\zeta_n, \eta_s \right\rangle_{H_{01}} \left\langle \xi_m, A_1^* \sum_{t\in\mathbb{N}} \delta_{st}\vartheta_t \right\rangle_{H_{11}}$$

and, by Cauchy-Schwarz,

$$|\langle A_0 \otimes A_1 x, y\rangle_{H_{01}\otimes H_{11}}|^2$$
$$\le \left(\sum_{m,s\in\mathbb{N}} \left\langle A_0 \sum_{n\in\mathbb{N}} \gamma_{nm}\zeta_n, \eta_s \right\rangle_{H_{01}} \right)^2 \left(\sum_{m,s\in\mathbb{N}} \left\langle \xi_m, A_1^* \sum_{t\in\mathbb{N}} \delta_{st}\vartheta_t \right\rangle_{H_{11}} \right)^2$$

which yields (using Bessel's inequality and orthonormality of ζ and ϑ)

$$|\langle A_0 \otimes A_1 x, y\rangle_{H_{01}\otimes H_{11}}|^2$$
$$\le \left(\sum_{m\in\mathbb{N}} \left\| A_0 \sum_{n\in\mathbb{N}} \gamma_{nm}\zeta_n \right\|_{H_{01}}^2 \right) \left(\sum_{s\in\mathbb{N}} \left\| A_1^* \sum_{t\in\mathbb{N}} \delta_{st}\vartheta_t \right\|_{H_{11}}^2 \right)$$
$$\le \|A_0\|_{L(H_{00},H_{01})}^2 \|A_1\|_{L(H_{10},H_{11})}^2 \left(\sum_{m\in\mathbb{N}} \left\| \sum_{n\in\mathbb{N}} \gamma_{nm}\zeta_n \right\|_{H_{01}}^2 \right) \left(\sum_{s\in\mathbb{N}} \left\| \sum_{t\in\mathbb{N}} \delta_{st}\vartheta_t \right\|_{H_{11}}^2 \right)$$
$$\le \|A_0\|_{L(H_{00},H_{01})}^2 \|A_1\|_{L(H_{10},H_{11})}^2 \left(\sum_{m,n\in\mathbb{N}} |\gamma_{nm}|^2 \right) \left(\sum_{s,t\in\mathbb{N}} |\delta_{st}|^2 \right)$$
$$= \|A_0\|_{L(H_{00},H_{01})}^2 \|A_1\|_{L(H_{10},H_{11})}^2 \|x\|_{H_{00}\otimes H_{10}}^2 \|y\|_{H_{01}\otimes H_{11}}^2.$$

For $y = A_0 \otimes A_1 x$, this implies

$$\|A_0 \otimes A_1 x\|_{H_{01}\otimes H_{11}} \le \|A_0\|_{L(H_{00},H_{01})} \|A_1\|_{L(H_{10},H_{11})} \|x\|_{H_{00}\otimes H_{10}},$$

i.e.,

$$\|A_0 \otimes A_1\|_{L(H_{00}\otimes H_{10}, H_{01}\otimes H_{11})} \le \|A_0\|_{L(H_{00},H_{01})} \|A_1\|_{L(H_{10},H_{11})}.$$

On the other hand, let $x \in B_{H_{00}}^{\mathbb{N}}$ and $y \in B_{H_{10}}^{\mathbb{N}}$ with $\|A_0x_n\|_{H_{01}} \to \|A_0\|_{L(H_{00},H_{01})}$ and $\|A_1y_n\|_{H_{11}} \to \|A_1\|_{L(H_{10},H_{11})}$ for $n \to \infty$. Then, $(x_n \otimes y_n)_{n\in\mathbb{N}} \in B_{H_{00}\otimes H_{10}}^{\mathbb{N}}$ and

$$\|A_0 \otimes A_1 x_n \otimes y_n\|_{H_{01}\otimes H_{11}} = \|A_0x_n\|_{H_{01}} \|A_1y_n\|_{H_{11}} \to \|A_0\|_{L(H_{00},H_{01})} \|A_1\|_{L(H_{10},H_{11})}$$

completes the proof.

□

Observation 2.10. *For $i, j \in \{0,1\}$, let H_{ij} be a Hilbert space and $A_i \in L(H_{i0}, H_{i1})$. Then, $A_0 \otimes 1$ and $1 \otimes A_1$ commute. Furthermore, $(A_0 \otimes 1)(1 \otimes A_1)$ and $(1 \otimes A_1)(A_0 \otimes 1)$ are bounded operators.*

Proof. Boundedness of $(A_0 \otimes 1)(1 \otimes A_1)$ and $(1 \otimes A_1)(A_0 \otimes 1)$ follows directly from boundedness of $A_0 \otimes 1$ and $1 \otimes A_1$.

Let $x \in H_{00}$ and $y \in H_{01}$. Then,

$$\begin{aligned}(A_0 \otimes 1)(1 \otimes A_1)x \otimes y =&(A_0 \otimes 1)x \otimes A_1 y\\ =&A_0 x \otimes A_1 y\\ =&(1 \otimes A_1)A_0 x \otimes y\\ =&(1 \otimes A_1)(A_0 \otimes 1)x \otimes y\end{aligned}$$

shows that the commutator $[A_0 \otimes 1, 1 \otimes A_1]$ vanishes on all algebraic tensors, i.e., $[A_0 \otimes 1, 1 \otimes A_1] = 0$ by boundedness.

□

Observation 2.11. *Let H_0, H_1, and H_2 be Hilbert spaces, and $A \subseteq H_1 \oplus H_2$ a densely defined closable linear operator. Then,*

$$(1 \otimes A)^* = 1 \otimes A^*$$

and

$$(A \otimes 1)^* = A^* \otimes 1.$$

Proof. So far, we know

$$1 \otimes A^* \subseteq (1 \otimes \bar{A})^* = (1 \otimes A)^*.$$

To show the missing inclusion let $x \in D((1 \otimes A)^*) \subseteq H_0 \otimes H_2$ and $S_i \subseteq H_i$ an orthonormal basis for $i \in \{0,1,2\}$. Then, $[S_0] \otimes [S_j]$ is an orthonormal basis of $H_0 \otimes H_j$ for $j \in \{1,2\}$. Hence, there are sequences $\xi \in S_0^{\mathbb{N}}$, $\zeta \in S_1^{\mathbb{N}}$, and $\eta \in S_2^{\mathbb{N}}$ such that

$$x = \sum_{n \in \mathbb{N}} \langle \xi_n \otimes \eta_n, x \rangle_{H_0 \otimes H_2} \xi_n \otimes \eta_n$$

and

$$(1 \otimes A)^* x = \sum_{n \in \mathbb{N}} \langle \xi_n \otimes \zeta_n, (1 \otimes A)^* x \rangle_{H_0 \otimes H_1} \xi_n \otimes \zeta_n.$$

For $s \in S_0$ and $u \in D(A)$, we obtain

$$\langle (1 \otimes A)(s \otimes u), x \rangle_{H_0 \otimes H_2} = \sum_{n \in \mathbb{N}} \langle \xi_n \otimes \zeta_n, (1 \otimes A)^* x \rangle_{H_0 \otimes H_1} \langle s \otimes u, \xi_n \otimes \zeta_n \rangle_{H_0 \otimes H_1}$$

which is equivalent to

$$\begin{aligned}&\sum_{n \in \mathbb{N}} \langle \xi_n \otimes \eta_n, x \rangle_{H_0 \otimes H_2} \langle s \otimes Au, \xi_n \otimes \eta_n \rangle_{H_0 \otimes H_2}\\ =&\sum_{n \in \mathbb{N}} \langle \xi_n \otimes \zeta_n, (1 \otimes A)^* x \rangle_{H_0 \otimes H_1} \langle s \otimes u, \xi_n \otimes \zeta_n \rangle_{H_0 \otimes H_1}.\end{aligned}$$

With $s = \xi_i$, this implies

$$\begin{aligned}\langle Au, \langle \xi_i \otimes \eta_i, x \rangle_{H_0 \otimes H_2} \eta_i \rangle_{H_2} =&\langle \xi_i \otimes \eta_i, x \rangle_{H_0 \otimes H_2} \langle Au, \eta_i \rangle_{H_2}\\ =&\langle \xi_i \otimes \zeta_i, (1 \otimes A)^* x \rangle_{H_0 \otimes H_1} \langle u, \xi_i \otimes \zeta_i \rangle_{H_1}\\ =&\langle u, \langle \xi_i \otimes \zeta_i, (1 \otimes A)^* x \rangle_{H_0 \otimes H_1} \xi_i \otimes \zeta_i \rangle_{H_1}\end{aligned}$$

for all $u \in D(A)$, that is, $\langle \xi_i \otimes \eta_i, x\rangle_{H_0 \otimes H_2} \eta_i \in D(A^*)$ and

$$A^* \langle \xi_i \otimes \eta_i, x\rangle_{H_0 \otimes H_2} \eta_i = \langle \xi_i \otimes \zeta_i, (1 \otimes A)^* x\rangle_{H_0 \otimes H_1} \zeta_i$$

for every $i \in \mathbb{N}$. Thus,

$$\sum_{i=1}^{m} \langle \xi_i \otimes \eta_i, x\rangle_{H_0 \otimes H_2} \xi_i \otimes \eta_i \in H_0 \overset{a}{\otimes} D(A^*) \subseteq D(1 \otimes A^*)$$

and

$$(1 \otimes A^*) \sum_{i=1}^{m} \langle \xi_i \otimes \eta_i, x\rangle_{H_0 \otimes H_2} \xi_i \otimes \eta_i = \langle \xi_i \otimes \zeta_i, (1 \otimes A)^* x\rangle_{H_0 \otimes H_1} \xi_i \otimes \zeta_i$$

holds for all $m \in \mathbb{N}$. Since $1 \otimes A^*$ is closed, we conclude that $x \in D(1 \otimes A^*)$ and $(1 \otimes A^*)x = (1 \otimes A)^* x$.

The other identity follows similarly.

□

OBSERVATION 2.12. *Let $H_0 \neq \{0\}$, H_1, and H_2 be Hilbert spaces, and $A \subseteq H_1 \oplus H_2$ a densely defined closable operator. Then, $1 \otimes A$ is continuously invertible if and only if A is continuously invertible. In that case,*

$$(1 \otimes A)^{-1} = 1 \otimes A^{-1}$$

holds.

PROOF. Let A be continuously invertible. Then, $1 \otimes A^{-1}$ is a bounded operator. Let $x \in H_0$ and $y \in H_1$. Then,

$$(1 \otimes A^{-1})(1 \otimes A) x \otimes y = (1 \otimes A^{-1}) x \otimes Ay = x \otimes A^{-1}Ay = x \otimes y$$

shows

$$(1 \otimes A^{-1})(1 \otimes A)|_{D(1 \dot{\otimes} A)} = 1|_{D(1 \dot{\otimes} A)}.$$

Let $x \in D(1 \otimes A)$ and $(x_n)_{n \in \mathbb{N}} \in D(1 \dot{\otimes} A)$ such that $x_n \to x$ in $H_0 \otimes H_1$ and $(1 \otimes A)x_n \to (1 \otimes A)x$ in $H_0 \otimes H_2$. Then, continuity of $1 \otimes A^{-1}$ implies

$$x_n = (1 \otimes A^{-1})(1 \otimes A)x_n \to (1 \otimes A^{-1})(1 \otimes A)x,$$

i.e.,

$$(1 \otimes A^{-1})(1 \otimes A)|_{D(1 \otimes A)} = 1|_{D(1 \otimes A)}.$$

Furthermore, for $y \in H_2$,

$$(1 \otimes A)(1 \otimes A^{-1}) x \otimes y = (1 \otimes A) x \otimes A^{-1} y = x \otimes AA^{-1} y = x \otimes y$$

shows

$$(1 \otimes A)(1 \otimes A^{-1})|_{H_0 \overset{a}{\otimes} H_2} = 1|_{H_0 \overset{a}{\otimes} H_2}.$$

Let $y \in H_0 \otimes H_2$ and $(y_n)_{n \in \mathbb{N}} \in \left(H_0 \overset{a}{\otimes} H_2\right)$ with $y_n \to y$ in $H_0 \otimes H_2$. Then, $(1 \otimes A^{-1})y_n \to (1 \otimes A^{-1})y$ in $H_0 \otimes H_1$ by continuity, and

$$(1 \otimes A^{-1})y_n \in D(1 \otimes A)$$

and

$$(1 \otimes A)(1 \otimes A^{-1})y_n = y_n$$

hold. Since $1 \otimes A$ is closed, it follows

$$(1 \otimes A^{-1})y \in D(1 \otimes A)$$

and

$$(1 \otimes A)(1 \otimes A^{-1})y = y.$$

Hence, $(1 \otimes A^{-1})$ is bounded left- and right-inverse of $1 \otimes A$, i.e.,

$$1 \otimes A^{-1} = (1 \otimes A)^{-1} \in L(H_0 \otimes H_2, H_0 \otimes H_1).$$

Let us now assume that $1 \otimes A$ is continuously invertible. Let $x \in H_0 \setminus \{0\}$ and $y \in [\{0\}]A$. Then, $1 \otimes Ax \otimes y = x \otimes Ay = x \otimes 0 = 0$. Since $1 \otimes A$ is injective, this implies $x \otimes y = 0$, i.e.,

$$\forall (\varphi, \psi) \in H_0 \times H_1 : \ \langle x, \varphi \rangle_{H_0} \langle y, \psi \rangle_{H_1} = x \otimes y(\varphi, \psi) = 0.$$

In particular, $\varphi = x$ and $\psi = y$ implies

$$\|x\|_{H_0}^2 \|y\|_{H_1}^2 = 0$$

and, hence, $y = 0$ since we assumed $x \neq 0$. In other words, A is injective.

Let $y \in A[H_1]^{\perp}$. Then, for all $z \in D(A)$, we obtain $\langle Az, y \rangle_{H_2} = 0$ and, therefore,

$$\forall \tilde{x} \in H_0 \ \forall z \in D(A) : \ \langle (1 \otimes A) \tilde{x} \otimes z, x \otimes y \rangle = 0$$

which implies

$$\forall \xi \in H_0 \overset{a}{\otimes} D(A) : \ \langle (1 \otimes A) \xi, x \otimes y \rangle = 0$$

and, by continuity of the inner product,

$$\forall \xi \in D(1 \otimes A) : \ \langle (1 \otimes A) \xi, x \otimes y \rangle = 0.$$

Hence, $x \otimes y \in (1 \otimes A)[H_0 \otimes H_1]^{\perp} = \{0\}$. Since x was assumed non-zero, this implies $y = 0$, i.e., A has dense range. Thus, it suffices to show continuity of A^{-1} to prove the assertion. Since

$$(1 \otimes A)^{-1}|_{[H_0] \otimes [A[H_1]]} = (1 \otimes A^{-1})|_{[H_0] \otimes [A[H_1]]},$$

we obtain, for $y \in A[H_1]$,

$$\begin{aligned}
\left\| A^{-1} y \right\|_{H_1} &= \frac{\|x\|_{H_0} \left\| A^{-1} y \right\|_{H_1}}{\|x\|_{H_0}} \\
&= \frac{1}{\|x\|_{H_0}} \left\| x \otimes A^{-1} y \right\|_{H_0 \otimes H_1} \\
&= \frac{1}{\|x\|_{H_0}} \left\| (1 \otimes A)^{-1} x \otimes y \right\|_{H_0 \otimes H_1} \\
&\leq \frac{\left\| (1 \otimes A)^{-1} \right\|_{L(H_0 \otimes H_2, H_0 \otimes H_1)}}{\|x\|_{H_0}} \|x \otimes y\|_{H_0 \otimes H_2} \\
&\leq \left\| (1 \otimes A)^{-1} \right\|_{L(H_0 \otimes H_2, H_0 \otimes H_1)} \|y\|_{H_2} .
\end{aligned}$$

□

Remark We will use the notation of tensor products in cases where the spaces involved are not Hilbert spaces themselves; e.g., $C^k(M; H_1) \otimes C^k(M; H_2)$ with H_1, H_2 Hilbert spaces. By writing this we mean to consider $C^k(M; H_1 \otimes H_2)$.

■

CHAPTER 3

L_p spaces on $C^{1,1}$-manifolds

Throughout these notes, unless explicitly stated otherwise, let (M, g) be an orientable real 3-dimensional Riemannian $C^{1,1}$-manifold[1] endowed with a connection ∇. Then, the tangent bundle TM is a Riemannian $(2\dim M)$-manifold and a Hausdorff space itself. Furthermore, $(g_i(x))_{i\in\mathbb{N}_{\leq \dim M}}$ will always be a local basis of T_xM and $(g^i(x))_{i\in\mathbb{N}_{\leq \dim M}}$ the corresponding dual basis in T_xM^*. $d\mathrm{vol}_M$ will denote the volume form on M, G the Gramian matrix and $\gamma := \sqrt{\det G}$.

Let $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$, $j, N \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0^N$. Then, using $f \in C^\omega(A; B) :\Leftrightarrow f : A \to B$ analytic wherever this makes any kind of sense, we define the following spaces

$\mathfrak{X}_k(M) := \{f \in C^{k,1}(M; TM);\ \forall x \in M :\ f(x) \in T_xM\}$ locally Lipschitz vector fields
$\mathfrak{X}_{k,c}(M) := \{f \in X_k(M);\ \operatorname{spt} f \text{ compact in } \operatorname{int} M\}$
$\mathfrak{T}_x(\alpha, \beta) := \bigotimes_{i=1}^N \left(\left(\bigotimes_{j=1}^{\alpha_i} T_xM\right) \otimes \left(\bigotimes_{l=1}^{\beta_i} T_xM^*\right)\right)$
$\mathfrak{T}(\alpha, \beta) := \dot\bigcup_{x\in M} \mathfrak{T}_x(\alpha, \beta) \cong \bigcup_{x\in M}\{x\} \times \mathfrak{T}_x(\alpha, \beta)$
$\mathfrak{T}^*(\alpha, \beta) := \dot\bigcup_{x\in M} \mathfrak{T}_x(\alpha, \beta)^* \cong \bigcup_{x\in M}\{x\} \times \mathfrak{T}_x(\alpha, \beta)^*$
$\mathfrak{M}_k^{(\alpha,\beta)}(M) := \{f \in C^{k,1}(M; \mathfrak{T}^*(\alpha, \beta));\ \forall x \in M :\ f(x) \in \mathfrak{T}_x(\alpha, \beta)^*\}$ (α, β)-tensor fields
$\mathfrak{M}_{k,c}^{(\alpha,\beta)}(M) := \{f \in M_k^{(\alpha,\beta)}(M);\ \operatorname{spt} f \text{ compact in } \operatorname{int} M\}$
$S^j := \{\sigma;\ \sigma : \mathbb{N}_{\leq j} \to \mathbb{N}_{\leq j} \text{ injective}\}$ permutations
$\mathfrak{A}^j(x) := \{f \in (T_xM^j)^*;\ \forall X \in T_xM^j\ \forall \sigma \in S^j :\ f(X) = \operatorname{sgn} \sigma f(X \circ \sigma)\}$ alternating linear forms
$\mathfrak{A}^j(M) := \dot\bigcup_{x\in M} \mathfrak{A}^j(x)$
$\Lambda_k^j(M) := \{f \in C^{k,1}(M; \mathfrak{A}^j(M));\ \forall x \in M :\ f(x) \in \mathfrak{A}^j(x)\}$ C^k-j-forms

Remark The tensor bundles $\mathfrak{T}(\alpha, \beta)$ and $\mathfrak{T}^*(\alpha, \beta)$ are topological spaces. Unfortunately, the topologies are far from adorable. But it is possible to show that if $U \subseteq M$ is open and contractible[2], then there is a diffeomorphism from $\dot\bigcup_{x\in U} \mathfrak{T}_x(\alpha, \beta)$ (or $\dot\bigcup_{x\in U} \mathfrak{T}_x(\alpha, \beta)^*$, respectively) to $U \times \mathbb{R}^m$ with $m = (\dim M)^{\sum_{i=1}^N \alpha_i + \beta_i}$. This is strongly liked to local trivializations of $\mathfrak{T}(\alpha, \beta)$ and $\mathfrak{T}^*(\alpha, \beta)$. For notational simplicity we will only consider the case $\mathfrak{T}(m, 0)$ as all other cases can be constructed using Riesz identifications. Then, the bundle projection $\pi : \mathfrak{T}(m, 0) \to M$ is defined by

$$\forall (x, v) \in \{x\} \times \mathfrak{T}_x(m, 0) :\ \pi(x, v) = x$$

and, given an atlas $(U_i, \psi_i)_{i\in I}$, we define

$$\forall i \in I :\ \varphi_i :\ [U_i]\pi \to U_i \times \mathbb{R}^m;\ (x, v_i g^i) \mapsto (x, (v_1, \ldots, v_m)).$$

[1] $f \in C^{1,1}$ means f is Fréchet-differentiable and its derivative f' is Hölder continuous with Hölder exponent one, i.e., f' is locally Lipschitz.

[2] A topological space is called contractible if and only if the identity map is null-holomorphic.

These φ_i are vector space isomorphisms and the (U_i, φ_i) locally trivialize $\mathfrak{T}(m,0)$. In fact, this property is very important as for a vector bundle to be locally trivializable ensures the existence of global cross sections with maximal regularity, i.e., $\mathfrak{M}_k^{(\alpha,\beta)}(M)$ is non-trivial if M is a C^k-manifold.

■

LEMMA 3.1. *$\mathfrak{M}_k^{(0,0)}(M) \cong C^{k,1}(M;\mathbb{R})$, $\mathfrak{M}_k^{(1,0)}(M) \cong \Lambda_k^1(M)$, $\mathfrak{M}_k^{(0,1)}(M) \cong \mathfrak{X}_k(M)$ and $\mathfrak{M}_k^{(\alpha,\beta)}(M) \otimes \mathfrak{M}_k^{(\alpha',\beta')}(M) = \mathfrak{M}_k^{(\alpha\oplus\alpha',\beta\oplus\beta')}(M)$*

PROOF. (i)

$$\begin{aligned}\mathfrak{M}_k^{(0,0)}(M) =& \{f \in C^{k,1}(M;\mathfrak{T}(0,0)^*);\ \forall x \in M:\ f(x) \in \mathfrak{T}_x(0,0)^*\}\\ =& \left\{f \in C^{k,1}(M;\mathfrak{T}(0,0)^*);\ \forall x \in M:\ f(x) \in \left(\bigotimes_{\varnothing} \mathfrak{T}_x M \otimes \bigotimes_{\varnothing} T_x M^*\right)^*\right\}\\ \cong& \{f \in C^{k,1}(M;\mathfrak{T}(0,0)^*);\ \forall x \in M:\ f(x) \in (\mathbb{R}\otimes\mathbb{R})^*\}\\ \cong& C^{k,1}(M;\mathbb{R})\end{aligned}$$

(ii)

$$\begin{aligned}\mathfrak{M}_k^{(1,0)}(M) =& \{f \in C^{k,1}(M;\mathfrak{T}(1,0)^*);\ \forall x \in M:\ f(x) \in \mathfrak{T}_x(1,0)^*\}\\ \cong& \{f \in C^{k,1}(M;\mathfrak{T}(1,0)^*);\ \forall x \in M:\ f(x) \in \mathfrak{T}_x M^*\}\\ =& \{f \in C^{k,1}(M;\mathfrak{A}^1(M));\ \forall x \in M:\ f(x) \in \mathfrak{A}^1(x)\}\\ =& \Lambda_k^1(M)\end{aligned}$$

(iii)

$$\begin{aligned}\mathfrak{M}_k^{(0,1)}(M) =& \{f \in C^{k,1}(M;\mathfrak{T}(0,1)^*);\ \forall x \in M:\ f(x) \in \mathfrak{T}_x(0,1)^*\}\\ \cong& \{f \in C^{k,1}(M;\mathfrak{T}(0,1)^*);\ \forall x \in M:\ f(x) \in \mathfrak{T}_x M^{**}\}\\ \cong& \{f \in C^{k,1}(M;TM);\ \forall x \in M:\ f(x) \in \mathfrak{T}_x M\}\\ =& \mathfrak{X}_k(M)\end{aligned}$$

(iv)

$$\begin{aligned}\mathfrak{M}_k^{(\alpha,\beta)}(M) \otimes \mathfrak{M}_k^{(\alpha',\beta')}(M) =& \{f \in C^{k,1}(M;\mathfrak{T}(\alpha,\beta)^* \otimes \mathfrak{T}(\alpha',\beta')^*);\\ & \forall x \in M:\ f(x) \in \mathfrak{T}_x(\alpha,\beta)^* \otimes \mathfrak{T}_x(\alpha',\beta')^*\}\\ =& \{f \in C^{k,1}(M;\mathfrak{T}(\alpha,\beta)^* \otimes \mathfrak{T}(\alpha',\beta')^*);\\ & \forall x \in M:\ f(x) \in \mathfrak{T}_x(\alpha\oplus\alpha',\beta\oplus\beta')^*\}\\ =& \{f \in C^{k,1}(M;\mathfrak{T}(\alpha\oplus\alpha',\beta\oplus\beta')^*);\\ & \forall x \in M:\ f(x) \in \mathfrak{T}_x(\alpha\oplus\alpha',\beta\oplus\beta')^*\}\\ =& \mathfrak{M}_k^{(\alpha\oplus\alpha',\beta\oplus\beta')}(M)\end{aligned}$$

□

Recall that the volume form $d\mathrm{vol}_M$ defines a measure on the Borel sets $\mathcal{B}(M)$ by

$$\forall B \in \mathcal{B}(M):\ \mu(B) := \int_B d\mathrm{vol}_M.$$

Using this interpretation we are in the realm of Lebesgue-integrals.

DEFINITION 3.2. *Let $p \in \mathbb{R}_{\geq 1}$, $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}_0^n$ and $\langle \cdot, \cdot \rangle_{(\alpha,\beta)}$ be the canonical scalar form on $\mathfrak{M}_0^{(\alpha,\beta)}(M)$, i.e. for $x, y \in \mathfrak{M}_0^{(\alpha,\beta)}(M)$*

$$\forall p \in M: \ \langle x, y \rangle_{(\alpha,\beta)}(p) = \langle x(p), y(p) \rangle_{\mathfrak{T}_p(\alpha,\beta)^*}$$

holds. We define

$$\|\cdot\|_{p,\alpha,\beta} : \mathfrak{M}_0^{(\alpha,\beta)}(M) \to \mathbb{R}; \ x \mapsto \left(\int_M \left| \langle x, x \rangle_{(\alpha,\beta)} \right|^{\frac{p}{2}} \mathrm{dvol}_M \right)^{\frac{1}{p}}$$

and

$$L_p^{(\alpha,\beta)}(M) := \overline{\mathfrak{M}_0^{(\alpha,\beta)}(M)}^{\|\cdot\|_{p,\alpha,\beta}}.$$

We will denote measurable and p-integrable functions, i.e. those functions being $\|\cdot\|_{p,\alpha,\beta}$-limits of continuous functions, by $\mathcal{L}_p^{(\alpha,\beta)}(M)$.

Remark Fischer-Riesz's theorem (theorem 3.3 below) allows us identify elements of $L_p^{(\alpha,\beta)}(M)$ with a functions.

■

Obviously all $L_p^{(\alpha,\beta)}(M)$ are Banach spaces and $L_2^{(\alpha,\beta)}(M)$ are Hilbert spaces, since, $\langle x, x \rangle_{(\alpha,\beta)}$ is non-negative and $\langle x, y \rangle_{L_2^{(\alpha,\beta)}(M)} = \int_M \langle x, y \rangle_{(\alpha,\beta)} \mathrm{dvol}_M$ is a scalar product.

THEOREM 3.3 (Fischer-Riesz). *Let $p \in \mathbb{R}_{\geq 1}$ and $(f_n)_{n \in \mathbb{N}} \in \mathcal{L}_p^{(\alpha,\beta)}(M)^{\mathbb{N}}$ converging to $f \in \mathcal{L}_p^{(\alpha,\beta)}(M)$ in $\mathcal{L}_p^{(\alpha,\beta)}(M)$. Then there is $g \in \mathcal{L}_p(\mathrm{vol}_M)$ and a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$, such that*

(i) *$f_{n_j} \to f$ μ-almost everywhere*

and

(ii) *$\forall j \in \mathbb{N}: \ \left| \langle f_{n_j}, f_{n_j} \rangle_{(\alpha,\beta)} \right|^{\frac{1}{2}} \leq g$*

hold.

PROOF. Choose any subsequence $(f_{n_j})_{j \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ satisfying

$$\forall j \in \mathbb{N}: \ \left\| f_{n_{j+1}} - f_{n_j} \right\|_{\mathcal{L}_p^{(\alpha,\beta)}(M)} \leq 2^{-j}.$$

For $j \in \mathbb{N}$ let $\tilde{f}_j := f_{n_{j+1}} - f_{n_j}$. Then, for $k \in \mathbb{N}$,

$$\begin{aligned}
\left(\int \left(\sum_{j=1}^k \left| \langle \tilde{f}_j, \tilde{f}_j \rangle_{(\alpha,\beta)} \right|^{\frac{1}{2}} \right)^p \mathrm{dvol}_M \right)^{\frac{1}{p}} &= \left\| \sum_{j=1}^k \left| \langle \tilde{f}_j, \tilde{f}_j \rangle_{(\alpha,\beta)} \right|^{\frac{1}{2}} \right\|_{L_p(\mathrm{vol}_M)} \\
&\leq \sum_{j=1}^k \left\| \left| \langle \tilde{f}_j, \tilde{f}_j \rangle_{(\alpha,\beta)} \right|^{\frac{1}{2}} \right\|_{L_p(\mathrm{vol}_M)} \\
&= \sum_{j=1}^k \left\| f_{n_{j+1}} - f_{n_j} \right\|_{\mathcal{L}_p^{(\alpha,\beta)}(M)} \\
&\leq \sum_{j \in \mathbb{N}} 2^{-j} \\
&= 1
\end{aligned}$$

and

$$\left(\sum_{j=1}^{k}\left|\langle\tilde f_j,\tilde f_j\rangle_{(\alpha,\beta)}\right|^{\frac12}\right)^p \nearrow \left(\sum_{j\in\mathbb{N}}\left|\langle\tilde f_j,\tilde f_j\rangle_{(\alpha,\beta)}\right|^{\frac12}\right)^p$$

hold. Thus, using dominated convergence, we find

$$\int\left(\sum_{j\in\mathbb{N}}\left|\langle\tilde f_j,\tilde f_j\rangle_{(\alpha,\beta)}\right|^{\frac12}\right)^p d\mathrm{vol}_M < \infty.$$

Hence, $\tilde g := \lim_{k\to\infty}\sum_{j=1}^{k}\left|\langle\tilde f_j,\tilde f_j\rangle_{(\alpha,\beta)}\right|^{\frac12} \in \mathcal{L}_p(\mathrm{vol}_M)$ exists with $\tilde g < \infty$ vol_M-almost everywhere. Since $\mathcal{L}_p^{(\alpha,\beta)}(M)$ is a Banach space, $\sum_{j\in\mathbb{N}}\tilde f_j$ converges vol_M-almost everywhere absolutely; $\sum_{j\in\mathbb{N}}\tilde f_j =: \tilde f$. By definition of $(\tilde f_j)_{j\in\mathbb{N}}$ we find

$$\tilde f \leftarrow \sum_{j=1}^{k}\tilde f_j = \sum_{j=1}^{k}(f_{n_{j+1}} - f_{n_j}) = f_{n_{k+1}} - f_{n_1} \to f - f_{n_1}$$

and, hence, $f_{n_{k+1}} = f_{n_1} + \sum_{j=1}^{k}\tilde f_j \to f$ vol_M-almost everywhere. Furthermore, for $j\in\mathbb{N}$

$$\tilde g^2 \geq \langle\tilde f_j,\tilde f_j\rangle_{(\alpha,\beta)} = \langle f_{n_{j+1}}, f_{n_{j+1}}\rangle_{(\alpha,\beta)} - 2\langle f_{n_{j+1}}, f_{n_j}\rangle_{(\alpha,\beta)} + \langle f_{n_j}, f_{n_j}\rangle_{(\alpha,\beta)}$$

holds. Hence,

$$0 \leq \langle f_{n_j}, f_{n_j}\rangle_{(\alpha,\beta)} \leq \underbrace{\tilde g^2}_{\in\mathcal{L}_{\frac p2}(\mathrm{vol}_M)} \underbrace{-\langle f_{n_{j+1}}, f_{n_{j+1}}\rangle_{(\alpha,\beta)}}_{\in\mathcal{L}_{\frac p2}(\mathrm{vol}_M)} + \underbrace{2\langle f_{n_{j+1}}, f_{n_j}\rangle_{(\alpha,\beta)}}_{\in\mathcal{L}_{\frac p2}(\mathrm{vol}_M)} \in \mathcal{L}_{\frac p2}(\mathrm{vol}_M)$$

yields (ii) with $g := \left(\tilde g^2 - \langle f_{n_{j+1}}, f_{n_{j+1}}\rangle_{(\alpha,\beta)} + 2\langle f_{n_{j+1}}, f_{n_j}\rangle_{(\alpha,\beta)}\right)^{\frac12} \in \mathcal{L}_p(\mathrm{vol}_M)$.

□

Remark Let $f \in L_p^{(\alpha,\beta)}(M)$. Then, there is a representative g such that $g(x) \in \mathfrak{T}_x(\alpha,\beta)^*$ holds for every $x\in M$ since it holds where a subsequence converges. Since the complement of this set is a null-set, we may choose g to be zero there. Furthermore, $L_2^{(\alpha,\beta)}(M)$ may be isometrically embedded in $L_2(M)\otimes\mathfrak{T}(\alpha,\beta)^*$.

■

Remark Note that we may restrict all considerations to $\beta = 0$, for all spaces with $\beta \neq 0$ can be generated using Riesz identifications.

■

Definition 3.4. *Let $p\in\mathbb{R}_{\geq1}$, $m\in\mathbb{N}$, and $\alpha\in\mathbb{N}_0$. Then, we define*

$$\begin{aligned}
D_c\left(\|\cdot\|_{L_p^{(\alpha,0)}(M)}\right) &:= \left\{x\in\mathfrak{M}_{0,c}^{(\alpha,0)}(M);\ \int_M\left|\langle x,x\rangle_{(\alpha,0)}\right|^{\frac p2} d\mathrm{vol}_M < \infty\right\},\\
D\left(\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}\right) &:= \Big\{x\in\mathfrak{M}_1^{(\alpha,0)}(M)\cap D\left(\|\cdot\|_{L_p^{(\alpha,0)}(M)}\right);\\
&\qquad\qquad \int_M\left|\langle\nabla x,\nabla x\rangle_{(\alpha+1,0)}\right|^{\frac p2} d\mathrm{vol}_M < \infty\Big\},\\
D_c\left(\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}\right) &:= \Big\{x\in\mathfrak{M}_{1,c}^{(\alpha,0)}(M)\cap D\left(\|\cdot\|_{L_p^{(\alpha,0)}(M)}\right);\\
&\qquad\qquad \int_M\left|\langle\nabla x,\nabla x\rangle_{(\alpha+1,0)}\right|^{\frac p2} d\mathrm{vol}_M < \infty\Big\},
\end{aligned}$$

$$\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}:\ D\left(\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}\right)\to\mathbb{R};\ x\mapsto\left(\|x\|_{L_p^{(\alpha,0)}(M)}^p + \|\nabla x\|_{L_p^{(\alpha+1,0)}(M)}^p\right)^{\frac1p},$$

as well as,

$$L_{p,0}^{(\alpha,0)}(M) := \overline{D_c\left(\|\cdot\|_{L_p^{(\alpha,0)}(M)}\right)}^{\|\cdot\|_{L_p^{(\alpha,0)}(M)}},$$
$$W_p^{1,(\alpha,0)}(M) := \overline{D\left(\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}\right)}^{\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}},$$
$$W_{p,0}^{1,(\alpha,0)}(M) := \overline{D_c\left(\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}\right)}^{\|\cdot\|_{W_p^{1,(\alpha,0)}(M)}}.$$

Remark A priori, we cannot define Sobolev spaces in the same manner as we would define $W_p^k(\Omega)$ for $\Omega \subseteq_{\text{open}} \mathbb{R}^n$. However, using Sobolev chains, it is still possible to show that they exists and are dense in $L_p(M)$. It is important to keep in mind that this does not imply non-triviality of $C^k(M)$ because the Sobolev embedding theorems do not hold, in general.

■

DEFINITION 3.5. *Let $\alpha \in \mathbb{N}_0$. Then we define gradient and divergence to be*

$$\operatorname{grad}_{c,\alpha} : \mathfrak{M}_{1,c}^{(\alpha,0)}(M) \to \mathfrak{M}_{0,c}^{(\alpha+1,0)}(M);\ x \mapsto \nabla x,$$
$$\operatorname{div}_{c,\alpha} : \mathfrak{M}_{1,c}^{(\alpha+1,0)}(M) \to \mathfrak{M}_{0,c}^{(\alpha,0)}(M);\ x \mapsto \operatorname{tr} \nabla x$$

where tr *denotes the trace acting on the first two components.*[3]

Before we can show that the gradient and divergence are formally adjoint, let us recall the Gauss divergence theorem.

DEFINITION 3.6. *Let V be a closed subset of M. V has smooth boundary if and only if for each $a \in V$ there is an open neighborhood $U \subseteq V$ of a and a function $g \in C^1(U;\mathbb{R})$ such that*

$$V \cap U = \{x \in U;\ g(x) \le 0\}$$

and $\nabla g(x) \ne 0$ for all $x \in U$ hold. Then we define

$$\partial V \cap U := \{x \in U;\ g(x) = 0\}$$

and ∂V the union of every such $\partial V \cap U$.

Let a be in ∂V. Then we call

$$\nu(a) := \frac{1}{\|\nabla g(a)\|} \nabla g(a)$$

the outward-pointing normal at a.

Remark Having smooth boundary as defined above means having C^1 boundary, i.e., the boundary is locally a C^1 manifold.

Sketch of proof Let $V \subseteq M$ be a closed subset of M with smooth boundary. Let $p \in \partial V$ and φ a chart with $p \in \varphi\left[\mathbb{R}^{\dim M}\right]$. Let $r \in \mathbb{R}_{>0}$ such that $U_0 := B_{\mathbb{R}^{\dim M}}(\varphi^{-1}(p), r) \subseteq [M]\varphi$ and such that there exists g according to the definition with respect to $U := \varphi[U_0]$. Note that $g \in C^1(U)$ means, per definitionem, $\tilde{g} := g \circ \varphi \in C^1(U_0)$ and $\nabla g(x) \ne 0$ is equivalent to $(g \circ \varphi)'(x) \ne 0$ since $\nabla_{g_i} f = \partial_i (g \circ \varphi)$.

[3]that is, for instance, $\operatorname{tr}(a_{ijk} g^i \otimes g^j \otimes g^k) = a_{ijk} g^{ij} g^k$

THEOREM 3.7 (Level Set Criterion). *A set $S \subseteq \mathbb{R}^n$ is an m-dimensional C^l-manifold if and only if for ever $p \in S$ there is an open neighborhood U_p of p and a function $g_p \in C^l(U_p, \mathbb{R}^k)$ with $m + k = n$ such that $S \cap U_p = [\{0\}]g_p$ and $\operatorname{rank} g_p' = l$ in U_p.*

Since U_0 is an open neighborhood of $\varphi^{-1}(p)$ and $\tilde{g} \in C^1(U_0, \mathbb{R})$ with $[\partial V]\varphi \cap U_0 = [\{0\}]\tilde{g}$ and $\operatorname{rank} \tilde{g}' = 1$, we obtain that $\varphi[[\partial V]\varphi \cap U_0]$ is a $(\dim M - 1)$-dimensional C^1-manifold. Since $p \in \partial V$ was arbitrarily chosen, we conclude that ∂V is a $(\dim M - 1)$-dimensional C^1-manifold.

■

THEOREM 3.8 (Gauss divergence Theorem). *Let V be a compact subset of M with smooth boundary and $\nu \in \mathfrak{M}_0^{(1,0)}(M)$ such that $\nu|_{\partial V}$ is the outward-pointing normal vector field on ∂V. Let $\mathrm{dvol}_{\partial V}$ be the surface form on ∂V. Let F be a continuous vector field on V and continuously differentiable in the interior, i.e., $F \in \mathfrak{M}_0^{(1,0)}(V) \cap \mathfrak{M}_1^{(1,0)}(V \setminus \partial V)$. Then*

$$\int_V \operatorname{tr} \nabla F \mathrm{dvol}_M = \int_{\partial V} \langle F, \nu \rangle \mathrm{dvol}_{\partial V}$$

holds.

OBSERVATION 3.9. $-\operatorname{div}_{c,\alpha} \subseteq \left(\operatorname{grad}_{c,\alpha}\right)^*$ *holds in* $L_2^{(\alpha+1,0)}(M) \oplus L_2^{(\alpha,0)}(M)$.

PROOF. Let $\varphi \in D(\operatorname{grad}_{c,\alpha})$ and $\tau \in D(\operatorname{div}_{c,\alpha})$. Then

$$\begin{aligned}
\langle -\operatorname{tr} \nabla \tau, \varphi \rangle_{(\alpha,0)} &= \left\langle -\operatorname{tr}\left(\nabla_{g_i} \tau_{j_1 \ldots j_{\alpha+1}} g^i \otimes g^{j_1} \otimes \ldots \otimes g^{j_{\alpha+1}}\right), \varphi \right\rangle_{(1,0)} \\
&= -\left\langle \nabla_{g_i} \tau_{j_1 \ldots j_{\alpha+1}} g^{ij_1} g^{j_2} \otimes \ldots \otimes g^{j_{\alpha+1}}, \varphi_{k_1 \ldots k_\alpha} g^{k_1} \otimes \ldots \otimes g^{k_\alpha} \right\rangle_{(1,0)} \\
&= -\nabla_{g_i} \tau_{j_1 \ldots j_{\alpha+1}} g^{ij_1} g^{j_2 k_1} \ldots g^{j_{\alpha+1} k_\alpha} \varphi_{k_1 \ldots k_\alpha} \\
&= -\nabla_{g_i} \left(\tau_{j_1 \ldots j_{\alpha+1}} g^{ij_1} g^{j_2 k_1} \ldots g^{j_{\alpha+1} k_\alpha} \varphi_{k_1 \ldots k_\alpha}\right) \\
&\quad + \tau_{j_1 \ldots j_{\alpha+1}} g^{ij_1} g^{j_2 k_1} \ldots g^{j_{\alpha+1} k_\alpha} \nabla_{g_i} \varphi_{k_1 \ldots k_\alpha} \\
&= -\nabla_{g_i} \left(\tau_{j_1 \ldots j_{\alpha+1}} g^{ij_1} g^{j_2 k_1} \ldots g^{j_{\alpha+1} k_\alpha} \varphi_{k_1 \ldots k_\alpha}\right) + \langle \tau, \nabla \varphi \rangle_{(\alpha+1,0)}
\end{aligned}$$

and, hence, using a $\tilde{M} \subseteq_{\text{compact}} M$ such that φ and τ are compactly supported in $\tilde{M}$,

$$\begin{aligned}
\int_{\tilde{M}} \langle -\operatorname{tr} \nabla \tau, \varphi \rangle_{(\alpha,0)} \mathrm{dvol}_M &= \int_{\tilde{M}} \langle \tau, \nabla \varphi \rangle_{(\alpha+1,0)} \mathrm{dvol}_M \\
&\quad - \int_{\tilde{M}} \nabla_{g_i} \left(\tau_{j_1 \ldots j_{\alpha+1}} g^{ij_1} g^{j_2 k_1} \ldots g^{j_{\alpha+1} k_\alpha} \varphi_{k_1 \ldots k_\alpha}\right) \mathrm{dvol}_M \\
&= \int_{\tilde{M}} \langle \tau, \nabla \varphi \rangle_{(\alpha+1,0)} \mathrm{dvol}_M \\
&\quad - \int_{\tilde{M}} \operatorname{tr} \nabla \left(\tau_{j_1 \ldots j_{\alpha+1}} g^{j_2 k_1} \ldots g^{j_{\alpha+1} k_\alpha} \varphi_{k_1 \ldots k_\alpha} g^{j_1}\right) \mathrm{dvol}_M \\
&= \int_{\tilde{M}} \langle \tau, \nabla \varphi \rangle_{(\alpha+1,0)} \mathrm{dvol}_M \\
&\quad - \int_{\partial \tilde{M}} \left\langle \tau_{j_1 \ldots j_{\alpha+1}} g^{j_2 k_1} \ldots g^{j_{\alpha+1} k_\alpha} \varphi_{k_1 \ldots k_\alpha} g^{j_1}, \nu \right\rangle \mathrm{dvol}_{\partial \tilde{M}}
\end{aligned}$$

holds according to the Gauss divergence theorem. However, we have assumed that τ and φ are compactly supported in $\tilde{M} \setminus \partial \tilde{M}$, i.e., the integral over $\partial \tilde{M}$ vanishes which, thence, reduces to

$$\int_M \langle -\operatorname{tr} \nabla \tau, \varphi \rangle_{(\alpha,0)} \mathrm{dvol}_M = \int_M \langle \tau, \nabla \varphi \rangle_{(\alpha+1,0)} \mathrm{dvol}_M,$$

i.e., the assertion.

□

LEMMA 3.10. *Let X, Y be reflexive Banach spaces, and $A \subseteq X \oplus Y$, $B \subseteq Y' \oplus X'$ densely defined, linear operators where X' and Y' denote the dual spaces of X and Y, respectively. If A and B are formally adjoint, i.e., $A \subseteq B^*$, then $B \subseteq A^*$ holds and both operators are closable, where A^* and B^* denote the respective dual or adjoint operators depending on whether or not X and Y are Hilbert spaces.*

PROOF. A^* and B^* are closed operators since A and B are densely defined which directly implies that A is closable. Thus,

$$B \subseteq \bar{B} = B^{**} \subseteq A^*$$

shows $B \subseteq A^*$ and, therefore, closability of B, too.

□

The lemma above enables us to define

$$\mathrm{grad}_{0,\alpha} := \overline{\mathrm{grad}_{c,\alpha}} \quad , \quad \mathrm{div}_{0,\alpha} := \overline{\mathrm{div}_{c,\alpha}}$$

as well as,

$$\mathrm{grad}_\alpha := -\left(\mathrm{div}_{c,\alpha}\right)^* \quad , \quad \mathrm{div}_\alpha := -\left(\mathrm{grad}_{c,\alpha}\right)^* .$$

From this point on, we will drop the index α as it is uniquely determined by the context.

Remarks on Sobolev Spaces

Since we are on a $C^{1,1}$-manifold M, we only know that the space $C^1(M)$ is non-trivial and dense in $C(M)$. For $k \geq 2$ the spaces $C^k(M)$ may very well be trivial. Hence, the usual approach to defining Sobolev spaces fails. However, we may use the notion of a Sobolev chain; for further detail, please, refer to [**15**].

LEMMA 3.11 ([**15**]; Lemma 2.1.3). *Let H be a Hilbert space and $A \subseteq H \oplus H$ a closed, densely defined, linear operator with zero in the resolvent set $\varrho(A)$. Then, A^n is a closed, densely defined operator for every $n \in \mathbb{N}$ with $0 \in \varrho(A^n)$ and*

$$\forall x \in D(A^n): \ \|A^n x\|_H \geq \left\|A^{-1}\right\|_{L(H)}^{-n} \|x\|_H .$$

Let $H_n(A)$ be the Hilbert space $D(A^n)$ equipped with the norm $x \mapsto \|A^n x\|_H$. Then,

$$A_{n+1,n}: \ H_{n+1}(A) \to H_n(A); \ x \mapsto Ax$$

is unitary for every $n \in \mathbb{N}_0$.

If A is a closed, densely defined, linear operator with $0 \in \varrho(A)$, then $A^* = (-A^{-1})^\perp$ is closed linear operator. Furthermore, closedness of A implies that A^* is densely defined, and $0 \in \varrho(A^*)$ follows from $\varrho(A^*) = \varrho(A)^*$. Hence, $H_n(A^*)$ is well-defined, as well, and we can extend the family $(H_n(A))_{n \in \mathbb{N}_0}$.

DEFINITION 3.12. *Let H be a Hilbert space, $A \subseteq H \oplus H$ a closed, densely defined, linear operator with $0 \in \varrho(A)$, and $n \in \mathbb{Z}$. Then, we define*

$$H_n(A) := \begin{cases} (D(A^n), \|A^n \cdot\|_H); \ n \in \mathbb{N}_0 \\ H_{-n}(A^*)^*; \ n \in -\mathbb{N} \end{cases}$$

where $H_{-n}(A^*)^*$ denotes the topological dual of $H_{-n}(A^*)$. Then, we call

(i) *$(H_n(A))_{n \in \mathbb{N}_0}$ the positive Sobolev chain associated with A,*
(ii) *$(H_n(A))_{n \in -\mathbb{N}_0}$ the negative Sobolev chain associated with A, and*
(iii) *$(H_n(A))_{n \in \mathbb{Z}}$ the (long) Sobolev chain associated with A.*

LEMMA 3.13 ([**15**]; Lemma 2.1.6). *Let $(H_n(A))_{n\in\mathbb{Z}}$ the Sobolev chain associated with the operator A. Then, we obtain that the embedding*

$$H_{n+k}(A) \hookrightarrow H_n(A)$$

is dense and continuous (in the sense of canonical embeddings) for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. Furthermore, the operators

$$D\left(A^{|n|+1}\right) \subseteq H_{n+1}(A) \to H_n(A);\ x \mapsto Ax$$

have unitary closures $A_{n+1,n} \subseteq H_{n+1}(A) \oplus H_n(A)$ for every $n \in \mathbb{Z}$.

It is often convenient to define the "closures" of the Sobolev chain

$$H_\infty(A) \coloneqq \bigcap_{n\in\mathbb{Z}} H_n(A)$$

which is a Fréchet space if equipped with the family of semi-norms $\left(\|\cdot\|_{H_n(A)}\right)_{n\in\mathbb{Z}}$ and dense in all $H_n(A)$ (in fact, $H_\infty(A)$ is a core of $A_{(k)}:\ H_{k+1}(A) \subseteq H_k(A) \to H_k(A);\ x \mapsto Ax$; cf., Lemma 2.1.15 in [**15**]) and

$$H_{-\infty}(A) \coloneqq \bigcup_{n\in\mathbb{Z}} H_n(A)$$

which is complete if equipped with the topology induced by saying that x is a Cauchy-sequence/convergent in $H_{-\infty}(A)$ if and only if there exists an $n \in \mathbb{Z}$ such that x is a Cauchy-sequence/convergent in $H_n(A)$ (cf., Lemma 2.1.11 in [**15**]).

As for the definition of Sobolev spaces on the $C^{1,1}$-manifold M, we know that the gradient grad is a well-defined, closed, densely defined, linear operator on the Hilbert space $L_2(M)$. Furthermore, we have the following theorem which can be obtained from the first representation theorem (Theorem VI.2.1 in [**11**]) applied to the closed, positive, symmetric form τ with $D(\tau) \coloneqq D(A)$ and $\forall x, y \in D(\tau):$ $\tau(x,y) \coloneqq \langle Ax, Ay\rangle_{H_2}$.

THEOREM 3.14 (von Neumann). *Let H_1 and H_2 be Hilbert spaces and $A \subseteq H_1 \oplus H_2$ a closed and densely defined operator. Then A^*A is self-adjoint in H_1 and its domain is a core of A.*

Hence, $\operatorname{grad}^* \operatorname{grad}$ is a well-defined, closed, densely defined, linear operator and, additionally, self-adjoint, i.e., so is $|\operatorname{grad}| \coloneqq \sqrt{\operatorname{grad}^* \operatorname{grad}}$ which allows us to define the Sobolev spaces

$$\forall k \in \mathbb{Z}:\ W_2^k(M) \coloneqq H_k(1 + |\operatorname{grad}|).$$

We may also define $W_2^s(M)$ for $s \in \mathbb{R}$ to be the closure of $H_\infty(1 + |\operatorname{grad}|)$ with respect to the scalar product

$$\forall x, y \in H_\infty(1 + |\operatorname{grad}|):\ \langle x, y\rangle_{W_2^s(M)} \coloneqq \langle (1 + |\operatorname{grad}|)^s x, (1 + |\operatorname{grad}|)^s y\rangle_{L_2(M)}.$$

Alternatively, we may consider the spaces $\tilde{W}_2^s(M)$ defined by the closure of $H_\infty(1 + |\operatorname{grad}|)$ with respect to the scalar product

$$\forall x, y \in H_\infty(1 + |\operatorname{grad}|):\ \langle x, y\rangle_{\tilde{W}_2^s(M)} \coloneqq \langle (1 + \operatorname{grad}^* \operatorname{grad})^s x, y\rangle_{L_2(M)}$$

for $s \in \mathbb{R}$ which are equivalent but sometimes more suitable. None the less, both families, $(W_2^s(M))_{s\in\mathbb{R}}$ and $\left(\tilde{W}_2^s(M)\right)_{s\in\mathbb{R}}$, satisfy the interpolation property, that is, for every $s, t \in \mathbb{R}$ and $\vartheta \in [0,1]$

$$W_2^{(1-\vartheta)s+\vartheta t}(M) = \left[W_2^s(M), W_2^t(M)\right]_\vartheta$$
$$\tilde{W}_2^{(1-\vartheta)s+\vartheta t}(M) = \left[\tilde{W}_2^s(M), \tilde{W}_2^t(M)\right]_\vartheta$$

in the sense of complex interpolation; cf., e.g., section 4.2 in [**18**]. In fact, if A is strictly positive, then we can extend the Sobolev chain $(H_n(A))_{n\in\mathbb{Z}}$ to $(H_s(A))_{s\in\mathbb{R}}$ by setting

$$H_s(A) := (D(A^s), \|A^s\cdot\|_H)$$

for $s \geq 0$ and by duality for $s < 0$. Similarly, we might simply use the interpolation property directly to define the $H_s(A)$ for $s \in \mathbb{R} \setminus \mathbb{Z}$ via

$$\forall s,t \in \mathbb{R}_{\geq 0}\ \forall \vartheta \in [0,1]:\ H_{(1-\vartheta)s+\vartheta t}(A) = [H_s(A), H_t(A)]_\vartheta\,.$$

Finally, we'd like to note that not all Sobolev embeddings fail to hold. For instance, we still obtain the following theorem.

THEOREM 3.15 (Sobolev Embedding). *Let X be a Banach space, $S,T \in \mathbb{R}$ and $S < T$. Then*

$$\mathrm{id}:\ W_2^1([S,T];X) \to C([S,T];X);\ f \mapsto f$$

is continuous and injective.

PROOF. Let $s,t \in [S,T]$ and $f \in C^\infty([S,T];X)$ (mind that $C^\infty([S,T];X)$ is a dense subset of $W_2^1([S,T];X)$). Then

$$\|f(t)\|_X \leq \|f(s)\|_X + \left|\int_s^t \|f'(\tau)\|_X\, d\tau\right| \leq \|f(s)\|_X + \sqrt{|t-s|}\left|\int_s^t \|f'(\tau)\|_X^2\, d\tau\right|^{\frac{1}{2}}$$

holds. Integrating s yields

$$\begin{aligned}(T-S)\,\|f(t)\|_X &\leq \int_S^T \|f(s)\|_X\, ds + \int_S^T \underbrace{\sqrt{|t-s|}}_{\leq\sqrt{T-S}}\underbrace{\left|\int_s^t \|f'(\tau)\|_X^2\, d\tau\right|^{\frac{1}{2}}}_{\leq\left|\int_S^T \|f'(\tau)\|_X^2\, d\tau\right|^{\frac{1}{2}}} ds \\ &\leq \sqrt{T-S}\left|\int_S^T \|f(s)\|_X^2\, ds\right|^{\frac{1}{2}} + (T-S)^{\frac{3}{2}}\left|\int_S^T \|f'(s)\|_X^2\, ds\right|^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\|f\|_{C([S,T];X)} \leq \max\left\{(T-S)^{\frac{1}{2}}, (T-S)^{-\frac{1}{2}}\right\}\sqrt{2}\,\|f\|_{W_2^1([S,T];X)}$$

holds, too, where we used

$$\sqrt{a}+\sqrt{b} = \left\|\begin{pmatrix}\sqrt{a}\\ \sqrt{b}\end{pmatrix}\right\|_1 \leq \left\|\begin{pmatrix}1\\ 1\end{pmatrix}\right\|_2 \left\|\begin{pmatrix}\sqrt{a}\\ \sqrt{b}\end{pmatrix}\right\|_2 = \sqrt{2}\sqrt{a+b}.$$

Thus, any $W_2^1([S,T];X)$-Cauchy sequence in $C^\infty([S,T];X)$ is also a $C([S,T];X)$-Cauchy sequence and, therefore, $W_2^1([S,T];X) \subseteq C([S,T];X)$.

Furthermore the identities

$$\begin{aligned}\mathrm{id}_1 &: W_2^1([S,T];X) \to L_2([S,T];X);\ f \mapsto f\\ \mathrm{id}_2 &: C([S,T];X) \to L_2([S,T];X);\ f \mapsto f\end{aligned}$$

are injective. Thus, $\mathrm{id} = \mathrm{id}_2^{-1} \circ \mathrm{id}_1$ is injective.

□

CHAPTER 4

The Analytic Implicit Function Theorem

Before we prove the analytic implicit function theorem, we will recall a few facts about analytic operators. A more extensive account can be found in [**3**].

Definition 4.1. *Let X and Y be Banach spaces and $k \in \mathbb{N}_0$. A k-linear mapping $m_k : X^k \to Y$ is called symmetric if and only if for every permutation $\sigma \in S_k$ and $x_1, \dots, x_k \in X$*

$$m_k x_1 \cdots x_k := m_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = m_k(x_1, \dots, x_k)$$

holds.

Definition 4.2. *Let X and Y be Banach spaces, $U \subseteq X$ open, and $x_0 \in U$. A mapping $F : U \to Y$ is called analytic at x_0 if and only if there exist $r \in \mathbb{R}_{>0}$ and k-linear and symmetric operators $m_k : X^k \to Y$ $(k \in \mathbb{N}_0)$ such that for every $x \in B_X(x_0, r) \subseteq U$*

$$F(x) = \sum_{k \in \mathbb{N}_0} m_k (x - x_0)^k \tag{1}$$

and

$$\sup_{k \in \mathbb{N}_0} r^k \, \|m_k\|_{\mathrm{Lip}} =: M < \infty \tag{2}$$

hold. The series $\sum_{k \in \mathbb{N}_0} m_k (x - x_0)^k$ in (1) is called power series and F is said to be analytic in U if and only if it is analytic at every point of U.

Due to (2) we observe for $x \in B_X(x_0, r)$

$$\sum_{k \in \mathbb{N}_0} \left\| m_k (x - x_0)^k \right\|_Y \le M \sum_{k \in \mathbb{N}_0} \frac{\|x - x_0\|_X^k}{r^k} = M \frac{1}{1 - \frac{\|x - x_0\|_X}{r}} = \frac{Mr}{r - \|x - x_0\|_X} < \infty.$$

Hence, the power series converges absolutely.

Observation 4.3. *Let X, Y_1, and Y_2 be Banach spaces, $U \subseteq X$ open, and $(F_1, F_2) : U \to Y_1 \times Y_2$ analytic. Then, F_1 and F_2 are analytic.*

Proof. Since (F_1, F_2) is analytic, there is a representation

$$(F_1, F_2)(x) = \sum_{k \in \mathbb{N}_0} m_k (x - x_0)^k$$

with $\sup_{k \in \mathbb{N}_0} r^k \, \|m_k\|_{\mathrm{Lip}} =: M < \infty$ for some $r \in \mathbb{R}_{>0}$ and every $x_0 \in U$. Let $i \in \{1, 2\}$. The projection $\mathrm{pr}_i : Y_1 \times Y_2 \to Y_i;\ (y_1, y_2) \mapsto y_i$ is continuous with norm 1. Hence, we observe

$$F_i(x) = \mathrm{pr}_i(F_1, F_2)(x) = \sum_{k \in \mathbb{N}_0} \mathrm{pr}_i\, m_k(x - x_0)^k$$

and

$$\sup_{k \in \mathbb{N}_0} r^k \left\| \mathrm{pr}_i\, m_k \right\|_{\mathrm{Lip}} \le \sup_{k \in \mathbb{N}_0} r^k \left\| \mathrm{pr}_i \right\|_{\mathrm{Lip}} \left\| m_k \right\|_{\mathrm{Lip}} = M < \infty.$$

□

PROPOSITION 4.4. *Let F be defined by (1) such that (2) holds. Then, F is analytic at every point $x \in B_X(x_0, r) =: U_0$, $F \in C^\infty(U_0; Y)$ and for every $k \in \mathbb{N}_0$*

$$m_k = \frac{\partial^k F(x_0)}{k!}$$

holds. For every $k \in \mathbb{N}_0$ and $x \in B_X(x_0, r)$ the k^{th} derivative of F, $\partial^k F$, is analytic at x. Furthermore for every $x_1, \dots, x_k \in X$

$$\partial^k F(x)(x_1, \dots, x_k) = \sum_{j \in \mathbb{N}_0} \frac{(j+k)!}{j!} m_{j+k}(x - x_0)^j x_1 x_2 \cdots x_k$$

holds and there are $C \in \mathbb{R}_{>1}$ and $R \in (0,1)$ such that for every $x \in B_X\left(x_0, \frac{r}{2}\right)$ and $k \in \mathbb{N}$

$$\left\| \partial^k F(x) \right\|_{\mathrm{Lip}} \le C \frac{k!}{R^k} \tag{3}$$

holds, too. In particular, if $K \subseteq U$ is compact then C and R exist such that (3) holds for every $x \in K$.

PROOF. see **[3]** □

DEFINITION 4.5. *Let X, Y, and Z be Banach spaces, $U \subseteq X \times Y$ open, and $(x_0, y_0) \in U$. A mapping $F\colon U \to Z$ is called analytic at (x_0, y_0) if and only if there exist $r \in \mathbb{R}_{>0}$ and k-linear and symmetric operators $m_k\colon (X \times Y)^k \to Z$ $(k \in \mathbb{N}_0)$ such that for every $(x, y) \in B_X((x_0, y_0), r) \subseteq U$*

$$F(x, y) = \sum_{k \in \mathbb{N}_0} m_k(x - x_0, y - y_0)^k \tag{4}$$

and

$$\sup_{k \in \mathbb{N}_0} r^k \left\| m_k \right\|_{\mathrm{Lip}} =: M < \infty \tag{5}$$

hold.

F is said to be analytic in U if and only if it is analytic at every point of U.

DEFINITION 4.6. *$m_{p,q}\colon X^p \times Y^q \to Z$ is p-q-linear and symmetric if and only if there is a k-linear and symmetric $m_k\colon (X \times Y)^k \to Z$ with $k = p + q$ such that for all $x_1, \dots, x_p \in X$ and $y_1, \dots, y_q \in Y$*

$$m_{p,q}(x_1, \dots, x_p, y_1, \dots, y_q) = m_k((x_1, 0), \dots, (x_p, 0), (0, y_1), \dots, (0, y_q))$$

holds.

It is possible to express (cf., e.g., §4.4 in **[3]**) F in (4) as

$$F(x,y) = \sum_{(p,q)\in\mathbb{N}_0^2} \frac{(p+q)!}{p!q!} m_{p,q}(x-x_0)^p(y-y_0)^q$$

with

$$m_{p,q} = \frac{\partial_1^p \partial_2^q F(x_0,y_0)}{(p+q)!}$$

and

$$\sup_{p,q\in\mathbb{N}_0} r^{p+q} \|m_{p,q}\| < \infty.$$

In particular, the power series converges absolutely again.

Now we will prove the analytic implicit function theorem. We will start by proving the implicit function theorem for up to C^∞ functions with an adaptation of the standard approach in finite dimensional spaces. This has the advantage that it is constructive, i.e., the solutions of Navier-Stokes will be constructable. The prove of analyticity, however, is not "constructable" and, even though it is possible to prove the theorem directly with analyticity, we chose this approach since constructibility of the solution is quite a nice feature.

Proposition 4.7 (Chain Rule). *Let X, Y, and Z be Banach spaces, $U \subseteq X$ open, $V \subseteq Y$ open, $a \in U$, $f : U \to V$ Fréchet-differentiable in a, and $g : V \to Z$ Fréchet-differentiable in $f(a)$. Then, $g \circ f : U \to Z$ is Fréchet-differentiable in a and satisfies*

$$(g \circ f)'(a) = g'(f(a))f'(a)) \in L(X,Z).$$

Proof. Let $A := f'(a)$ and $B := g'(f(a))$. Then, we observe for $x \in U$ and $y \in V$

$$f(x) = f(a) + A(x-a) + \|x-a\|_X \varphi(x)$$

and

$$g(y) = g(f(a)) + B(y - f(a)) + \|y - f(a)\|_Y \psi(y)$$

for some $\varphi \in C(U,Y)$ and $\psi \in C(V,Z)$ with $\varphi(x) \to 0$ $(x \to a)$ and $\psi(y) \to 0$ $(y \to f(a))$. Therefore,

$$\begin{aligned}(g\circ f)(x) =& g\left(f(a) + A(x-a) + \|x-a\|_X \varphi(x)\right)\\ =& g(f(a)) + B\left(A(x-a) + \|x-a\|_X \varphi(x)\right)\\ &+ \|A(x-a) + \|x-a\|_X \varphi(x)\|_Y \,\psi\left((f(a) + A(x-a) + \|x-a\|_X \varphi(x))\right)\\ =& g(f(a)) + BA(x-a) + \|x-a\|_X \,\omega(x)\end{aligned}$$

with

$$\omega(x) := B\varphi(x) + \frac{\|A(x-a) + \|x-a\|_X \varphi(x)\|_Y \,\psi\left(f(x)\right)}{\|x-a\|_X}$$

which satisfies

$$\|\omega(x)\|_Z \le \|B\|_{L(Y,Z)} \underbrace{\|\varphi(x)\|_Y}_{\to 0} + \left(\|A\|_{L(X,Y)} + \|\varphi(x)\|_Y\right) \underbrace{\psi(f(x))}_{\to \psi(f(a))=0} \to 0 \ (x \to a),$$

thus, showing the assertion.

∎

PROPOSITION 4.8 (Mean Value Inequality). *Let X be a Banach space, $a, b \in \mathbb{R}$, $a < b$, and $f \in C([a,b], X)$ differentiable from the right on (a,b). Then, there is a $t \in (a,b)$ such that*

$$\|f(b) - f(a)\|_X \leq \|f_r'(t)\|_X (b - a)$$

holds where $f_r'(t)$ denotes the right-hand side derivative of f at t.

PROOF. (i) Let $\varphi \in C([a,b], \mathbb{R})$ with $\varphi(a) = \varphi(b) = 0$. Then, the intermediate value theorem yields the existence of a_1 and b_1 with $a < a_1 < b_1 < b$ satisfying $\varphi(a_1) = \varphi(b_1)$. By the extreme value theorem, there exists $t \in [a_1, b_1)$ such that

$$\forall r \in [a_1, b_1) : \ \varphi(t) \leq \varphi(r)$$

holds. Hence, there is $h \in \mathbb{R}_{>0}$ such that

$$\forall s \in [t, t+h] : \ \varphi(t) \leq \varphi(s)$$

holds.

(ii) Without loss of generality, let $a = 0$ and $f(0) = 0$. For $s \in [0, b]$, we define

$$\varphi(s) := \|f(s)\|_X - s \left\| \frac{1}{b} f(b) \right\|_X$$

and observe that φ is continuous with $\varphi(0) = \varphi(b) = 0$. According to (i), there exists a $t \in (0,b)$ and $h \in (0, b-t)$ such that

$$\forall s \in [t, t+h] : \ \varphi(s) \geq \varphi(t)$$

holds. Hence,

$$\begin{aligned} 0 \leq & \frac{\varphi(s) - \varphi(t)}{s - t} \\ = & \frac{\|f(s)\|_X - \|f(t)\|_X}{s - t} - \left\| \frac{1}{b} f(b) \right\|_X \\ \leq & \frac{\|f(s) - f(t)\|_X}{s - t} - \left\| \frac{1}{b} f(b) \right\|_X \\ \\ = & \left\| \frac{f(s) - f(t)}{s - t} \right\|_X - \left\| \frac{1}{b} f(b) \right\|_X \\ \to & \|f_r'(t)\|_X - \left\| \frac{1}{b} f(b) \right\|_X \end{aligned}$$

holds for $s \searrow t$. In other words,

$$\|f(b) - f(a)\|_X = \|f(b)\|_X \leq \|f_r'(t)\|_X \, b = \|f_r'(t)\|_X (b - a)$$

shows the assertion.

□

COROLLARY 4.9. *Let X and Y be Banach spaces, $U \subseteq X$ open, $f : U \to Y$ Fréchet-differentiable, and $a, b \in U$ such that their convex hull $\operatorname{conv}\{a, b\}$ is a subset of U. Then, there is a $t \in (0, a)$ such that*

$$\|f(b) - f(a)\|_Y \leq \|f'((1-t)a + tb)\|_{L(X,Y)} \|b - a\|_X$$

holds.

PROOF. Let

$$g : \ [0,1] \to Y; \ t \mapsto f((1-t)a + tb).$$

Then, g is differentiable and the chain rule yields

$$g'(t) = f'((1-t)a + tb)(b-a).$$

Furthermore, there is $t \in (0,1)$ such that

$$\|f(b) - f(a)\|_Y = \|g(1) - g(0)\|_Y \leq \|g'(t)\|_Y \leq \|f'((1-t)a + tb)\|_{L(X,Y)} \|b-a\|_X .$$

□

PROPOSITION 4.10 (Modified Newton's Method). *Let X and Y be Banach spaces, $U \subseteq X$ open, $G \subseteq U$ convex and closed in X, $f \in C^1(U,Y)$, and $B \in L(Y,X)$. Let $\Phi\colon\ U \to X;\ x \mapsto x - Bf(x)$ be such that G is Φ-invariant, i.e., $\Phi[G] \subseteq G$, and*

$$k := \sup_{x \in G} \|1 - Bf'(x)\|_{L(X)} < 1.$$

Then, Φ is a strict contraction on G and its unique fixed point $x^* \in G$ satisfies $Bf(x^*) = 0$.

PROOF. Clearly, $\Phi'(x) = 1 - Bf'(x)$. Hence, for $a, b \in G$, we obtain

$$\|\Phi(x) - \Phi(a)\|_X \leq \sup_{t \in [0,1]} \|\Phi'((1-t)a + tb)\|_{L(X)} \|b-a\|_X \leq k \|b-a\|_X .$$

Hence, Φ is a strict contraction and Banach's fixed point theorem implies the other assertions.

□

THEOREM 4.11 (Implicit Function Theorem). *Let M be a topological space, Y and Z Banach spaces, $U \subseteq Y$ open, $F\colon\ M \times U \to Z$ continuous and Fréchet-differentiable with respect to the second variable, as well as, $(a,b) \in M \times U$ such that $f(a,b) = 0$, $\partial_2 F$ is continuous at (a,b), and $\partial_2 F(a,b)$ is an isomorphism.*

Then, there are open neighborhoods $V_1 \subseteq M$ of a and $V_2 \subseteq U$ of b such that there exists a unique function $g\colon\ V_1 \to V_2$ with $\forall x \in V_1\colon\ F(x, g(x)) = 0$. Furthermore, g is continuous.

PROOF. Let $B := \partial_2 F(a,b)^{-1} \in L(Z,Y)$ and

$$\Phi\colon\ M \times U \to Y;\ (x,y) \mapsto y - BF(x,y).$$

Then, clearly, $1 - B\partial_2 F(a,b) = 0$ holds and, since $\partial_2 F$ is continuous at (a,b), there are $W_1 \subseteq_{\text{open}} M$ and $W_2 \subseteq_{\text{open}} U$ with $a \in W_1$ and $b \in W_2$ such that

$$\forall (x,y) \in W_1 \times W_2\colon\ \|1 - B\partial_2 F(x,y)\|_{L(Y)} < \frac{1}{2}$$

holds. Let $r \in \mathbb{R}_{>0}$ such that $B_Y[b,r] \subseteq W_2$. Since $F(a,b) = 0$ and F is continuous, there is an open neighborhood $V_1 \subseteq W_1$ of a such that

$$\sup_{x \in V_1} \|BF(x,b)\|_Y < \frac{r}{2}.$$

For $x \in V_1$ and $y \in B_Y[b,r]$ we, thus, observe

$$\begin{aligned}\|\Phi(x,y) - b\|_Y &\leq \|\Phi(x,y) - \Phi(x,b)\|_Y + \|\Phi(x,b) - b\|_Y \\ &\leq \underbrace{\sup_{t \in [0,1]} \|1 - B\partial_2 F(x,(1-t)y + tb)\|_{L(Y)}}_{<\frac{1}{2}} \underbrace{\|y-b\|_Y}_{\leq r} + \underbrace{\|BF(x,b)\|_Y}_{<\frac{r}{2}} \\ &< r,\end{aligned}$$

i.e., $\Phi(x,\cdot)[B_Y[b,r]] \subseteq B_Y(b,r) =: V_2$. Hence, $\Phi(x,\cdot)$ has a unique fixed point $g(x) \in V_2$ for every $x \in V_1$.

Concerning continuity of g, we observe for $x, x' \in V_1$ sufficiently close

$$\begin{aligned}\|g(x)-g(x')\|_Y &= \|\Phi(x,g(x))-\Phi(x',g(x'))\|_Y \\ &\le \|\Phi(x,g(x))-\Phi(x',g(x))\|_Y + \|\Phi(x',g(x))-\Phi(x',g(x'))\|_Y \\ &\le \|\Phi(x,g(x))-\Phi(x',g(x))\|_Y \\ &\quad + \underbrace{\sup_{t\in[0,1]} \|1-B\partial_2 F(x',(1-t)g(x)+tg(x'))\|_{L(Y)}}_{<\frac{1}{2}} \|g(x)-g(x')\|_Y \\ &< \|\Phi(x,g(x))-\Phi(x',g(x))\|_Y + \frac{1}{2}\|g(x)-g(x')\|_Y\end{aligned}$$

and, therefore,

$$\|g(x)-g(x')\|_Y < 2\|\Phi(x,g(x))-\Phi(x',g(x))\|_Y = 2\|B(F(x',g(x))-F(x,g(x)))\|_Y$$

which converges to zero as $x' \to x$ because F and B are continuous.

□

COROLLARY 4.12. *Using the notation of Theorem 4.11, let* $g_0 := b$ *and*

$$\forall n \in \mathbb{N}\ \forall x \in V_1 :\ g_n(x) := g_{n-1}(x) - \partial_2 F(a,b)^{-1} F(x, g_{n-1}(x)).$$

Then, g_n *converges to the implicit function* g *pointwise in* Y.

PROOF. In the proof of Theorem 4.11 we constructed g to be the unique fixed point of

$$g(x) = \Phi(x,g(x)) = g(x) - \partial_2 F(a,b)^{-1} F(x,g(x))$$

using Banach's fixed point theorem. Now defined $g_{n+1}(x) = \Phi(x, g_n(x))$. Thus, $b = g_0(x) \in V_2 = B_Y(b,r)$ for every $x \in V_1$ implies pointwise convergence of $(g_n)_{n\in\mathbb{N}}$ to g.

□

Now, that we can construct implicit functions, the remainder of the chapter will show that the solutions are sufficiently smooth if the function F is and we will state the inverse function theorem since this is the theorem we will end up using.

PROPOSITION 4.13. *Let* X, Y, *and* Z *be Banach spaces,* $U_1 \subseteq X$ *open,* $U_2 \subseteq Y$ *open,* $F:\ U_1 \times U_2 \to Z$, $(a,b) \in U_1, \times U_2$, $F(a,b) = 0$, F *Fréchet-differentiable at* (a,b), *and* $\partial_2 F(a,b)$ *an isomorphism. Let* $g:\ U_1 \to U_2$ *be continuous at* a, $g(a) = b$ *and* $\forall x \in U_1 :\ F(x,g(x)) = 0$.

Then, g *is Fréchet-differentiable at* a *satisfying*

$$g'(a) = -\partial_2 F(a,b)^{-1} \partial_1 F(a,b).$$

PROOF. Without loss of generality, let $a = 0$ and $b = 0$. Let $A := \partial_1 F(0,0) \in L(X,Z)$ and $B := \partial_2 F(0,0) \in L(Y,Z)$. Then, B is an isomorphism and there exists a function $\varphi :\ U_1 \times U_2 \to Z$ with $\varphi(x,y) \to 0$ $((x,y) \to 0)$ and

$$F(x,y) = \underbrace{F(0,0)}_{=0} + Ax + By + (\|x\|_X + \|y\|_Y)\,\varphi(x,y).$$

Thus,

$$0 = F(x,g(x)) = Ax + Bg(x) + (\|x\|_X + \|g(x)\|_Y)\,\varphi(x,g(x))$$

implies

$$g(x) = -B^{-1}Ax - (\|x\|_X + \|g(x)\|_Y)\,B^{-1}\varphi(x,g(x)) = -B^{-1}Ax + \|x\|_X\,\psi(x)$$

with

$$\psi(x) := -\left(1 + \frac{\|g(x)\|_Y}{\|x\|_X}\right) B^{-1}\varphi(x, g(x))$$

for $x \neq 0$. Hence, we will have to show $\psi(x) \to 0$ $(x \to 0)$. Let $\delta \in \mathbb{R}_{>0}$ be such that $B_X(0,\delta) \subseteq U_1$ and

$$\forall x \in B_Y(0,\delta): \ \left\|B^{-1}\varphi(x, g(x))\right\|_Y \leq \frac{1}{2}.$$

Then,

$$\|g(x)\|_Y \leq \left\|B^{-1}A\right\|_{L(X,Y)} \|x\|_X + \frac{\|x\|_X + \|g(x)\|_Y}{2}$$

implies

$$\|g(x)\|_Y \leq \underbrace{\left(2\left\|B^{-1}A\right\|_{L(X,Y)} + 1\right)}_{=:K} \|x\|_X ,$$

i.e.,

$$\|\psi(x)\|_Y \leq (1+K)\left\|B^{-1}\varphi(x, g(x))\right\|_Y \to 0 \qquad (x \to 0).$$

□

Remark *on the Neumann series*

Let X be a Banach space, $T \in L(X)$, and $\|T\|_{L(X)} < 1$. Then, $\sum_{k\in\mathbb{N}_0} T^k$ converges absolutely since

$$\sum_{k\in\mathbb{N}_0} \left\|T^k\right\|_{L(X)} \leq \sum_{k\in\mathbb{N}_0} \|T\|_{L(X)}^k = \frac{1}{1 - \|T\|_{L(X)}}.$$

Furthermore, we obtain

$$(1-T)\sum_{k\in\mathbb{N}_0} T^k = \left(\sum_{k\in\mathbb{N}_0} T^k\right)(1-T) = 1$$

holds, i.e., $1-T$ is a homeomorphism with $(1-T)^{-1} = \sum_{k\in\mathbb{N}_0} T^k$.

■

LEMMA 4.14. *Let X and Y be Banach spaces, $S, T \in L(X,Y)$, $0 \in \varrho(T)$, and*

$$\|S-T\|_{L(X,Y)} < \left\|T^{-1}\right\|_{L(Y,X)}^{-1}.$$

Then, $0 \in \varrho(S)$ and

$$B_{L(X,Y)}\left(T, \left\|T^{-1}\right\|_{L(Y,X)}^{-1}\right) \ni S \mapsto S^{-1} \in L(Y,X)$$

is continuous. In particular, the set of isomorphism in $L(X,Y)$ is open.

PROOF. Since $\left\|T^{-1}(S-T)\right\|_{L(X)} \leq \left\|T^{-1}\right\|_{L(Y,X)} \|S-T\|_{L(X,Y)} < 1$, the Neumann series yields that $1 + T^{-1}(S-T): X \to X$ is boundedly invertible. Using

$$S = T(1 + T^{-1}(S-T))$$

we obtain the assertion from

$$S^{-1} = (1 + T^{-1}(S-T))^{-1}T^{-1} \in L(Y,X).$$

□

COROLLARY 4.15. *With the assumptions of Theorem 4.11, M being an open subset of a Banach space X, and $F \in C^m(M \times U, Z)$ for some $m \in \mathbb{N} \cup \{\infty\}$, the set V_1 in Theorem 4.11 can be chosen such that $g: V_1 \to V_2$ is in $C^m(V_1, V_2)$.*

PROOF. It is possible to choose V_1 such that $\forall x \in V_1 : \ \partial_2 F(x, g(x))$ is an isomorphism. Thus, Proposition 4.13 yields continuity of g' with

$$g'(x) = -\partial_2 F(x, g(x))^{-1} \partial_1 F(x, g(x)).$$

For $m \geq 2$ the right-hand side is Fréchet-differentiable and $g \in C^2(V_1, V_2)$, therefore. Inductively, we obtain $g \in C^m(V_1, V_2)$.

□

THEOREM 4.16 (Inverse Function Theorem). *Let X and Y be Banach spaces, $U \subseteq X$ open, $m \in \mathbb{N} \cup \{\infty\}$, $f \in C^m(U, Y)$, $a \in U$, and $f'(a)$ an isomorphism. Then, there are open neighborhoods U_1 of a and U_2 of $b := f(a)$ such that $f : U_1 \to U_2$ is a C^m-diffeomorphism. Furthermore, $(f^{-1})'(b) = f'(a)^{-1}$.*

PROOF. Let

$$F : \ Y \times U \to Y; \ (y, x) \mapsto f(x) - y.$$

Then, $F \in C^m(Y \times U, Y)$, $F(b, a) = 0$, and $\partial_2 F(b, a) = f'(a)$ is an isomorphism. Hence, there are open neighborhoods $U_2 \subseteq Y$ of b and $V \subseteq U$ of a such that $g : U_2 \to V$ is uniquely determined by $F(y, g(y)) = 0$ and $g \in C^m(U_2, V)$. Since for $x \in C$ and $y \in U_2$

$$x = g(y) \ \Leftrightarrow \ F(y, x) = 0 \ \Leftrightarrow \ y = f(x)$$

holds, $U_1 := g[U_2] = [U_2]f \cap V$ is open and $f : U_1 \to U_2$ is bijective with $g = (f|_{U_1})^{-1}$. Hence, $g \circ f|_{U_1} = \mathrm{id}|_{U_1}$ implies $(g' \circ f) f' = 1$, i.e.,

$$g'(b) = f'(a)^{-1}.$$

□

At this point we have shown the implicit function theorem and inverse function theorem for C^m-function with $m \in \mathbb{N} \cup \{\infty\}$. Now we will show that they are also true for analytic functions (C^ω).

Let X be a Banach space and $r \in (0, 1)$. We define $B_r := B_X(0, r^2) \times B_X(0, r) \subseteq X^2$, as well as, E_r to be the set of all $u = \left((x, y) \mapsto \sum_{m,n \in \mathbb{N}_0} u_{m,n} x^m y^n\right) \in C^\omega(B_r, X)$ satisfying

$$\|u\|_{E_r} := \sum_{m,n \in \mathbb{N}_0} \|u_{m,n}\|_{\mathrm{Lip}} \, r^{2m+n} < \infty$$

which itself defines a norm on E_r.

LEMMA 4.17. *$(E_r, \|\cdot\|_{E_r})$ is a Banach space.*

PROOF. Let $(u^{(n)})_{n \in \mathbb{N}} \in E_r^{\mathbb{N}}$ be a Cauchy sequence. Then, all $(u_{i,j}^{(n)})_{n \in \mathbb{N}} \in L(X^{i+j}, X)^{\mathbb{N}}$ are Cauchy and, since X is complete, so is $L(X^{i+j}, X)$, i.e., $u_{i,j}^{(n)} \to: u_{i,j}$ in $L(X^{i+j}, X)$ for every $i, j \in \mathbb{N}_0$.

Let $x_1, \ldots, x_{i+j} \in B_r$, $\alpha \in \mathbb{R}$, and $y \in B_r$ sufficiently small such that $u_k + \alpha y \in B_r$ for every $k \in \mathbb{N}_{\leq i+j}$. Then,

$$\begin{aligned} u_{i,j}(x_1, \ldots, x_{k-1}, x_k + \alpha y, x_{k+1}, \ldots, x_{i+j}) \leftarrow & u_{i,j}^{(n)}(x_1, \ldots, x_{k-1}, x_k + \alpha y, x_{k+1}, \ldots, x_{i+j}) \\ = & u_{i,j}^{(n)}(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{i+j}) \\ & + \alpha u_{i,j}^{(n)}(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{i+j}) \\ \to & u_{i,j}(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{i+j}) \\ & + \alpha u_{i,j}^{(n)}(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{i+j}) \end{aligned}$$

shows multi-linearity of $u_{i,j}$.

Let

$$\tilde{u}_n : B_r \to X; \ (x,y) \mapsto \sum_{i,j=0}^{n} u_{i,j} x^i y^j$$

and $m \in \mathbb{N}$ sufficiently large such that $\forall i,j \in \mathbb{N}_{0,\le n}: \ \left\| u_{i,j} - u_{i,j}^{(m)} \right\|_{\mathrm{Lip}} \le \frac{\varepsilon}{n^2 r^{2i+j}}$. Then,

$$\begin{aligned}\sum_{i,j=0}^{n} \|u_{i,j}\|_{\mathrm{Lip}} r^{2i+j} &\le \sum_{i,j=0}^{n} \left\| u_{i,j} - u_{i,j}^{(m)} \right\|_{\mathrm{Lip}} r^{2i+j} + \sum_{i,j=0}^{n} \left\| u_{i,j}^{(m)} \right\|_{\mathrm{Lip}} r^{2i+j} \\ &\le \varepsilon + \underbrace{\sup_{k\in\mathbb{N}} \left\| u^{(k)} \right\|_{E_r}}_{<\infty}.\end{aligned}$$

Hence, the pointwise limit $\tilde{u}_n \to: u \ (n \to \infty)$ exists and is an element of E_r.

In order to show that $\|u - \tilde{u}_n\|_{E_r} = \sum_{i,j \in \mathbb{N}_{>n}} \left\| u_{i,j} - u_{i,j}^{(n)} \right\|_{\mathrm{Lip}} r^{2i+j}$ converges to zero, let $\varepsilon \in \mathbb{R}_{>0}$.

(i) Choose $m_1 \in \mathbb{N}$ such that $\forall m \in \mathbb{N}_{\ge m_1}: \ \|u - \tilde{u}_m\|_{E_r} < \frac{\varepsilon}{4}$.
(ii) Choose $n_1 \in \mathbb{N}$ such that $\forall n', n'' \in \mathbb{N}_{\ge n_1}: \ \left\| u^{(n')} - u^{(n'')} \right\|_{E_r} < \frac{\varepsilon}{4}$.
(iii) Choose $m_2 \in \mathbb{N}_{\ge m_1}$ such that $\sum_{i,j \in \mathbb{N}_{>m_2}} \left\| u_{i,j}^{(n_1)} \right\|_{\mathrm{Lip}} r^{2i+j} < \frac{\varepsilon}{4}$.
(iv) Since all $\left\| u_{i,j}^{(n)} \right\|_{\mathrm{Lip}} r^{2i+j}$ converge to zero, let $n_2 \in \mathbb{N}_{\ge n_1}$ be such that $\forall n \in \mathbb{N}_{\ge n_2} \ \forall i,j \in \mathbb{N}_{0,\le m_2}: \ \left\| u_{i,j}^{(n)} \right\|_{\mathrm{Lip}} r^{2i+j} < \frac{\varepsilon}{4(m_2+1)^2}$.

Then, we observe for $n \in \mathbb{N}_{\ge n_2}$

$$\begin{aligned}\left\| u - u^{(n)} \right\|_{E_r} &\le \underbrace{\|u - \tilde{u}_{m_2}\|_{E_r}}_{<\frac{\varepsilon}{4}} + \underbrace{\sum_{i,j=0}^{m_2} \left\| u_{i,j} - u_{i,j}^{(n)} \right\|_{\mathrm{Lip}} r^{2i+j}}_{<\frac{\varepsilon}{4}} + \sum_{i,j \in \mathbb{N}_{>m_2}} \left\| u_{i,j}^{(n)} \right\|_{\mathrm{Lip}} r^{2i+j} \\ &< \frac{\varepsilon}{2} + \underbrace{\sum_{i,j \in \mathbb{N}_{>m_2}} \left\| u_{i,j}^{(n)} - u_{i,j}^{(n_1)} \right\|_{\mathrm{Lip}} r^{2i+j}}_{\le \left\| u^{(n)} - u^{(n_1)} \right\|_{E_r} < \frac{\varepsilon}{4}} + \underbrace{\sum_{i,j \in \mathbb{N}_{>m_2}} \left\| u_{i,j}^{(n_1)} \right\|_{\mathrm{Lip}} r^{2i+j}}_{<\frac{\varepsilon}{4}} \\ &< \varepsilon\end{aligned}$$

which completes the proof.

□

Let us also consider the subspace

$$F_r := \{ u \in E_r; \ \forall m \in \mathbb{N}_0 : \ u_{m,0} = 0 \}.$$

Clearly, F_r is a closed subspace, i.e., a Banach space itself. Furthermore, let us define $L \in L(F_r)$ by

$$\forall (x,y) \in B_r : \ Lu(x,y) := \sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}} \frac{1}{n} u_{m,n} x^m y^n$$

and for $w \in E_r$

$$\forall (x,y) \in B_r : \ L_w u(x,y) := \partial_2 u(x,y) w(x,y) - \partial_2 u(x,0) w(x,0).$$

Obviously, we obtain $\|L\|_{L(F_r)} = 1$ and, for $w_0 : \ B_r \to X; \ (x,y) \mapsto y$, $L_{w_0} \circ L = \mathrm{id}_{F_r}$.

LEMMA 4.18. *$L_w \circ L$ is in $L(F_r)$ and satisfies* $\|L_w \circ L\|_{L(F_r)} \le \frac{\|w\|_{E_r}}{r}$.

PROOF. Let w be decomposed as

$$w(x,y) = \sum_{m,n\in\mathbb{N}_0} w_{m,n}x^m y^n.$$

Then, for $u = \left((x,y)\mapsto \sum_{(m,n)\in\mathbb{N}_0\times\mathbb{N}} u_{m,n}x^m y^n\right)\in F_r$,

$$\begin{aligned}
L_w Lu(z,0) &= \left((x,y)\mapsto L_w \sum_{(m,n)\in\mathbb{N}_0\times\mathbb{N}} \frac{1}{n}u_{m,n}x^m y^n\right)(z,0)\\
&= \left(\sum_{m,n\in\mathbb{N}_0} u_{m,n+1}x^m y^n\right)\left(\sum_{m,n\in\mathbb{N}_0} w_{m,n}x^m y^n\right)\Bigg|_{(x,y)=(z,0)}\\
&\quad - \left(\sum_{m\in\mathbb{N}_0} u_{m,1}x^m\right)\left(\sum_{m\in\mathbb{N}_0} w_{m,0}x^m\right)\Bigg|_{(x,y)=(z,0)}\\
&= \left(\sum_{m\in\mathbb{N}_0} u_{m,1}z^m\right)\left(\sum_{m\in\mathbb{N}_0} w_{m,0}x^m\right) - \left(\sum_{m\in\mathbb{N}_0} u_{m,1}z^m\right)\left(\sum_{m\in\mathbb{N}_0} w_{m,0}z^m\right)\\
&=0
\end{aligned}$$

shows $L_w Lu \in F_r$. Furthermore,

$$\begin{aligned}
&L_w Lu(x,y)\\
=&L_w \sum_{(m,n)\in\mathbb{N}_0\times\mathbb{N}} \frac{1}{n}u_{m,n}x^m y^n\\
=&\left(\sum_{m,n\in\mathbb{N}_0} u_{m,n+1}x^m y^n\right)\left(\sum_{m,n\in\mathbb{N}_0} w_{m,n}x^m y^n\right) - \left(\sum_{m\in\mathbb{N}_0} u_{m,1}x^m\right)\left(\sum_{m\in\mathbb{N}_0} w_{m,0}x^m\right)\\
=&\sum_{(M,N)\in\mathbb{N}_0\times\mathbb{N}}\ \sum_{(m,n)\in\mathbb{N}^2_{0,\le(M,N)}} \left(u_{m,n+1}x^m y^n\right)\left(w_{M-m,N-n}x^{M-m}y^{N-n}\right)
\end{aligned}$$

implies

$$\begin{aligned}
&\|L_w Lu\|_{E_r}\\
\le&\sum_{(M,N)\in\mathbb{N}_0\times\mathbb{N}}\left(\sum_{(m,n)\in\mathbb{N}^2_{0,\le(M,N)}} \|u_{m,n+1}\|_{\mathrm{Lip}}\|w_{M-m,N-n}\|_{\mathrm{Lip}}\right) r^{2M+N}\\
=&\frac{1}{r}\sum_{(M,N)\in\mathbb{N}_0\times\mathbb{N}}\left(\sum_{(m,n)\in\mathbb{N}^2_{0,\le(M,N)}} \|u_{m,n+1}\|_{\mathrm{Lip}} r^{2m+n+1}\|w_{M-m,N-n}\|_{\mathrm{Lip}} r^{2(M-m)+(N-n)}\right)\\
=&\frac{1}{r}\left(\sum_{m,n\in\mathbb{N}_0}\|u_{m,n+1}\|_{\mathrm{Lip}} r^{2m+n+1}\right)\left(\sum_{m,n\in\mathbb{N}_0}\|w_{m,n}\|_{\mathrm{Lip}} r^{2m+n)}\right)\\
=&\frac{\|w\|_{E_r}}{r}\|u\|_{E_r}.
\end{aligned}$$

□

LEMMA 4.19. *Let X be a Banach space, $U\subseteq X$ an open neighborhood of zero, $F\in C^\omega(U,X)$, $F(0)=0$, and $F'(0)=1$. Let $V\subseteq X$ be an open neighborhood of zero and $G\colon V\to X$ a local inverse of F at zero. Then, G is analytic in an open neighborhood of zero.*

PROOF. For $r\in\mathbb{R}_{>0}$ sufficiently small, let $v,w\in E_r$ be defined by

$$\forall (x,y)\in B_r:\ v(x,y) := F(y)-x$$

and

$$\forall (x,y) \in B_r :\ w(x,y) := v(x,y) - w_0(x,y) = F(y) - x - y.$$

Then,

$$\forall (x,y) \in B_r :\ w(x,y) = -x + \sum_{n \in \mathbb{N}_{\geq 2}} \frac{1}{n!} \partial^n F(0) y^n$$

and

$$\|w\|_{E_r} \leq r^2 + \sum_{n \in \mathbb{N}_{\geq 2}} \frac{\|\partial^n F(0)\|_{\mathrm{Lip}}}{n!} r^n \leq r^2 C_F$$

holds where $C_F \in \mathbb{R}_{>0}$ is a constant solely dependent on F. From the definitions of v and w, we obtain

$$L_v \circ L - 1 = (L_v - L_{w_0}) \circ L = L_w \circ L$$

and, hence, for $r < \frac{1}{C_F}$,

$$\|L_v \circ L - 1\|_{L(F_r)} \leq r C_F < 1$$

and, according to Lemma 4.14, $L_v \circ L$ is an isomorphism on F_r. Let $u_0 := (L_v \circ L)^{-1} w_0$. Then, we obtain for all $(x,y) \in B_r$

$$(*) \quad y = w_0(x,y) = (L_v \circ L) u_0(x,y) = \partial_2(Lu_0)(x,y) v(x,y) - \partial_2(Lu_0)(x,0) v(x,0).$$

In particular, we observe for $y \in B_X(0,r)$ and $t \in (0,1)$

$$\begin{aligned} ty =& \partial_2(Lu_0)(0,ty)v(0,ty) - \partial_2(Lu_0)(0,0)v(0,0) \\ =& \partial_2(Lu_0)(0,ty)v(0,ty) - \partial_2(Lu_0)(0,ty)v(0,0) \\ &+ \partial_2(Lu_0)(0,ty)v(0,0) - \partial_2(Lu_0)(0,0)v(0,0) \\ =& \partial_2(Lu_0)(0,ty)(F(ty) - F(0)) + \left(\partial_2(Lu_0)(0,ty) - \partial_2(Lu_0)(0,0)\right) F(0) \\ =& \partial_2(Lu_0)(0,ty)(F(ty) - F(0)) \end{aligned}$$

which (dividing by t and $t \searrow 0$) shows

$$\forall y \in B_X(0,r) :\ y = \partial_2(Lu_0)(0,0) F'(0) y = \partial_2(Lu_0)(0,0) y,$$

i.e., $\partial_2(Lu_0)(0,0) = 1 = \mathrm{id}_X$. Hence, there exists $\varepsilon \in (0,r)$ such that $\partial_2(Lu_0)(x,y)$ is a bijection on X for every $(x,y) \in B_X(0,\varepsilon^2) \times B_X(0,\varepsilon)$.

Defining

$$\tilde{G} :\ B_X(0,\varepsilon^2) \to X;\ x \mapsto \partial_2(Lu_0)(x,0)x$$

we observe $\tilde{G} \in C^\omega\left(B_X(0,\varepsilon^2), X\right)$ and

$$\begin{aligned} \tilde{G}(x) =& \partial_2(Lu_0)(x,0)x \\ =& -\partial_2(Lu_0)(x,0)(F(0) - x) \\ =& -\partial_2(Lu_0)(x,0)v(x,0) \\ \overset{(*)}{=}& y - \partial_2(Lu_0)(x,y)v(x,y) \\ =& y - \partial_2(Lu_0)(x,y)(F(y) - x), \end{aligned}$$

i.e., for $y = G(x)$,

$$\tilde{G}(x) = G(x) - \underbrace{\partial_2(Lu_0)(x,G(x))}_{\text{linear}} \underbrace{(F(G(x)) - x)}_{=0} = G(x).$$

Hence, G is analytic on $V \cap B_X(0,\varepsilon^2)$ which is an open neighborhood of zero.

□

THEOREM 4.20 (Analytic Inverse Function Theorem). *Let X and Y be Banach spaces, $U \subseteq X$ open, $m \in \mathbb{N} \cup \{\infty, \omega\}$, $f \in C^m(U, Y)$, $a \in U$, and $f'(a)$ an isomorphism. Then, there are open neighborhoods U_1 of a and U_2 of $b := f(a)$ such that $f : U_1 \to U_2$ is a C^m-diffeomorphism. Furthermore, $(f^{-1})'(b) = f'(a)^{-1}$.*

PROOF. The inverse function theorem (Theorem 4.16) yields the assertion for $m \in \mathbb{N} \cup \{\infty\}$, i.e., it suffices to show the assertion for $m = \omega$ knowing that $f : V_1 \to V_2$ is a C^∞-diffeomorphism for some open neighborhoods V_1 of a and V_2 of b. Let $\tilde{U} := U - a$ and

$$\tilde{f} : \tilde{U} \to X; \; x \mapsto f'(a)^{-1}(f(x+a) - f(a)).$$

Then, $\tilde{f}(0) = 0$ and $\tilde{f}'(0) = 1$. Thus, Lemma 4.19 yields that $\tilde{f}$ is a $C^\omega\left(\tilde{U}_1, \tilde{U}_2\right)$-diffeomorphism for some neighborhoods $\tilde{U}_1$ and $\tilde{U}_2$ of zero. Finally,

$$\forall x \in \left(a + \tilde{U}_1\right) \cap V_1 : \; f(x) = f(a) + f'(a)\tilde{f}(x - a)$$

implies the assertion for $U_1 := \left(a + \tilde{U}_1\right) \cap V_1$ and $U_2 := f[U_1]$.

□

THEOREM 4.21 (Analytic Implicit Function Theorem). *Let X, Y, and Z be Banach spaces, $U_1 \subseteq X$ open, $U_2 \subseteq Y$ open, $m \in \mathbb{N} \cup \{\infty, \omega\}$, $F \in C^m(U_1 \times U_2, Z)$, $(a, b) \in U_1 \times U_2$, $F(a, b) = 0$, and $\partial_2 F(a, b)$ an isomorphism. Then, there are open neighborhoods $V_1 \subseteq X$ of a and $V_2 \subseteq Y$ of b such that there is a unique function $g : V_1 \to V_2$ with $\forall x \in V_1 : \; F(x, g(x)) = 0$. Furthermore, $g \in C^m(V_1, V_2)$.*

PROOF. We already know the assertion for $m \in \mathbb{N} \cup \{\infty\}$. Hence, let $m = \omega$ and $g \in C^\infty(W_1, W_2)$ the implicit function with open neighborhoods $W_1 \subseteq X$ of a and $W_2 \subseteq Y$ of b. Let

$$G : \; U_1 \times U_2 \to Z \times X; \; (x, y) \mapsto (F(x, y), x).$$

Then,

$$G'(a, b)(x, y) = (\partial_1 F(a, b)x + \partial_2 F(a, b)y, x)$$

holds for every $(x, y) \in X \times Y$. Thus, $G'(a, b)$ has the bounded inverse

$$G'(a, b)^{-1} : \; Z \times X \mapsto X \times Y; \; (z, x) \mapsto \left(x, \partial_2 F(a, b)^{-1}\left(z - \partial_1 F(a, b)x\right)\right)$$

and the analytic inverse function theorem (Theorem 4.20) yields open sets $\tilde{U}_1 \subseteq U_1$, $\tilde{U}_2 \subseteq U_2$, and $V_0 \subseteq Z \times X$ such that $(a, b) \in \tilde{U}_1 \times \tilde{U}_2$ and G is a $C^\omega\left(\tilde{U}_1 \times \tilde{U}_2, V_0\right)$-diffeomorphism. Furthermore, we observe

$$g(x) = \mathrm{pr}_2(x, g(x)) = \left(\mathrm{pr}_2 \circ G^{-1}\right)(F(x, g(x)), x) = \left(\mathrm{pr}_2 \circ G^{-1}\right)(0, x)$$

for every $x \in W_1 \cap \tilde{U}_1 =: V_1$. Observation 4.3, thus, yields that

$$V_1 \ni x \mapsto \left(\mathrm{pr}_2 \circ G^{-1}\right)(0, x) \in Y$$

is analytic, i.e., g is a $C^\omega(V_1, V_2)$-diffeomorphism where $V_2 := g[V_1]$.

□

To conclude this chapter we will prove the incredibly handy fact that composition of analytic functions yields an analytic function.

PROPOSITION 4.22. *Let X, Y, and Z be Banach spaces, $U \subseteq X$ open, $V \subseteq Y$ open, $F \in C^\omega(U, V)$, and $G \in C^\omega(V, Z)$. Then, $G \circ F \in C^\omega(U, Z)$.*

PROOF. Let $W := U \times (V \times Z)$ and $H : W \to Y \times Z;\ (x,(y,z)) \mapsto (F(x) - y, G(y) - z)$. Let $x_0 \in U$, $y_0 := F(x_0)$, and $z_0 := G(y_0)$. Then, we observe

$$H(x_0,(y_0,z_0)) = 0$$

and the equation

$$(\hat{y},\hat{z}) = \partial_2 H(x_0,(y_0,z_0))(y,z) = (y,z)\begin{pmatrix} -1 & G'(y_0) \\ 0 & -1 \end{pmatrix} = (-y, G'(y_0)y - z)$$

is equivalent to $y = -\hat{y}$ and $z = -\hat{z} - G'(y_0)\hat{y}$. Thus, $\partial_2 H(x_0,(y_0,z_0))$ is an isomorphism, and, by the analytic implicit function theorem, there is an analytic implicit function $\left(\hat{Y},\hat{Z}\right)$ solving

$$H\left(x, \left(\hat{Y}(x), \hat{Z}(x)\right)\right) = 0$$

in an open neighborhood of x_0. But $H(x,(y,z)) = 0$ implies $G(F(x)) = z$ and, thus, $G \circ F = \hat{Z}$. Observation 4.3 yields analyticity of $G \circ F$ at x_0 and, since x_0 was arbitrarily chosen in U, $G \circ F$ is analytic.

□

CHAPTER 5

Fredholm Operators

At last, we will state a few facts about Fredholm operators as we will use them quite extensively in the proof of well-posedness of the Navier-Stokes equations.

DEFINITION 5.1. *Let X and Y be Banach spaces. $T \in L(X,Y)$ is called a Fredholm operator if and only if* $\dim[\{0\}]T$ *and* $\operatorname{codim} T[X]$ *are finite.*

The number $\operatorname{ind}(T) := \dim[\{0\}]T - \operatorname{codim} T[X]$ *is called the index of T.*

DEFINITION 5.2. *Let X and Y be Banach spaces. A linear operator $T \subseteq X \oplus Y$ has finite rank if and only if* $\dim T[X]$ *is finite.*

COROLLARY 5.3. *Every bounded finite rank operator is compact. In particular, if X and Y are Banach spaces, one of which is finite dimensional, then every $T \in L(X,Y)$ has finite rank, i.e., is compact.*

LEMMA 5.4. *Let H be a Hilbert space, $M \subseteq H$ a closed subspace, and $V \subseteq H$ a finite dimensional subspace. Then, $M + V$ is closed.*

In particular, if $\operatorname{codim} M \in \mathbb{N}_0$ *and $W \subseteq H$ is a subspace with $M \subseteq W$, then W is closed and* $\operatorname{codim} W \in \mathbb{N}_0$.

PROOF. Let $P: \ H \to M$ be the orthogonal projection and $V_\perp := (1-P)V$. Then, $M + V = M \oplus V_\perp$ where $M \oplus V_\perp$ is an orthogonal direct sum, i.e., $M + V$ is closed since a sequence $((x_n, y_n))_{n\in\mathbb{N}} \in (M \oplus V_\perp)^{\mathbb{N}}$ converges if and only if $(x_n)_{n\in\mathbb{N}}$ converges in M and $(y_n)_{n\in\mathbb{N}}$ converges in $V_\perp$ and both spaces are closed (M by assumption and $V_\perp$ since it is finite dimensional).

Let $\operatorname{codim} M \in \mathbb{N}_0$. Then, there is a finite dimensional subspace $\tilde{V}$ such that $W = M + \tilde{V}$, i.e., W is closed by the previous part of the proof, and $\operatorname{codim} W \leq \operatorname{codim} M \in \mathbb{N}_0$ is trivial.

□

PROPOSITION 5.5. *Let X and Y be Banach spaces, and $T \in L(X,Y)$ a Fredholm operator.*

(i) *If* $\operatorname{ind}(T) = 0$ *and T is injective, then T is continuously invertible, i.e.,* $0 \in \varrho(T)$.

(ii) *The range $T[X]$ of T is closed. Furthermore, the equation $Tx = y$ has a solution $x \in X$ for given $y \in Y$ if and only if $\forall x^* \in [\{0\}]T^*: \ \langle x^*, y\rangle = 0$ where T^* denotes the dual operator.*

(iii) *Let $S \in L(X,Y)$ be compact. Then, $T + S$ is a Fredholm operator with* $\operatorname{ind}(T+S) = \operatorname{ind}(T)$.

(iv) *The dual operator T^* is a Fredholm operator with*

$$\dim[\{0\}]T^* = \operatorname{codim} T[X] \text{ and } \operatorname{codim} T^*[Y'] = \dim[\{0\}]T.$$

In particular, $\operatorname{ind}(T) = -\operatorname{ind}(T^*)$ *and the equation $T^*y^* = x^*$ has a solution $y^* \in Y'$ for given $x^* \in X'$ if and only if $\forall x \in [\{0\}]T: \ \langle x^*, x\rangle = 0$.*

PROOF. **[21]** Proposition 8.14 □

Remark The range of a Fredholm operator being closed is non-trivial. Let X be a Banach space and $X_0 \subsetneq X$ a dense subspace. Let $x_0 \in X \setminus X_0$ and $V := \{x \in X;\ (x, x_0) \text{ linearly independent}\} \cup \{0\}$. Then, $\operatorname{codim} V = 1$ and $X_0 \subseteq V \subsetneq X$. Since X_0 is dense, so is V. However, V cannot be closed since $V \neq X$.

■

PROPOSITION 5.6. *Let H_1 and H_2 be Hilbert spaces and $F \in L(H_1, H_2)$. Then, the following are equivalent.*

(i) *F is a Fredholm operator.*
(ii) *There exists $A \in L(H_2, H_1)$ such that $AF - 1$ and $FA - 1$ are both compact.*
(iii) *There exists $A \in L(H_2, H_1)$ such that $AF - 1$ and $FA - 1$ are both of finite rank.*

PROOF. "(i)⇒(ii)" Let F be a Fredholm operator, $x, y \in [\{0\}]F^{\perp}$, and $Fx = Fy$. Then, $F(x - y) = 0$, i.e., $x - y \in [\{0\}]F \cap [\{0\}]F^{\perp} = \{0\}$. Thus, $F:\ [\{0\}]F^{\perp} \to F[H_1]$ is bijective. Let $G:\ F[H_1] \to [\{0\}]F^{\perp}$ be the inverse of F on $F[H_1]$, $P:\ H_1 \to [\{0\}]F^{\perp}$ and $Q:\ H_2 \to F[H_1]$ the orthoprojections, and $A := GQ$. Then,

$$AF - 1 = GQF - 1 = GF - 1 = P - 1$$

and

$$FA - 1 = FGQ - 1 = Q - 1$$

hold. Since $P - 1$ and $Q - 1$ are of finite rank (they are the orthoprojections on $[\{0\}]F$ and $F[H_1]^{\perp}$), they are, in particular, compact.

"(ii)⇒(iii)" Since $AF - 1$ is compact, there are $G_1 \in L(H_1)$ of finite rank and $\Delta_1 \in B_{L(H_1)}(0, 1)$ such that $AF - 1 = G_1 + \Delta_1$ because compact operators are limits of finite rank operators. Let $A_1 := (1 - \Delta_1)^{-1} A$. Then, we observe

$$A_1 F = (1 - \Delta_1)^{-1} AF = (1 - \Delta_1)^{-1} (1 + G_1 + \Delta_1) = 1 + (1 - \Delta_1)^{-1} G_1$$

where $(1 - \Delta_1)^{-1} G_1 =: B_1$ is another operator of finite rank. Similarly, we can choose A_2, G_2, Δ_2, and B_2 (with the same properties as the operators with index $_1$) such that $FA_2 = 1 + B_2$. Since

$$A_1 + A_1 B_2 = A_1 F A_2 = A_2 + B_1 A_2$$

holds, we may define the finite rank operator

$$J := A_1 - A_2 = B_1 A_2 - A_1 B_2$$

and observe

$$FA_1 - 1 = FA_1 - (FA_2 - B_2) = FJ + B_2$$

which is of finite rank, as well as,

$$A_1 F - 1 = 1 + B_1 - 1 = B_1.$$

Hence, the operator A can be modified to A_1 such that $A_1 F - 1$ and $FA_1 - 1$ are both of finite rank.

"(iii)⇒(i)" Let $AF - 1 =: G_1$ and $FA - 1 =: G_2$. Since G_1 and G_2 have finite rank, $AF = 1 + G_1$ and $FA = 1 + G_2$ are Fredholm operators. Thus,

$$[\{0\}]F \subseteq [\{0\}]AF,$$

i.e.,

$$\dim[\{0\}]F \leq \dim[\{0\}]AF \in \mathbb{N}_0,$$

and

$$F[H_1] \supseteq FA[H_2],$$

i.e.,

$$\operatorname{codim} F[H_1] \le \operatorname{codim} FA[H_2] \in \mathbb{N}_0,$$

yield the assertion.

□

OBSERVATION 5.7. *Let H_1 and H_2 be Hilbert spaces and $F \in L(H_1, H_2)$ be a Fredholm operator. Then, F^* is a Fredholm operator with* $\operatorname{ind}(F^*) = -\operatorname{ind}(F)$.

PROOF. The observation follows directly from

$$[\{0\}]F^* = F[H_1]^\perp \text{ and } F^*[H_2]^\perp = [\{0\}]F.$$

□

PROPOSITION 5.8. *Let H_1 and H_2 be Hilbert spaces and $F \in L(H_1, H_2)$. Then, F is a Fredholm operator if and only if there are orthogonal decompositions $H_1 = H_{11} \oplus H_{12}$ and $H_2 = H_{21} \oplus H_{22}$ such that*

- *H_{11} and H_{21} are closed,*
- *H_{12} and H_{22} are finite dimensional, and*
- *F has the block decomposition*

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} : \ H_{11} \oplus H_{12} \to H_{21} \oplus H_{22}$$

with $F_{11} \in L(H_{11}, H_{21})$ boundedly invertible.

Furthermore, given this decomposition, $\operatorname{ind}(F) = \dim H_{12} - \dim H_{22}$.

PROOF. [**4**] Lemma 16.34 □

PROPOSITION 5.9. *Let H_1 and H_2 be Hilbert spaces, $F_1, C, \Delta \in L(H_1, H_2)$, $F_2 \in L(H_2, H_1)$, F_1 and F_2 Fredholm operators, C compact, and $\|\Delta\|_{L(H_1,H_2)}$ sufficiently small. Then, $F_1 + \Delta$, $F_1 + C$. and $F_1 F_2$ are Fredholm operators with* $\operatorname{ind}(F_1 + \Delta) = \operatorname{ind}(F_1 + C) = \operatorname{ind}(F_1)$ *and* $\operatorname{ind}(F_1 F_2) = \operatorname{ind}(F_1) + \operatorname{ind}(F_2)$.

In particular, the set of Fredholm operators in $L(H_1, H_2)$ is open.

PROOF. [**4**] Proposition 16.35 □

PROPOSITION 5.10. *Let H_1 and H_2 be Hilbert spaces. The index of a Fredholm operator is constant on connected components of the set of Fredholm operators in $L(H_1, H_2)$ and is a bijection between $\mathbb{Z}$ and the connected components.*

PROOF. [**7**] Theorem 1.4 (b) □

Remark In fact, it can be shown that the set of Fredholm operators in $L(H)$ of a given index is path connected where H is a Hilbert space. To prove this, recall that if A_1 and A_2 have index k, then $A_1^* A_2$ has index zero. If $A_1^* A_2$ can be connected to the identity by γ_1, then $A_1\gamma_1$ connects A_1 with $A_1 A_1^* A_2$. On the other hand, if the operator $A_1 A_1^*$, which is also of index zero, can be connected to the identity using γ_2, then $\gamma_2 A_2$ connects A_2 with $A_1 A_1^* A_2$. Hence, it suffices to show that the set of operators of index zero are path connected. In that case, $[\{0\}]A$ and $A[H]^\perp$ are isomorphic since they are finite dimensional spaces of the same dimension. Thus, for such an isomorphism I, which we extend by zero on $[\{0\}]A^\perp$, the path $[0,1] \ni t \mapsto A + tI \in L(H)$ connects A with an isomorphism (cf., Observation 5.11 below) and $GL(H)$ is known to be path connected (even more so, Kuiper's Theorem states that $GL(H)$ is contractible).

■

Observation 5.11. *Let H_1 and H_2 be Hilbert spaces, $GL(H_1,H_2)$ the set of isomorphisms mapping H_1 to H_2, and $F_k(H_1,H_2)$ the set of Fredholm operators of index $k\in\mathbb{Z}$ mapping H_1 to H_2. Then, $GL(H_1,H_2)$ is dense in $F_0(H_1,H_2)$.*

Proof. Obviously $GL(H_1,H_2)\subseteq F_0(H_1,H_2)$ holds since every isomorphism is bijective. Let $A\in F_0(H_1,H_2)$. Then, $\dim[\{0\}]A=\dim A[H_1]^\perp\in\mathbb{N}_0$ holds, i.e., $[\{0\}]A$ and $A[H_1]^\perp$ are isomorphic. Let I be such an isomorphism, decompose $H_1=[\{0\}]A^\perp\oplus[\{0\}]A$, and define for $t\in[0,1]$

$$A_t:\ [\{0\}]A^\perp\oplus[\{0\}]A\to A[H_1]\oplus A[H_1]^\perp;\ x+y\mapsto Ax+tIy$$

which, by definition, is bijective for $t>0$ (note that $A|_{[\{0\}]A^\perp}$ is injective and surjective on to $A[H_1]$). Thus, observing $A_t\to A$ $(t\searrow 0)$ completes the proof.

□

Observation 5.12. *Let X and Y be Banach spaces, $S,T\in L(X,Y)$, S bijective, and T compact. Then, $S+T$ is a Fredholm operator of index zero.*

In particular, the following are equivalent.

(i) *$S+T$ is injective.*
(ii) *$S+T$ is surjective.*
(iii) *$S+T$ is bijective.*

Proof. Since S is bounded and bijective, S is Fredholm of index zero. Hence, Proposition 5.9 implies that $S+T$ is a Fredholm operator of index zero, as well.

In particular, we have $\dim[\{0\}](S+T)=\operatorname{codim}(S+T)[X]$, i.e., (i)⇔(ii), which implies (i)⇒(ii)⇒(i)∧(ii)⇒(iii)⇒(i).

□

Part 2

The Navier-Stokes Equations

CHAPTER 6

Modeling Navier-Stokes

From now on, we will require M to satisfy the Rellich-Kondrachov condition and .

Definition 6.1 (Rellich-Kondrachov condition). *Let $(\tilde{M},\tilde{g})$ be a finite dimensional Riemannian $C^{1,1}$-manifold. Then, we say $(\tilde{M},\tilde{g})$ satisfies the Rellich-Kondrachov condition if and only if* $\forall q \in [1, \dim \tilde{M})\ \forall \alpha \in \mathbb{N}_0\ \forall p \in \left[1, \frac{q \dim \tilde{M}}{\dim \tilde{M} - q}\right)$:

$$W_q^{1,(\alpha,0)}(\tilde{M}) \hookrightarrow_{\text{compact}} L_p^{(\alpha,0)}(\tilde{M}).$$

We will start modeling the Navier-Stokes equations on $[0,\tau]\times M$ with the mass flow ϱu where ϱ is the density of the fluid and u the velocity field. Until we identify the Hilbert spaces, we will assume that all functions are sufficiently smooth. For $V \subseteq M$ open with smooth boundary, we obtain by the Gauss divergence theorem

$$\forall t \in [0,\tau]: \int_V \operatorname{tr} \nabla(\varrho u(t)) d\mathrm{vol}_M = \int_{\partial V} \langle \varrho u(t), \nu \rangle d\mathrm{vol}_{\partial V}.$$

But, since the right-hand side is nothing else than the mass transported out of V, we observe

$$\forall t \in [0,\tau]: \int_V \operatorname{tr} \nabla(\varrho(t) u(t)) d\mathrm{vol}_M = \int_{\partial V} \langle \varrho u(t), \nu \rangle d\mathrm{vol}_{\partial V} = -\partial_t \int_V \varrho(t) d\mathrm{vol}_M$$

and, since this holds for ever $V \subseteq M$ with smooth boundary,

$$\forall t \in [0,\tau]: \operatorname{tr} \nabla(\varrho(t)u(t)) = -\partial_t \varrho(t).$$

We want to consider fluids only, that is, an incompressible medium, i.e., $\partial_t \varrho = 0$. Hence, this last equation yields the continuity equation

$$\text{(continuity)} \qquad \operatorname{tr} \nabla u = 0.$$

Next, we will have a look at the stress term. Let $\nabla^{\mathrm{sym}} := \mathrm{sym}\nabla$ be the symmetrized[1] co-variant derivative on $(1,0)$-tensors and η, ζ positive, bounded, and bounded from below (these are the two scalar dynamic viscosities in hydrodynamics). A fluid is called isotropic if and only if the viscous stress tensor

$$\sigma := 2\eta \nabla^{\mathrm{sym}} u + \zeta \operatorname{tr}^* \operatorname{tr} \nabla u$$

satisfies

$$\operatorname{tr} \sigma = 0.$$

In this case, we call η the shear viscosity and $\zeta + \frac{2\eta}{\dim M}$ the bulk viscosity. Using the continuity equation, we obtain

$$\sigma = 2\eta \nabla^{\mathrm{sym}} u$$

which we generalize to the non-Newtonian case

$$\sigma = C \nabla^{\mathrm{sym}} u$$

where C is a viscosity operator.

[1] Let τ be a $(2,0)$-tensor. Then, $\mathrm{sym}T(x,y) := \frac{1}{2}(T(x,y) + T(y,x)) = \frac{T_{ij}+T_{ji}}{2} g^i \otimes g^j$.

Definition 6.2 (Viscosity Operator). *Let* $C \in L\left(L_2\left([0,\tau]; L_2^{(2,0)}(M)\right)\right)$ *be a positive operator that furthermore satisfies*

(i) *C is an isomorphism on the symmetric tensor fields, i.e.,*

$$0 \in \varrho\left(C\Big|_{L_2\left([0,\tau];\mathrm{sym}\left[L_2^{(2,0)}(M)\right]\right)}^{L_2\left([0,\tau];\mathrm{sym}\left[L_2^{(2,0)}(M)\right]\right)}\right).$$

To simplify notation, let C^{-1} *denote* $\left(C\Big|_{L_2\left([0,\tau];\mathrm{sym}\left[L_2^{(2,0)}(M)\right]\right)}^{L_2\left([0,\tau];\mathrm{sym}\left[L_2^{(2,0)}(M)\right]\right)}\right)^{-1}$.

(ii) *C vanishes on anti-symmetric tensor fields, i.e.,*

$$N(C) = [\{0\}]C = L_2\left([0,\tau]; \mathrm{asym}\left[L_2^{(2,0)}(M)\right]\right).$$

(iii) *C and C^{-1} preserve differentiability classes, i.e.,*

$$x \in W_2^k\left([0,\tau]; \mathrm{sym}\left[W_2^{k',(2,0)}(M)\right]\right)$$

implies

$$Cx, C^{-1}x \in W_2^k\left([0,\tau]; \mathrm{sym}\left[W_2^{k',(2,0)}(M)\right]\right).$$

(iv) *C is a (timely) causal operator, i.e.,*

$$\forall x \in L_2\left([0,\tau]; \mathrm{sym}\left[L_2^{(2,0)}(M)\right]\right): \ \inf \mathrm{spt}_0\, x \le \inf \mathrm{spt}_0\, Cx$$

where spt_0 *denotes the support in* $L_2([0,\tau])$.[2] *In other words, if x is zero on some interval $[0,\tau']$ then so is Cx; viz., the viscosity of the fluid does not depend on the future.*

(v) $\mathrm{tr}\, C^{-1}\, \mathrm{tr}^*$ *is boundedly invertible.*

Remark The "classical" Navier-Stokes problem, cf. [**5**],

$$\partial_t u + \langle u, \nabla\rangle u = \nu\Delta u - \nabla p + f, \ \mathrm{div}\, u = 0, \ u(0) = u_0$$

can be retrieved choosing the viscosity operator $C = 2\nu\mathrm{sym}$.

■

In order to obtain the entire stress tensor, we will have to take the pressure p into account. The stress tensor T is, then, defined as

$$T := \sigma - \mathrm{tr}^*\, p.$$

Observation 6.3. *Let* $t \in L_2^{(2,0)}(M)$ *be anti-symmetric. Then,* $\mathrm{tr}\, t = 0$.

Proof. Let $t \in L_2^{(2,0)}(M)$ be anti-symmetric, i.e., $t = \frac{t_{ij}-t_{ji}}{2} g^i \otimes g^j$. Then,

$$\begin{aligned}\mathrm{tr}\, t = &\mathrm{tr}\left(\frac{1}{2}\left(t_{ij}g^i \otimes g^j - t_{ji}g^i \otimes g^j\right)\right)\\ =&\frac{1}{2}\left(t_{ij}g^{ij} - t_{ji}g^{ij}\right)\\ =&\frac{1}{2}\left(t_{ij}g^{ij} - t_{ij}g^{ji}\right)\\ =&\frac{1}{2}\left(t_{ij}g^{ij} - t_{ij}g^{ij}\right)\\ =&0\end{aligned}$$

holds.

□

[2] Mind that $L_2([0,\tau]; H) = L_2([0,\tau]) \otimes H$ holds for every Hilbert space H.

Thus, we also obtain the stress equation

$$\text{(continuity)} \qquad \operatorname{tr} \nabla^{\text{sym}} u = 0$$
$$\text{(stress)} \qquad C^{-1} T - \nabla^{\text{sym}} u + C^{-1} \operatorname{tr}^* p = 0$$

Finally, we add the initial condition $u(0) = u_0$ and Cauchy's momentum equation (which is Newton's law of motion written down for fluids)

$$\varrho \left(\partial_t u + \langle u, \nabla \rangle u \right) = \operatorname{tr} \nabla T + f$$

using $\langle u, \nabla \rangle u := u_i g^{ij} \nabla_{g_j} u = \nabla_u u$ where f is an external force. Without loss of generality, we may assume $\varrho = 1$ since we can replace C, p, and f by $\frac{1}{\varrho} C$, $\frac{1}{\varrho} p$, and $\frac{1}{\varrho} f$, respectively.

Now, the classical Navier-Stokes system is

$$\begin{aligned}
&\text{(continuity)} & \operatorname{tr} \nabla^{\text{sym}} u &= 0 \text{ in } (0,\tau) \times M, \\
&\text{(stress)} & C^{-1} T - \nabla^{\text{sym}} u + C^{-1} \operatorname{tr}^* p &= 0 \text{ in } (0,\tau) \times M, \\
&\text{(Cauchy)} & \partial_t u + \langle u, \nabla \rangle u - \operatorname{tr} \nabla T &= f \text{ in } (0,\tau) \times M, \\
&\text{(initial condition)} & u(0) &= u_0 \text{ in } M
\end{aligned}$$

where u is the velocity field, T the stress tensor (symmetric), p the pressure, C a viscosity operator, and f an external force. The objective is to find reasonable conditions for all these symbols to be physically senseful and interpretable in an L_2-sense.

First, let us observe for $\varphi \in \mathfrak{M}_1^{(0,0)}(M)$

$$\begin{aligned}
\operatorname{tr} \nabla \operatorname{tr}^* \varphi &= \operatorname{tr} \nabla \left(\varphi g_{jk} g^j \otimes g^k \right) \\
&= \operatorname{tr} \left(\nabla_{g_i} \varphi g_{jk} g^i \otimes g^j \otimes g^k \right) \\
&= \nabla_{g_i} \varphi g_{jk} g^{ij} g^k \\
&= \nabla_{g_k} \varphi g^k \\
&= \nabla \varphi
\end{aligned}$$

and, therefrom, for $u \in \mathfrak{M}_1^{(1,0)}(M)$

$$\begin{aligned}
\langle u, \nabla \rangle u &= u_i g^{ij} \nabla_{g_j} u_k g^k \\
&= u_i g^{ij} \left(\nabla_{g_j} u_k g^k + \nabla_{g_k} u_j g^k \right) - u_i g^{ij} \nabla_{g_k} u_j g^k \\
&= 2 \operatorname{tr} \left(u \otimes \operatorname{sym} \nabla u \right) - \frac{1}{2} \left(u_i g^{ij} \nabla_{g_k} u_j g^k + \nabla_{g_k} u_i g^{ij} u_j g^k \right) \\
&= 2 \operatorname{tr} \left(u \otimes \operatorname{sym} \nabla u \right) - \frac{1}{2} \nabla \langle u, u \rangle_{(1,0)} \\
&= 2 \operatorname{tr} \left(u \otimes \operatorname{sym} \nabla u \right) - \frac{1}{2} \operatorname{tr} \nabla \operatorname{tr}^* \langle u, u \rangle_{(1,0)}.
\end{aligned}$$

Defining

$$\tilde{p} := p - \frac{1}{2} \langle u, u \rangle_{(1,0)}$$

and

$$\tilde{T} := T + \operatorname{tr}^* \frac{1}{2} \langle u, u \rangle_{(1,0)} = C \nabla^{\text{sym}} u - \operatorname{tr}^* p + \operatorname{tr}^* \frac{1}{2} \langle u, u \rangle_{(1,0)} = C \nabla^{\text{sym}} u - \operatorname{tr}^* \tilde{p},$$

we observe

$$\begin{aligned}
\langle u, \nabla \rangle u - \operatorname{tr} \nabla T &= 2 \operatorname{tr} \left(u \otimes \nabla^{\text{sym}} u \right) - \frac{1}{2} \operatorname{tr} \nabla \operatorname{tr}^* \langle u, u \rangle_{(1,0)} - \operatorname{tr} \nabla \tilde{T} - \operatorname{tr} \nabla \operatorname{tr}^* \frac{1}{2} \langle u, u \rangle_{(1,0)} \\
&= 2 \operatorname{tr} \left(u \otimes \nabla^{\text{sym}} u \right) - \operatorname{tr} \nabla \tilde{T}
\end{aligned}$$

$$=2\operatorname{tr}\left(u\otimes\left(C^{-1}T+C^{-1}\operatorname{tr}^{*}p\right)\right)-\operatorname{tr}\nabla\tilde{T}$$
$$=2\operatorname{tr}\left(u\otimes\left(C^{-1}\tilde{T}+C^{-1}\operatorname{tr}^{*}\tilde{p}\right)\right)-\operatorname{tr}\nabla\tilde{T}.$$

Hence, Cauchy's momentum equation reduces to

$$\partial_t u-\operatorname{tr}\nabla\tilde{T}=f-2\operatorname{tr}\left(u\otimes\left(C^{-1}\tilde{T}+C^{-1}\operatorname{tr}^{*}\tilde{p}\right)\right)$$

and the model becomes

$$\begin{aligned}\operatorname{tr}\nabla^{\mathrm{sym}}u&=0\text{ in }(0,\tau)\times M,\\ C^{-1}T-\nabla^{\mathrm{sym}}u+C^{-1}\operatorname{tr}^{*}p&=0\text{ in }(0,\tau)\times M,\\ \partial_t u-\operatorname{tr}\nabla\tilde{T}&=f-2\operatorname{tr}\left(u\otimes\left(C^{-1}\tilde{T}+C^{-1}\operatorname{tr}^{*}\tilde{p}\right)\right)\text{ in }(0,\tau)\times M,\\ u(0)&=u_0\text{ in }M\end{aligned}$$

which, omitting the initial condition for the moment, is equivalent to

$$\begin{pmatrix}0 & \operatorname{tr}\operatorname{sym}^{\mathrm{sym}}\nabla & 0\\ 0 & \partial_t & -\operatorname{tr}\nabla\\ C^{-1}\operatorname{tr}^{*} & -\nabla^{\mathrm{sym}} & C^{-1}\end{pmatrix}\begin{pmatrix}\tilde{p}\\ u\\ \tilde{T}\end{pmatrix}=\begin{pmatrix}0\\ f-2\operatorname{tr}\left(u\otimes C^{-1}\left(\tilde{T}+\operatorname{tr}^{*}\tilde{p}\right)\right)\\ 0\end{pmatrix}$$

and, hence, equivalent to

$$\begin{pmatrix}\operatorname{tr}C^{-1}\operatorname{tr}^{*} & 0 & \operatorname{tr}C^{-1}\\ 0 & \partial_t & -\operatorname{tr}\nabla\\ C^{-1}\operatorname{tr}^{*} & -\nabla^{\mathrm{sym}} & C^{-1}\end{pmatrix}\begin{pmatrix}\tilde{p}\\ u\\ \tilde{T}\end{pmatrix}=\begin{pmatrix}0\\ f-2\operatorname{tr}\left(u\otimes C^{-1}\left(\tilde{T}+\operatorname{tr}^{*}\tilde{p}\right)\right)\\ 0\end{pmatrix},$$

as well. Using the (non-unitary) transformation

$$U:=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -C^{-1}\operatorname{tr}^{*}\left(\operatorname{tr}C^{-1}\operatorname{tr}^{*}\right)^{-1} & 0 & 1\end{pmatrix}$$

with

$$U^{*}=\begin{pmatrix}1 & 0 & -\left(\operatorname{tr}C^{-1}\operatorname{tr}^{*}\right)^{-1}\operatorname{tr}C^{-1}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\left(U^{*}\right)^{-1}=\begin{pmatrix}1 & 0 & \left(\operatorname{tr}C^{-1}\operatorname{tr}^{*}\right)^{-1}\operatorname{tr}C^{-1}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\begin{aligned}&U\begin{pmatrix}\operatorname{tr}C^{-1}\operatorname{tr}^{*} & 0 & \operatorname{tr}C^{-1}\\ 0 & \partial_t & -\operatorname{tr}\nabla\\ C^{-1}\operatorname{tr}^{*} & -\nabla^{\mathrm{sym}} & C^{-1}\end{pmatrix}U^{*}\\ &=\begin{pmatrix}\operatorname{tr}C^{-1}\operatorname{tr}^{*} & 0 & 0\\ 0 & \partial_t & -\operatorname{tr}\nabla\\ 0 & -\nabla^{\mathrm{sym}} & C^{-1}-C^{-1}\operatorname{tr}^{*}\left(\operatorname{tr}C^{-1}\operatorname{tr}^{*}\right)^{-1}\operatorname{tr}C^{-1}\end{pmatrix},\end{aligned}$$

and

$$\left(U^{*}\right)^{-1}\begin{pmatrix}\tilde{p}\\ u\\ \tilde{T}\end{pmatrix}=\begin{pmatrix}\tilde{p}+\left(\operatorname{tr}C^{-1}\operatorname{tr}^{*}\right)^{-1}\operatorname{tr}C^{-1}\tilde{T}\\ u\\ \tilde{T}\end{pmatrix}$$

yields

$$U\begin{pmatrix}\operatorname{tr}C^{-1}\operatorname{tr}^{*} & 0 & \operatorname{tr}C^{-1}\\ 0 & \partial_t & -\operatorname{tr}\nabla\\ C^{-1}\operatorname{tr}^{*} & -\nabla^{\mathrm{sym}} & C^{-1}\end{pmatrix}U^{*}\left(U^{*}\right)^{-1}\begin{pmatrix}\tilde{p}\\ u\\ \tilde{T}\end{pmatrix}=U\begin{pmatrix}0\\ f-2\operatorname{tr}\left(u\otimes C^{-1}\left(\tilde{T}+\operatorname{tr}^{*}\tilde{p}\right)\right)\\ 0\end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ f - 2\operatorname{tr}\left(u \otimes C^{-1}\left(\tilde{T} + \operatorname{tr}^* \tilde{p}\right)\right) \\ 0 \end{pmatrix}$$

allowing us to further reduce the system, since

$$\tilde{p} = -\left(\operatorname{tr} C^{-1} \operatorname{tr}^*\right)^{-1} \operatorname{tr} C^{-1}\tilde{T}$$

decouples, yielding

$$\begin{pmatrix} \partial_t & -\operatorname{tr}\nabla \\ -\nabla^{\mathrm{sym}} & C^{-1} - C^{-1}\operatorname{tr}^*\left(\operatorname{tr} C^{-1}\operatorname{tr}^*\right)^{-1}\operatorname{tr} C^{-1} \end{pmatrix}\begin{pmatrix} u \\ \tilde{T} \end{pmatrix}$$
$$= \begin{pmatrix} f - 2\operatorname{tr}\left(u \otimes \left(1 - C^{-1}\operatorname{tr}^*\left(\operatorname{tr} C^{-1}\operatorname{tr}^*\right)^{-1}\operatorname{tr}\right) C^{-1}\tilde{T}\right) \\ 0 \end{pmatrix}.$$

Let

$$\mathbb{E} := 1 - C^{-\frac{1}{2}}\operatorname{tr}^*\left(\operatorname{tr} C^{-1}\operatorname{tr}^*\right)^{-1}\operatorname{tr} C^{-\frac{1}{2}}$$

and

$$\Theta := C^{-\frac{1}{2}}\tilde{T}.$$

Then, we observe

$$(1-\mathbb{E})^2 = C^{-\frac{1}{2}}\operatorname{tr}^*\left(\operatorname{tr} C^{-1}\operatorname{tr}^*\right)^{-1}\operatorname{tr} C^{-1}\operatorname{tr}^*\left(\operatorname{tr} C^{-1}\operatorname{tr}^*\right)^{-1}\operatorname{tr} C^{-\frac{1}{2}} = 1 - \mathbb{E}$$

and

$$\begin{aligned} \mathbb{E}^2 &= 1 - 2C^{-\frac{1}{2}}\operatorname{tr}^*\left(\operatorname{tr} C^{-1}\operatorname{tr}^*\right)^{-1}\operatorname{tr} C^{-\frac{1}{2}} + \left(C^{-\frac{1}{2}}\operatorname{tr}^*\left(\operatorname{tr} C^{-1}\operatorname{tr}^*\right)^{-1}\operatorname{tr} C^{-\frac{1}{2}}\right)^2 \\ &= 1 - 2(1-\mathbb{E}) + (1-\mathbb{E})^2 \\ &= \mathbb{E}. \end{aligned}$$

Hence, $\mathbb{E}$ is an orthogonal projection onto $[\{0\}]\operatorname{tr} C^{-\frac{1}{2}}$ and self-adjoint in $L_2^{(2,0)}(M)$. Furthermore, the Navier-Stokes system becomes

$$\begin{pmatrix} \partial_t & -\operatorname{tr}\nabla \\ -\nabla^{\mathrm{sym}} & C^{-\frac{1}{2}}\mathbb{E}C^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} u \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} f - 2\operatorname{tr}\left(u \otimes C^{-\frac{1}{2}}\mathbb{E}C^{-\frac{1}{2}}\tilde{T}\right) \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} \partial_t & -\operatorname{tr}\nabla C^{\frac{1}{2}} \\ -C^{\frac{1}{2}}\nabla^{\mathrm{sym}} & \mathbb{E} \end{pmatrix}\begin{pmatrix} u \\ \Theta \end{pmatrix} = \begin{pmatrix} f - 2\operatorname{tr}\left(u \otimes C^{-\frac{1}{2}}\mathbb{E}\Theta\right) \\ 0 \end{pmatrix}.$$

Since $C^{\frac{1}{2}}$ vanishes on anti-symmetric tensors, this last equation is equivalent to

$$\begin{pmatrix} \partial_t & -\operatorname{tr}\nabla C^{\frac{1}{2}} \\ -C^{\frac{1}{2}}\nabla & \mathbb{E} \end{pmatrix}\begin{pmatrix} u \\ \Theta \end{pmatrix} = \begin{pmatrix} f - 2\operatorname{tr}\left(u \otimes C^{-\frac{1}{2}}\mathbb{E}\Theta\right) \\ 0 \end{pmatrix}.$$

However, this equation can be interpreted in an $L_2^{(\alpha,\beta)}(M)$ setting using the (partial) time derivative ∂_0 in $L_2([0,\tau])$ which yields

$$\begin{pmatrix} \partial_0 & -\operatorname{div}_{(0)} C^{\frac{1}{2}} \\ -C^{\frac{1}{2}}\operatorname{grad}_{(0)} & \mathbb{E} \end{pmatrix}\begin{pmatrix} u \\ \Theta \end{pmatrix} = \begin{pmatrix} f - 2\operatorname{tr}\left(u \otimes C^{-\frac{1}{2}}\mathbb{E}\Theta\right) \\ 0 \end{pmatrix}$$

where

$$\operatorname{grad}_{(0)} = \begin{cases} \operatorname{grad}_0 & \text{, Dirichlet case} \\ \operatorname{grad} & \text{, Neumann or no boundary case} \end{cases}$$

and

$$\operatorname{div}_{(0)} = \begin{cases} \operatorname{div}_0 & \text{, Neumann case} \\ \operatorname{div} & \text{, Dirichlet or no boundary case} \end{cases}.$$

Recall that for all $t \in [0, \tau]$

$$u(t) \in N(\operatorname{tr} \nabla) = [\{0\}] \operatorname{tr} \nabla$$

shall hold and, hence, $C^{\frac{1}{2}} \nabla u$ takes values in $[\{0\}] \operatorname{tr} C^{-\frac{1}{2}}$ which is nothing else than the range of $\mathbb{E}$. Let

$$Y : [\{0\}] \operatorname{tr} \operatorname{grad} \subseteq \overline{[\{0\}] \operatorname{tr} \operatorname{grad}} \to [\{0\}] \operatorname{tr} C^{-\frac{1}{2}};\ x \mapsto -C^{\frac{1}{2}} \operatorname{grad} x$$

in case of no boundary or Neumann boundary conditions[3]. In case of Dirichlet boundary conditions let

$$Y : [\{0\}] \operatorname{tr} \operatorname{grad}_0 \subseteq \overline{[\{0\}] \operatorname{tr} \operatorname{grad}_0} \to [\{0\}] \operatorname{tr} C^{-\frac{1}{2}};\ x \mapsto -C^{\frac{1}{2}} \operatorname{grad}_0 x.$$

Then

$$Y = -\operatorname{pr}_{[\{0\}] \operatorname{tr} C^{-\frac{1}{2}}} C^{\frac{1}{2}} \operatorname{grad}_{(0)} \operatorname{pr}^*_{\overline{[\{0\}] \operatorname{tr} \operatorname{grad}_{(0)}}}$$

embedded in $L_2^{(1,0)}(M) \oplus L_2^{(2,0)}(M)$ and

$$Y^* = \operatorname{pr}_{\overline{[\{0\}] \operatorname{tr} \operatorname{grad}_{(0)}}} \operatorname{div}_{(0)} C^{\frac{1}{2}} \operatorname{pr}^*_{[\{0\}] \operatorname{tr} C^{-\frac{1}{2}}}$$

embedded in $L_2^{(2,0)}(M) \oplus L_2^{(1,0)}(M)$.

Remark Let $\Gamma \subseteq \partial M$ be Borel measurable,

$$D(a) := \left\{ u \in W_2^1(M);\ u|_\Gamma = 0 \right\},$$

and

$$a : D(a) \times D(a) \to \mathbb{R};\ (u, v) \mapsto \langle C \operatorname{grad} u, \operatorname{grad} v \rangle_{L_2^{(1,0)}(M)}.$$

Then, a generates a positive operator $Y^* Y$ which can be considered as a realization of the mixed boundary condition "Dirichlet on Γ and Neumann on $\partial M \setminus \Gamma$". Similarly, other boundary conditions can be introduced and the following would be virtually the same up to a few subtle changes which we will not address any further. ∎

Hence, the system reduces to

$$\begin{pmatrix} \partial_0 & -Y^* \\ Y & \mathbb{E} \end{pmatrix} \begin{pmatrix} u \\ \Theta \end{pmatrix} = \begin{pmatrix} f - 2 \operatorname{tr} \left(u \otimes C^{-\frac{1}{2}} \mathbb{E} \Theta \right) \\ 0 \end{pmatrix}$$

which yields

$$\partial_0 u + Y^* Y u - Y^* (1 - \mathbb{E}) \Theta = f + 2 \operatorname{tr} \left(u \otimes C^{-\frac{1}{2}} Y u \right).$$

From $Y^* = \operatorname{pr}_{\overline{[\{0\}] \operatorname{tr} \operatorname{grad}_{(0)}}} \operatorname{div}_{(0)} C^{\frac{1}{2}} \operatorname{pr}^*_{[\{0\}] \operatorname{tr} C^{-\frac{1}{2}}}$ and, since $\left([\{0\}] \operatorname{tr} C^{-\frac{1}{2}} \right)^\perp$ is the range of $(1 - \mathbb{E})$ (recall that $\mathbb{E}$ is an orthogonal projection), we directly deduce $Y^*(1 - \mathbb{E}) = 0$ and, therefore,

$$\partial_0 u + Y^* Y u = f + 2 \operatorname{tr} \left(u \otimes C^{-\frac{1}{2}} Y u \right).$$

[3]Note that these Neumann boundary conditions do not have vanishing normal derivative but con-normal derivative $\langle C \operatorname{grad} u, \nu \rangle = 0$ where ν is the exterior normal on the boundary. Furthermore, this is a generalization of vanishing con-normal derivative which only makes sense if the boundary is sufficiently smooth because for general boundary there is no reason why the trace on ∂M of $C \operatorname{grad} u$ should even exist due to the partial derivatives occurring.

Since the left-hand side takes values in $\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}$, so the right-hand side has to. Thus, we may state the system as

$$\partial_0 u + Y^*Yu - 2\,\mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\mathrm{tr}\left(u \otimes C^{-\frac{1}{2}}Yu\right) = f$$

where f takes values in $H := \mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\left[L_2^{(1,0)}(M)\right]$.

Remark Note that the operator Y^*Y is densely defined due to a theorem by von Neumann.

THEOREM 6.4 (von Neumann). *Let H_1 and H_2 be Hilbert spaces and $A \subseteq H_1 \oplus H_2$ a closed and densely defined operator. Then A^*A is self-adjoint in H_1 and its domain is a core of A.*

This theorem can be obtained from the first representation theorem (Theorem VI.2.1 in [**11**]) applied to the closed, positive, symmetric form τ with $D(\tau) := D(A)$ and $\forall x, y \in D(\tau):\ \tau(x,y) := \langle Ax, Ay\rangle_{H_2}$.

■

We define the space of maximal regularity

$$\mathfrak{MR}_\tau := W_2^1([0,\tau];H) \cap L_2\left([0,\tau]; D\left(Y^*Y\right)\right)$$

endowed with the norm

$$\|\cdot\|_{\mathfrak{MR}_\tau} :\ \mathfrak{MR}_\tau \to \mathbb{R};\ x \mapsto \left(\|x\|^2_{W_2^1([0,\tau];H)} + \|x\|^2_{L_2([0,\tau];D(Y^*Y))}\right)^{\frac{1}{2}}.$$

Then, the Sobolev Embedding Theorem (Theorem 3.15) yields

$$\mathfrak{MR}_\tau \hookrightarrow_{\text{continuous}} C([0,\tau];H).$$

The embedding $\mathfrak{MR}_\tau \hookrightarrow_{\text{continuous}} C([0,\tau];H)$ is, in fact, compact as can be shown using Arzelà-Ascoli's theorem. We, on the other hand, only need continuity since this ensures that

$$\mathfrak{MR}_{\tau,0} := \{x \in \mathfrak{MR}_\tau;\ x(0) = 0\}$$

and

$$\mathfrak{TR} := \mathfrak{MR}_\tau/_{\mathfrak{MR}_{\tau,0}} \cong \{x(0) \in H;\ x \in \mathfrak{MR}_\tau\}$$

are well-defined Hilbert spaces.

As of now, we have identified the abstract Cauchy problem we would like to consider and the spaces the equation should hold in. The only thing in question is whether the non-linearity behaves nicely. This is where we need the Rellich-Kondrachov condition. The Rellich-Kondrachov condition implies

$$\forall \alpha \in \mathbb{N}_0:\ W_2^{1,(\alpha,0)}(M) \subseteq L_4^{(\alpha,0)}(M)$$

which combined with $Y\left[D\left(Y^*Y\right)\right] \subseteq W_2^{1,(2,0)}(M)$ yields

$$\forall u \in \mathfrak{MR}_\tau:\ \mathrm{tr}\left(u \otimes C^{-\frac{1}{2}}Yu\right) \in L_2\left([0,\tau]; L_2^{(1,0)}(M)\right).$$

Now, we may actually state the Navier-Stokes problem we want to address.

PROBLEM 6.5 (Navier-Stokes). *Let $\tau \in \mathbb{R}_{>0}$, $f \in L_2([0,\tau];H)$ and $u_0 \in \mathfrak{TR}$. Find $u \in \mathfrak{MR}_\tau$ such that*

$$\begin{aligned} \partial_0 u + Y^*Yu - 2\,\mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\mathrm{tr}\left(u \otimes C^{-\frac{1}{2}}Yu\right) &= f \quad \text{in } (0,\tau)\times M, \\ u(0) &= u_0 \quad \text{in } M \end{aligned} \tag{Navier-Stokes}$$

holds.

CHAPTER 7

Construction of Solutions and Analytic Dependence

In order to solve the Navier-Stokes problem, let us define

$$B:\ \mathfrak{MR}_\tau \to L_2([0,\tau];H);\ x \mapsto -2\,\mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\,\mathrm{tr}\left(x\otimes C^{-\frac{1}{2}}Yx\right)$$

and

$$F_\tau:\ \mathfrak{MR}_\tau \to L_2([0,\tau];H)\times\mathfrak{TR};\ x\mapsto (\partial_0 x + Y^*Yx + B(x)\ ,\ x(0)).$$

These yield the nice and short notation

$$F_\tau(u) = (f,u_0)$$

for the Navier-Stokes equations. F_τ will, thus, be called the Navier-Stokes operator and our objective is to continuously invert F_τ locally in time and show that the inverse is an analytic operator on the reduced time interval.

Note that F_τ is an analytic operator since it is a polynomial of degree two and for $u,v\in\mathfrak{MR}_\tau$ we observe

$$F_\tau'(v)u = (d_0u + Y^*Yu + B'(v)u\ ,\ u(0))\,.$$

Hence, if we can find a $v\in\mathfrak{MR}_\tau$ such that $F_\tau'(v)$ is an isomorphism and our data (f,u_0) are sufficiently close to $F_\tau(v)$, then the analytic inverse function theorem yields existence and constructibility of solutions and analytic dependence on the data. We are going to achieve this by defining

$$g_{\tau',v} := f1_{[0,\tau']} + (\partial_0 v + Y^*Yv + B(v))\,1_{(\tau',\tau]}$$

for $\tau'\in(0,\tau)$ and $v\in\mathfrak{MR}_\tau$. Then we observe for $v\in\mathfrak{MR}_\tau$ with $v(0)=u_0$

$$\begin{aligned}&\|(g_{\tau',v},u_0) - F_\tau(v)\|^2_{L_2([0,\tau];H)\times\mathfrak{TR}}\\ &= \int_0^{\tau'} \|f(s) - \partial_0 v(s) - Y^*Yv(s) - B(v)(s)\|_H^2\,ds \searrow 0\end{aligned}$$

as $\tau'\searrow 0$, i.e., for τ' sufficiently small, we can solve a slightly Navier-Stokes system and τ' is even locally constant. In order for us to know that this solution with respect to $(g_{\tau',v},u_0)$ also solves the Navier-Stokes system with respect to (f,u_0) on $[0,\tau']$, we need to make sure that the solution on $[0,\tau']$ does not depend on the data on $(\tau',\tau]$. Hence, our to-do-list is:

- Find $v\in\mathfrak{MR}_\tau$ with $v(0)=u_0$ and $F_\tau'(v)$ being an isomorphism; in fact, we are going to show that $F_\tau'(v)$ is always an isomorphism and the existence of a v with $v(0)=u_0$ is trivial by definition of $\mathfrak{TR}$.
- Show injectivity of F_τ.
- Show causality of solutions.

CHAPTER 8

Linearized Navier-Stokes

This chapter is devoted to showing that $F_\tau'(v)$ is an isomorphism but, lucky us, will also yield injectivity of F_τ as a corollary. In order to show that $F_\tau'(v)$ is an isomorphism, we will prove that the Stokes operator[1]

$$I:\ \mathfrak{MR}_\tau \to L_2([0,\tau];H)\times\mathfrak{TR};\ x\mapsto(\partial_0 x+Y^*Yx\ ,\ x(0))$$

is an isomorphism, first, and then we will consider the perturbation

$$\mathcal{B}_v:\ \mathfrak{MR}_\tau \to L_2([0,\tau];H)\times\mathfrak{TR};\ x\mapsto(B'(v)x\ ,\ 0).$$

Lemma 8.1. *Let $\lambda\in\mathbb{R}_{>0}$. Then*

$$(\partial_0+Y^*Y+\lambda):\ \mathfrak{MR}_{\tau,0}\to L_2([0,\tau];H)$$

is an isomorphism.

Furthermore, if $\lambda\geq\lambda_0\in\mathbb{R}_{>0}$ is uniformly bounded away from zero then

$$\left\|(\partial_0+Y^*Y+\lambda)^{-1}\right\|_{L(L_2([0,\tau];H),\mathfrak{MR}_{\tau,0})}\ \text{and}\ \left\|(\partial_0+Y^*Y+\lambda)^{-1}\right\|_{L(L_2([0,\tau];H))}$$

are uniformly bounded (varying λ).

Proof. Since

$$Y^*Y:\ D(Y^*Y)\subseteq \mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\left[W_2^{1,(1,0)}(M)\right]\to H$$

is self-adjoint and non-negative, the spectral theorem warrants the existence of a measure space $(\Omega,\mathcal{A},\mu)$ and $a:\ \Omega\to\mathbb{R}_{\geq 0}$ measurable such that Y^*Y is unitarily equivalent to

$$a(m):\ D(a(m))\subseteq L_2(\mu)\to L_2(\mu);\ f\mapsto(\Omega\ni x\mapsto a(x)f(x)\in\mathbb{R}_{\geq 0})$$

with

$$D(a(m)):=\{f\in L_2(\mu);\ (\Omega\ni x\mapsto a(x)f(x)\in\mathbb{R}_{\geq 0})\in L_2(\mu)\}.$$

Without loss of generality, we may, hence, assume that $Y^*Y=a(m)$.

Let $f\in L_2([0,\tau];\ L_2(\mu))$ and $\tilde{f}$ be a representative. For $(t,x)\in[0,\tau]\times\Omega$ with $\tilde{f}(\cdot,x)\in L_1([0,\tau])$ define

$$S_\lambda f(t,x):=\int_0^t e^{(a(x)+\lambda)(s-t)}\tilde{f}(s,x)ds.$$

Note that S_λ is unitarily equivalent to $(\partial_0+Y^*Y+\lambda)^{-1}$. Let $b:\ \Omega\to\mathbb{R}_{\geq 0}$ be measurable and $c\in\mathbb{R}_{>0}$ such that for μ-almost every $x\in\Omega$

$$0\leq\frac{b(x)}{a(x)+\lambda}\leq c$$

[1]The name "Stokes operator" is ambiguous here. To be precise, the operator Y^*Y should be called Stokes operator whereas I is the operator associated with the Stokes system. However, we will always refer to Y^*Y as Y^*Y and chose the name "Stokes operator" for I for reasons of brevity.

holds. Then we observe

$$
\begin{aligned}
&\|b(m)S_\lambda f\|^2_{L_2([0,\tau];L_2(\mu))}\\
&= \int_0^\tau \|b(m)S_\lambda f(t)\|^2_{L_2(\mu)}\ dt\\
&= \int_0^\tau \int_\Omega |b(x)S_\lambda f(t,x)|^2\ d\mu(x)\ dt\\
&= \int_\Omega \int_0^\tau |b(x)|^2 \left|\int_0^t e^{(a(x)+\lambda)(s-t)} f(s,x)ds\right|^2\ dt\ d\mu(x)\\
&= \int_\Omega |b(x)|^2 \left\|t \mapsto \int_0^t e^{-(a(x)+\lambda)(t-s)} f(s,x)ds\right\|^2_{L_2([0,\tau])} d\mu(x)\\
&= \int_\Omega |b(x)|^2 \left\|t \mapsto \int_0^t e^{-(a(x)+\lambda)(t-s)} 1_{[0,\tau]}(t-s)f(s,x)1_{[0,\tau]}(s)ds\right\|^2_{L_2([0,\tau])} d\mu(x)\\
&= \int_\Omega |b(x)|^2 \left\|t \mapsto \int_{\mathbb{R}} e^{-(a(x)+\lambda)(t-s)} 1_{[0,\tau]}(t-s)f(s,x)1_{[0,\tau]}(s)ds\right\|^2_{L_2([0,\tau])} d\mu(x)\\
&= \int_\Omega |b(x)|^2 \left\|\left(e^{-(a(x)+\lambda)\cdot}1_{[0,\tau]}\right) * \left(f(\cdot,x)1_{[0,\tau]}\right)\right\|^2_{L_2([0,\tau])} d\mu(x)\\
&\le \int_\Omega |b(x)|^2 \left\|\left(e^{-(a(x)+\lambda)\cdot}1_{[0,\tau]}\right) * \left(f(\cdot,x)1_{[0,\tau]}\right)\right\|^2_{L_2(\mathbb{R})} d\mu(x)\\
&\le_{\text{Young}} \int_\Omega |b(x)|^2 \left\|e^{-(a(x)+\lambda)\cdot}1_{[0,\tau]}\right\|^2_{L_1(\mathbb{R})} \left\|f(\cdot,x)1_{[0,\tau]}\right\|^2_{L_2(\mathbb{R})} d\mu(x)\\
&= \int_\Omega |b(x)|^2 \left(\int_0^\tau e^{-(a(x)+\lambda)s}ds\right)^2 \left\|f(\cdot,x)1_{[0,\tau]}\right\|^2_{L_2(\mathbb{R})} d\mu(x)\\
&= \int_\Omega |b(x)|^2 \left(\frac{e^{-(a(x)+\lambda)\tau}-1}{-(a(x)+\lambda)}\right)^2 \left\|f(\cdot,x)1_{[0,\tau]}\right\|^2_{L_2(\mathbb{R})} d\mu(x)\\
&\le \int_\Omega \left(\frac{\left(e^{-(a(x)+\lambda)\tau}-1\right)b(x)}{a(x)+\lambda}\right)^2 \left\|f(\cdot,x)1_{[0,\tau]}\right\|^2_{L_2(\mathbb{R})} d\mu(x)\\
&\le c^2 \|f\|^2_{L_2([0,\tau];L_2(\mu))}.
\end{aligned}
$$

For $b = 1$ we may choose $c = \frac{1}{\lambda}$ and for $b = a$ we may choose $c = 1$. Then we obtain

$$
\begin{aligned}
\|S_\lambda f\|_{L_2([0,\tau];D(a(m)))} &\le \|S_\lambda f\|_{L_2([0,\tau];L_2(\mu))} + \|a(m)S_\lambda f\|_{L_2([0,\tau];L_2(\mu))}\\
&\le \left(\frac{1}{\lambda}+1\right)\|f\|_{L_2([0,\tau];L_2(\mu))}.
\end{aligned}
$$

Furthermore, for f continuous, the fundamental theorem of calculus for Bochner integrals implies $S_\lambda f \in C^1([0,\tau];L_2(\mu))$ and, thus, $S_\lambda f \in W_2^1([0,\tau];L_2(\mu))$ for $f \in L_2([0,\tau];L_2(\mu))$. Hence, $\partial_0 S_\lambda f = f - (a(m)+\lambda)S_\lambda f$ implies

$$
\begin{aligned}
&\|S_\lambda f\|_{W_2^1([0,\tau];L_2(\mu))}\\
&\le \|S_\lambda f\|_{L_2([0,\tau];L_2(\mu))} + \|\partial_0 S_\lambda f\|_{L_2([0,\tau];L_2(\mu))}\\
&\le \|S_\lambda f\|_{L_2([0,\tau];L_2(\mu))} + \|f-(a(m)+\lambda)S_\lambda f\|_{L_2([0,\tau];L_2(\mu))}\\
&\le \frac{\|f\|_{L_2([0,\tau];L_2(\mu))}}{\lambda} + \|f\|_{L_2([0,\tau];L_2(\mu))} + \|(a(m)+\lambda)S_\lambda f\|_{L_2([0,\tau];L_2(\mu))}\\
&\le \left(\frac{1}{\lambda}+1+1\right)\|f\|_{L_2([0,\tau];L_2(\mu))}.
\end{aligned}
$$

□

Remark To show $S_\lambda = (\partial_0 + Y^*Y + \lambda)^{-1}$ in the sense of unitary equivalence, we observe for $f \in L_2([0,\tau], L_2(\mu))$, $g \in \mathfrak{MR}_{\tau,0}$, and $\tilde{f}$ and $\tilde{g}$ representatives

$$\begin{aligned}(\partial_0 + Y^*Y + \lambda)S_\lambda f(t,x) =& \partial_0 \int_0^t e^{(a(x)+\lambda)(s-t)}\tilde{f}(s,x)ds \\ &+ \int_0^t (a(x)+\lambda)e^{(a(x)+\lambda)(s-t)}\tilde{f}(s,x)ds \\ =& \int_0^t -(a(x)+\lambda)e^{(a(x)+\lambda)(s-t)}\tilde{f}(s,x)ds + \tilde{f}(t,x) \\ &+ \int_0^t (a(x)+\lambda)e^{(a(x)+\lambda)(s-t)}\tilde{f}(s,x)ds \\ =& \tilde{f}(t,x)\end{aligned}$$

and (using Hille's theorem[2])

$$\begin{aligned}S_\lambda(\partial_0 + Y^*Y + \lambda)g(t,x) =& \int_0^t e^{(a(x)+\lambda)(s-t)}(\partial_0 + Y^*Y + \lambda)\tilde{g}(s,x)ds \\ =& \int_0^t e^{(a(x)+\lambda)(s-t)}\partial_0\tilde{g}(s,x)ds \\ &+ \int_0^t e^{(a(x)+\lambda)(s-t)}(Y^*Y + \lambda)\tilde{g}(s,x)ds \\ =& \int_0^t \partial_0\left(e^{(a(x)+\lambda)(s-t)}\tilde{g}(s,x)\right)ds \\ &- \int_0^t \partial_0 e^{(a(x)+\lambda)(s-t)}\tilde{g}(s,x)ds \\ &+ (Y^*Y + \lambda)\int_0^t e^{(a(x)+\lambda)(s-t)}\tilde{g}(s,x)ds \\ =& \int_0^t \partial_0\left(e^{(a(x)+\lambda)(s-t)}\tilde{g}(s,x)\right)ds \\ &- (Y^*Y + \lambda)\int_0^t e^{(a(x)+\lambda)(s-t)}\tilde{g}(s,x)ds \\ &+ (Y^*Y + \lambda)\int_0^t e^{(a(x)+\lambda)(s-t)}\tilde{g}(s,x)ds \\ =& \tilde{g}(t,x) - e^{-(a(x)+\lambda)t}\underbrace{\tilde{g}(0,x)}_{=0} \\ =& \tilde{g}(t,x)\end{aligned}$$

almost everywhere. Hence, $S_\lambda = (\partial_0 + Y^*Y + \lambda)^{-1}$ in the sense of unitary equivalence.

■

PROPOSITION 8.3. *Let $B_0 \in L(\mathfrak{MR}_\tau, L_2([0,\tau];H))$ with*

$$\forall \lambda \in \mathbb{R}_{>0} : \; B_0 e^{\lambda m_0} = e^{\lambda m_0} B_0$$

(m_0 is the multiplication operator with the "time" argument, i.e., in $L_2([0,\tau])$, with maximal domain) and $\forall \alpha \in (0,1)\ \exists C_\alpha \in \mathbb{R}_{>0}\ \forall u \in \mathfrak{MR}_\tau$:

$$\|B_0 u\|_{L_2([0,\tau];H)} \le C_\alpha \|u\|_{L_2([0,\tau];H)} + \alpha \|u\|_{\mathfrak{MR}_\tau}.$$

2

THEOREM 8.2 (Hille). *Let $I \subseteq \mathbb{R}$ be an interval, X and Y Banach spaces, $A \subseteq X \oplus Y$ a closed linear operator, $f : I \to X$ Bochner-integrable, $\forall t \in I : f(t) \in D(A)$, and $t \mapsto Af(t)$ Bochner-integrable. Then, $\int_I f(t)dt \in D(A)$ and $A\int_I f(t)dt = \int_I Af(t)dt$ holds.*

PROOF. see **[10]** □

Then

$$J:\ \mathfrak{MR}_\tau \to L_2([0,\tau];H)\times \mathfrak{TR};\ x\mapsto (\partial_0 x + Y^*Yx + B_0 x\ ,\ x(0))$$

is an isomorphism.

Proof. For $f \in L_2([0,\tau];H)$ and $u_0 \in \mathfrak{TR}$ we want to find a solution $u \in \mathfrak{MR}_\tau$ of

$$\begin{aligned}(\partial_0 + Y^*Y + B_0)\,u &= f\\ u(0) &= u_0.\end{aligned}$$

Case $u_0 = 0$: For $\lambda \in \mathbb{R}_{>0}$ consider

$$\Phi_\lambda:\ L_2([0,\tau];H)\to L_2([0,\tau];H);\ x\mapsto e^{-\lambda m_0} f - B_0\left(\partial_0 + Y^*Y + \lambda\right)^{-1} x.$$

The lemma above ensures that Φ_λ is well-defined and for $x,y \in L_2([0,\tau];H)$ we observe

$$\begin{aligned}&\|\Phi_\lambda(x)-\Phi_\lambda(y)\|_{L_2([0,\tau];H)}\\ =&\left\|B_0\left(\partial_0+Y^*Y+\lambda\right)^{-1}(x-y)\right\|_{L_2([0,\tau];H)}\\ \le& C_\alpha\left\|\left(\partial_0+Y^*Y+\lambda\right)^{-1}(x-y)\right\|_{L_2([0,\tau];H)} + \alpha\left\|\left(\partial_0+Y^*Y+\lambda\right)^{-1}(x-y)\right\|_{\mathfrak{MR}_\tau}\\ \le&\left(\frac{C_\alpha}{\lambda}+\alpha\left\|\left(\partial_0+Y^*Y+\lambda\right)^{-1}\right\|_{L(L_2([0,\tau];H),\mathfrak{MR}_\tau)}\right)\|x-y\|_{L_2([0,\tau];H)}.\end{aligned}$$

For α sufficiently small and subsequently λ large, the lemma above implies that

$$\lambda\mapsto\left\|\left(\partial_0+Y^*Y+\lambda\right)^{-1}\right\|_{L(L_2([0,\tau];H),\mathfrak{MR}_\tau)}$$

can be uniformly bounded and, hence, there are choices of α and λ such that

$$\left(\frac{C_\alpha}{\lambda}+\alpha\left\|\left(\partial_0+Y^*Y+\lambda\right)^{-1}\right\|_{L(L_2([0,\tau];H),\mathfrak{MR}_\tau)}\right)<1,$$

i.e., Φ_λ a contraction.

Let $x^* \in L_2([0,\tau];H)$ be the unique fixed point of Φ_λ, i.e.,

$$x^* = e^{-\lambda m_0} f - B_0\left(\partial_0+Y^*Y+\lambda\right)^{-1}x^*$$

holds. Considering $u^* := e^{\lambda m_0}(\partial_0+Y^*Y+\lambda)^{-1}x^* \in \mathfrak{MR}_0$ we observe

$$(\partial_0+Y^*Y+\lambda)e^{-\lambda m_0}u^* = e^{-\lambda m_0}f - B_0 e^{-\lambda m_0}u^*$$

and, therefore,

$$(\partial_0+Y^*Y+B_0)u^* = f.$$

Case $u_0 \neq 0$: Choose $w \in \mathfrak{MR}_\tau$ with $w(0) = u_0$ and consider

$$(*)\qquad (\partial_0+Y^*Y+B_0)(u-w) = f-(\partial_0+Y^*Y+B_0)w.$$

Then the first case yields a solution v of $(*)$ and $u^* := v+w \in \mathfrak{MR}_\tau$ solves the initial problem.

The two cases above show that $J:\ \mathfrak{MR}_\tau \to L_2([0,\tau];H)\times\mathfrak{TR}$ is a bijection and the bounded inverse theorem[3] yields that J is, in fact, an isomorphism.

□

3

Theorem 8.4 (Bounded Inverse Theorem). *Let X_1 and X_2 be Banach spaces and $T \in L(X_1,X_2)$ bijective. Then $T^{-1} \in L(X_2,X_1)$.*

COROLLARY 8.5. *The Stokes operator*

$$I:\ \mathfrak{MR}_\tau \to L_2([0,\tau];H)\times \mathfrak{TR};\ x\mapsto (\partial_0 x + Y^*Yx\ ,\ x(0))$$

is an isomorphism.

We are now going to prove that $\mathcal{B}_v$ is a compact operator. Therefore, we need to have a look at some compact embedding theorems, first.

LEMMA 8.6 (Aubin-Lions). *Let $X_0,\ X_1,\ X_2$ be Banach spaces, X_0 and X_1 be reflexive, and*

$$X_0 \hookrightarrow_{\text{compact}} X \hookrightarrow_{\text{continuous}} X_1.$$

Let $p,q\in\mathbb{R}_{\geq 1}$ and

$$W := \{f\in L_p([0,\tau];X_0);\ f'\in L_q([0,\tau];X_1)\}.$$

Then

$$W \hookrightarrow_{\text{compact}} L_p([0,\tau];X).$$

PROOF. see [**17**]; Proposition III.1.3 □

We will start by proving two embedding theorems.

LEMMA 8.7. $\mathfrak{MR}_\tau \hookrightarrow_{\text{compact}} L_2([0,\tau];H)$

PROOF. Clearly, $\mathfrak{MR}_\tau = W_2^1([0,\tau],H)\cap L_2([0,\tau],D(Y^*Y))$ is continuously embedded into

$$W := \{u\in L_2([0,\tau],D(Y^*Y));\ u'\in L_2([0,\tau],H)\}.$$

Using Aubin-Lions' Lemma (Lemma 8.6) with $X_0 := D(Y^*Y)$, $X := X_1 := H$, and $p := q := 2$, the assertion reduces to showing

$$D(Y^*Y)\subseteq \mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\left[W_2^{1,(1,0)}(M)\right]\hookrightarrow_{\text{compact}} H.$$

Let $(f_n)_{n\in\mathbb{N}}\in D(Y^*Y)^{\mathbb{N}}$ be a bounded sequence. Then, $\left(\mathrm{pr}^*_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}} f_n\right)_{n\in\mathbb{N}}$ is a bounded sequence in $W_2^{1,(1,0)}(M)$ which is compactly embedded in $L_2^{(1,0)}(M)$ by the Rellich-Kondrachov condition. In other words, there exists a subsequence $\left(\mathrm{pr}^*_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}} f_{n_k}\right)_{k\in\mathbb{N}}$ which converges in $L_2^{(1,0)}(M)$. Hence,

$$(f_{n_k})_{k\in\mathbb{N}} = \left(\mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\mathrm{pr}^*_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}} f_{n_k}\right)_{k\in\mathbb{N}}$$

converges in $\mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\left[L_2^{(1,0)}(M)\right] = H$.

□

Remark Using the theorem of Arzelà-Ascoli, it is possible to show that the embedding $\mathfrak{MR}_\tau \hookrightarrow C([0,\tau],H)$ is compact, as well.

■

LEMMA 8.8. $|Y|[\mathfrak{MR}_\tau]\hookrightarrow_{\text{compact}} L_2([0,\tau];H)$

PROOF. Note that $|Y| := \sqrt{Y^*Y}$ is non-negative, i.e., $-1\in\varrho(|Y|)$. Let

$$\forall x\in H:\ \|x\|_{H_{-1}} := \left\|(|Y|+1)^{-1}x\right\|_H$$

and

$$H_{-1} := \overline{H}^{\|\cdot\|_{H_{-1}}}.$$

Then, $|Y|+1$ maps H unitarily to X_1. Furthermore, $(\partial_0 \otimes 1)(1 \otimes |Y|)$ and $(1 \otimes |Y|)(\partial_0 \otimes 1)$ coincide in $L\left(\mathfrak{MR}_\tau, L_2\left([0,\tau], H_{-1}\right)\right)$. Thus,

$$\forall x \in \mathfrak{MR}_\tau:\ \partial_0 |Y| x = |Y| \partial_0 x \in L_2\left([0,\tau], H_{-1}\right)$$

implies that $|Y|\left[\mathfrak{MR}_\tau\right]$ is continuously embedded in

$$W := \left\{x \in L_2([0,\tau], D(|Y|));\ \partial_0 x \in L_2\left([0,\tau], H_{-1}\right)\right\}.$$

Furthermore, $D(|Y|) = \mathrm{pr}_{\overline{[\{0\}]\,\mathrm{grad}_{(0)}}}\left[W_2^{1,(1,0)}(M)\right]$ is compactly embedded in H by the Rellich-Kondrachov condition and the calculation in the proof of Lemma 8.7. Choosing $p := q := 2$, $X_0 := D(|Y|)$, $X := H$, and $X_1 := H_{-1}$ in Aubin-Lions' Lemma (Lemma 8.6), thus, yields that W is compactly embedded in $L_2([0,\tau], H)$ and, hence, the assertion.

□

Before proving compactness of $\mathcal{B}_v$, we will need one last lemma.

LEMMA 8.9. *Let H_1 and H_2 be Hilbert spaces and $T \in L(H_1, H_2)$. Then, the following are equivalent.*

(i) *T is compact.*
(ii) *T maps weakly-convergent sequences to norm-convergent sequences.*

PROOF. "(i)⇒(ii)" Let $(x_n)_{n\in\mathbb{N}} \in H_1^{\mathbb{N}}$ be weakly convergent to $x \in H_1$. Then, $(Tx_n)_{n\in\mathbb{N}} \in H_2^{\mathbb{N}}$ converges weakly to Tx since

$$\forall y \in H_2:\ \langle Tx_n, y\rangle_{H_2} = \langle x_n, T^* y\rangle_{H_1} \to \langle x, T^* y\rangle_{H_1} = \langle Tx, y\rangle_{H_2}.$$

Suppose $(Tx_n)_{n\in\mathbb{N}}$ does not converge in norm. Then, there exists $\delta \in \mathbb{R}_{>0}$ and a subsequence $(Tx_{n_k})_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}:\ \left\|Tx_{n_k} - Tx\right\|_{H_2} > \delta.$$

The uniform boundedness principle[4] for $F = \{y \mapsto \langle x_n, y\rangle_{H_1};\ n \in \mathbb{N}\}$ yields that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence. Therefore, $(Tx_{n_k})_{k\in\mathbb{N}}$ is bounded and, since T is compact, there is a norm-convergent subsequence $(Tx_{n_{k_j}})_{j\in\mathbb{N}}$ with $Tx_{n_{k_j}} \to: \hat{y}$ $(j \to \infty)$. Since norm-convergence implies weak convergence, we obtain that $(Tx_{n_{k_j}})_{j\in\mathbb{N}}$ converges weakly to $\hat{y}$, as well. But the weak limit was Tx, i.e., $\hat{y} = Tx$ by the Highlander principle[5] which is a contradiction.

"(ii)⇒(i)" Since H_1 is a Hilbert space, the unit ball B_{H_1} is weakly compact (Banach-Alaoglu). Let $(x_n)_{n\in\mathbb{N}} \in B_{H_1}^{\mathbb{N}}$. Then, $(x_n)_{n\in\mathbb{N}}$ contains a weakly convergent subsequence which, by (ii), is mapped to a norm-convergent subsequence of $(Tx_n)_{n\in\mathbb{N}}$; hence, T is compact.

□

PROPOSITION 8.11. *Let $v \in \mathfrak{MR}_\tau$. Then $\mathcal{B}_v$ is compact.*

PROOF. Note that it suffices to show that $B'(v)$ maps weakly convergent sequences in $\mathfrak{MR}_\tau$ to norm-convergent sequences in $L_2([0,\tau]; H)$. Let $w \in \mathfrak{MR}_\tau^{\mathbb{N}}$ be weakly convergent to $w_0 \in \mathfrak{MR}_\tau$.

4

THEOREM 8.10 (uniform boundedness principle). *Let X be a Banach space and N a normed vector space. Let $F \subseteq L(X, N)$ be such that $\forall x \in X:\ \sup_{T\in F} \|Tx\|_N < \infty$. Then,* $\sup_{T\in F} \|T\|_{L(X,N)} < \infty$.

5 There can only be one [limit].

(i) Using polar decomposition $Y = V|Y|$ and observing that

$$L_2^{(1,0)} \ni x \mapsto \operatorname{tr}\left(v(t) \otimes C^{-\frac{1}{2}} V x\right) \in L_2^{(1,0)}(M)$$

is continuous with

$$\sup_{t\in[0,\tau]} \left\| L_2^{(1,0)}(M) \ni x \mapsto \operatorname{tr}\left(v(t) \otimes C^{-\frac{1}{2}} V x\right) \in L_2^{(1,0)}(M) \right\|_{L(L_2^{(1,0)}(M))} < \infty,$$

since $\mathfrak{MR}_\tau \subseteq C([0,\tau];H)$, it remains to show that $(|Y| w_n)_{n\in\mathbb{N}}$ is norm-convergent in $L_2([0,\tau];H)$ which follows directly from $|Y|[\mathfrak{MR}_\tau] \hookrightarrow_{\text{compact}} L_2([0,\tau];H)$.

(ii) Note, $\mathfrak{MR}_\tau \subseteq C([0,\tau];H) \cap L_2([0,\tau];D(Y^*Y))$ also implies that every continuous representative of v takes values in $D(Y^*Y)$ almost everywhere, i.e., for almost every t we obtain

$$C^{-\frac{1}{2}} Y v(t) \in W_2^{1,(2,0)}(M) \subseteq L_2^{(2,0)}(M) =: H_2.$$

Let $D := C^{-\frac{1}{2}} Y v \in L_\infty([0,\tau];H_2)$. Introducing the abbreviations $x_n := w_n - w_0$ and $E := D_{\beta\gamma} D_{\varepsilon\delta} g^{\gamma\delta} g^\beta \otimes g^\varepsilon \in L_\infty([0,\tau];H_2)$, we observe

$$\begin{aligned}
&\|\operatorname{tr}(x_n \otimes D)\|^2_{L_2([0,\tau];H)} \\
&= \int_0^\tau \left\| x_n(t)_\alpha g^{\alpha\beta} D(t)_{\beta\gamma} g^\gamma \right\|_H^2 dt \\
&= \int_0^\tau x_n(t)_\alpha g^{\alpha\beta} D(t)_{\beta\gamma} g^{\gamma\delta} D(t)_{\varepsilon\delta} g^{\zeta\varepsilon} x_n(t)_\zeta dt \\
&= \int_0^\tau \left| \left\langle x_n(t)_\alpha x_n(t)_\zeta g^\alpha \otimes g^\zeta, D(t)_{\beta\gamma} D(t)_{\varepsilon\delta} g^{\gamma\delta} g^\beta \otimes g^\varepsilon \right\rangle_{(2,0)} \right| dt \\
&\leq \int_0^\tau \|(x_n \otimes x_n)(t)\|_{L_2^{(2,0)}(M)} \|E(t)\|_{L_2^{(2,0)}(M)} dt \\
&\leq \|E\|_{L_\infty([0,\tau];H_2)} \int_0^\tau \|x_n(t)\|_H^2 dt \\
&= \|E\|_{L_\infty([0,\tau];H_2)} \|w_n - w_0\|^2_{L_2([0,\tau];H)}
\end{aligned}$$

which converges to zero since $\mathfrak{MR}_\tau \hookrightarrow_{\text{compact}} L_2([0,\tau];H)$.

□

Hence, $F'_\tau(v)$ is a Fredholm operator of index zero, i.e., injective if and only if its range is dense. Since the range is also closed we obtain the following corollary.

Corollary 8.12. *Let $v \in \mathfrak{MR}_\tau$. Then, $F'_\tau(v)$ is an isomorphism if and only if $F'_\tau(v)$ is injective.*

For $x, y, z \in \mathfrak{MR}_\tau$ let

$$\beta(x,y,z) := \left\langle -2 \operatorname{pr}_{\overline{[\{0\}] \operatorname{tr} \operatorname{grad}_{(0)}}} \operatorname{tr}\left(x \otimes C^{-\frac{1}{2}} Y y\right), z \right\rangle_H.$$

Note that

$$\begin{aligned}
\beta(x,y,z) &= -2 \left\langle \operatorname{tr}(x \otimes \operatorname{sym} \operatorname{grad}_{(0)} y), z \right\rangle_{(1,0)} \\
&= \int_M x_i \left(\nabla_{g_j} y_k + \nabla_{g_k} y_j \right) g^{ij} g^{km} z_m \mathrm{dvol}_M.
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta(x,y,y) &= \int_M x_i \left(\nabla_{g_j} y_k + \nabla_{g_k} y_j \right) g^{ij} g^{km} y_m \mathrm{dvol}_M \\
&= -\int_M x_i g^{ij} \left(\nabla_{g_j} y_k g^{km} y_m + \nabla_{g_k} y_j g^{km} y_m \right) \mathrm{dvol}_M \\
&= -\int_M x_i g^{ij} \frac{\nabla_{g_j} \langle y, y \rangle_{(1,0)}}{2} + x_i g^{ij} \left\langle \operatorname{grad}_{(0)} y_j, y \right\rangle_{(1,0)} \mathrm{dvol}_M
\end{aligned}$$

$$
\begin{aligned}
&= -\int_M \frac{1}{2}\langle \underbrace{\operatorname{div}_{(0)} x}_{=0}, \langle y, y\rangle_{(1,0)}\rangle_{(0,0)} + \langle x, \langle y_j, \underbrace{\operatorname{div}_{(0)} y}_{=0}\rangle_{(1,0)}\rangle_{(0,0)} d\mathrm{vol}_M \\
&= 0,
\end{aligned}
$$

as well as,

$$0 = \beta(x, y+z, y+z) = \beta(x,y,y) + \beta(x,y,z) + \beta(x,z,y) + \beta(x,z,z),$$

i.e.,

$$\beta(x,y,z) = -\beta(x,z,y).$$

The last ingredient we need to prove injectivity of $F'_\tau(v)$ is Gronwall's lemma.

Lemma 8.13 (Gronwall's Lemma). *Let $f, g, h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be measurable with*

$$f(t) \leq g(t) + \int_0^t f(s)h(s)ds$$

for almost every $t \in \mathbb{R}_{\geq 0}$. Then

$$f(t) \leq g(t) + \int_0^t g(s)h(s)\exp\left(\int_s^t h(r)dr\right)ds$$

holds for almost every $t \in \mathbb{R}_{\geq 0}$.

Proof. see [**16**]; Theorem A.43 □

Proposition 8.14. *Let $v \in \mathfrak{MR}_\tau$. Then, $F'_\tau(v)$ is injective. In particular, F_τ is locally a diffeomorphism.*

Proof. Let $x \in \mathfrak{MR}_\tau$ and $F'_\tau(v)x = 0$. To show: $x = 0$. First, note that $F'_\tau(v)x = 0$ is equivalent to

$$x(0) = 0 \ \wedge \ \partial_0 x + Y^* Y x + B'(v)x = 0.$$

Multiplying the latter scalarly with x in H yields

$$
\begin{aligned}
0 &= \langle \partial_0 x, x\rangle_H + \langle |Y|x, |Y|x\rangle_H + \beta(x,v,x) + \beta(v,x,x) \\
&= \frac{1}{2}\left(\|x\|_H^2\right)' + \||Y|x\|_H^2 + \beta(x,v,x).
\end{aligned}
$$

With $\tilde{v} := \left(\nabla_{g_j} v_k + \nabla_{g_k} v_j\right) g^{kn} \left(\nabla_{g_m} v_n + \nabla_{g_n} v_m\right) g^j \otimes g^m$ this last equation yields

$$
\begin{aligned}
&\left(\|x\|_H^2\right)' + \||Y|x\|_H^2 \\
&\leq 2\left|\beta(x,v,x)\right| \\
&= 2\left|\left\langle -2\,\mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}} \mathrm{tr}\left(x \otimes C^{-\frac{1}{2}} Y v\right), x\right\rangle_H\right| \\
&\leq 4\left\|\mathrm{tr}(x \otimes \mathrm{sym}\,\mathrm{grad}_{(0)} v)\right\|_H \|x\|_H \\
&= 4\|x\|_H \left(\int_M x_i g^{ij} \frac{1}{2}\left(\nabla_{g_j} v_k + \nabla_{g_k} v_j\right) g^{kn} x_l g^{lm} \frac{1}{2}\left(\nabla_{g_m} v_n + \nabla_{g_n} v_m\right) d\mathrm{vol}_M\right)^{\frac{1}{2}} \\
&= 2\|x\|_H \left|\left\langle x \otimes x, \left(\nabla_{g_j} v_k + \nabla_{g_k} v_j\right) g^{kn} \left(\nabla_{g_m} v_n + \nabla_{g_n} v_m\right) g^j \otimes g^m\right\rangle_{L_2^{(2,0)}(M)}\right|^{\frac{1}{2}} \\
&\leq 2\|x\|_H \left(\|x \otimes x\|_{L_2^{(2,0)}(M)} \|\tilde{v}\|_{L_2^{(2,0)}(M)}\right)^{\frac{1}{2}} \\
&= 2\|\tilde{v}\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}} \|x\|_H^2 \\
&\leq 2\|\tilde{v}\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}} \|x\|_H^2 + \||Y|x\|_H^2
\end{aligned}
$$

and, hence,

$$\left(\|x\|_H^2\right)' \leq 2\,\|\tilde{v}\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\,\|x\|_H^2\,.$$

Thus, integration yields

$$\|x(t)\|_H^2 \leq \|x(0)\|_H^2 + \int_0^t 2\,\|\tilde{v}(s)\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\,\|x(s)\|_H^2\,ds$$

for almost every t and, investing Gronwall's lemma, gives

$$\|x(t)\|_H^2 \leq \underbrace{\|x(0)\|_H^2}_{=0} + \int_0^t 2\,\underbrace{\|x(0)\|_H^2}_{=0}\|\tilde{v}(s)\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\,e^{\int_s^t 2\|\tilde{v}(r)\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}dr}ds = 0.$$

□

The very same mechanism also yields the following proposition.

Proposition 8.15. *F_τ is injective. In particular, F_τ is a diffeomorphism.*

Proof. Let $x,y\in\mathfrak{MR}_\tau$ with $F_\tau(x)=F_\tau(y)$. Then $z:=x-y$ satisfies

$$\begin{aligned}\partial_0 z + Y^*Yz =& B(y)-B(x) \text{ in } (0,\tau)\times M,\\ z(0) =& 0 \text{ in } M.\end{aligned}$$

Just as before, but now in z, we obtain

$$\begin{aligned}\left(\|z\|_H^2\right)' + 2\,\||Y|\,z\|_H^2 =& -2\beta(x,x,z)+2\beta(y,y,z)\\ =& -2(z,x,z)-2\beta(y,x,z)+2\beta(y,y,z)\\ =& -2\beta(z,x,z)-2\underbrace{\beta(y,z,z)}_{=0}\\ \leq& 2\,|\beta(z,x,z)|\,.\end{aligned}$$

Choosing $\tilde{x}:=\left(\nabla_{g_j}x_k+\nabla_{g_k}x_j\right)g^{kn}\left(\nabla_{g_m}x_n+\nabla_{g_n}x_m\right)g^j\otimes g^m$ yields

$$\begin{aligned}&\left(\|z\|_H^2\right)' + \||Y|\,z\|_H^2\\ \leq& 2\,|\beta(z,x,z)|\\ =& 2\left|\left\langle -2\,\mathrm{pr}_{\overline{[\{0\}]\,\mathrm{tr}\,\mathrm{grad}_{(0)}}}\,\mathrm{tr}\left(z\otimes C^{-\frac{1}{2}}Yx\right),z\right\rangle_H\right|\\ \leq& 4\left\|\mathrm{tr}(z\otimes\mathrm{sym}\,\mathrm{grad}_{(0)}\,x)\right\|_H\|z\|_H\\ =& 4\,\|z\|_H\left(\int_M z_i g^{ij}\frac{1}{2}\left(\nabla_{g_j}x_k+\nabla_{g_k}x_j\right)g^{kn}z_l g^{lm}\frac{1}{2}\left(\nabla_{g_m}x_n+\nabla_{g_n}x_m\right)\mathrm{dvol}_M\right)^{\frac{1}{2}}\\ =& 2\,\|z\|_H\left|\left\langle z\otimes z,\left(\nabla_{g_j}x_k+\nabla_{g_k}x_j\right)g^{kn}\left(\nabla_{g_m}x_n+\nabla_{g_n}x_m\right)g^j\otimes g^m\right\rangle_{L_2^{(2,0)}(M)}\right|^{\frac{1}{2}}\\ \leq& 2\,\|z\|_H\left(\|z\otimes z\|_{L_2^{(2,0)}(M)}\|\tilde{x}\|_{L_2^{(2,0)}(M)}\right)^{\frac{1}{2}}\\ =& 2\,\|\tilde{x}\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\|z\|_H^2\\ \leq& 2\,\|\tilde{x}\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\|z\|_H^2+\||Y|\,z\|_H^2\end{aligned}$$

and, hence,

$$\left(\|z\|_H^2\right)' \leq 2\,\|\tilde{x}\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\,\|z\|_H^2\,.$$

Thus, integration yields

$$\|z(t)\|_H^2 \leq \|z(0)\|_H^2 + \int_0^t 2\,\|\tilde{x}(s)\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\,\|z(s)\|_H^2\,ds$$

for almost every t and, investing Gronwall's lemma, gives

$$\|z(t)\|_H^2 \leq \underbrace{\|z(0)\|_H^2}_{=0} + \int_0^t 2\,\underbrace{\|z(0)\|_H^2}_{=0}\,\|\tilde{x}(s)\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}\,e^{\int_s^t 2\|\tilde{x}(r)\|_{L_2^{(2,0)}(M)}^{\frac{1}{2}}dr}\,ds = 0.$$

□

CHAPTER 9

Causality and Well-posedness

As of the end of chapter 8 we know that F_τ is injective and locally an analytic diffeomorphism, i.e., if we have a solution u of the Navier-Stokes problem then it is unique with respect to the data $F_\tau(u)$, changing $F_\tau(u)$ slightly does not destroy unique solvability, and the corresponding solutions depend analytically on the data. But for our construction of solutions with arbitrary data to work, the notion of causality is needed.

Definition 9.1. *Let $R \subseteq L_2([0,\tau];X_1) \oplus L_2([0,\tau];X_2)$ where X_1 and X_2 are Banach spaces.*

(i) *The relation R is called weakly causal if and only if*

$$\forall (u, f_u), (v, f_v) \in R: \ \inf \operatorname{spt}_0(u - v) \le \inf \operatorname{spt}_0(f_u - f_v)$$

where spt_0 denotes the support with respect to time, i.e., in $L_2([0,\tau])$.

(ii) *The relation R is called strongly causal if and only if R is weakly causal and*

$$\forall (u, f) \in R: \ \inf \operatorname{spt}_0 u \le \inf \operatorname{spt}_0 f.$$

(iii) *R is said to have weakly causal solutions if and only if R^{-1} is weakly causal.*

(iv) *R is said to have strongly causal solutions if and only if R^{-1} is strongly causal.*

Additionally, let R be linear.

(v) *The linear relation R is called causal if and only if*

$$\forall (u, f) \in R: \ \inf \operatorname{spt}_0 u \le \inf \operatorname{spt}_0 f.$$

(vi) *The linear relation R is said to have causal solutions if and only if R^{-1} is causal.*

Corollary 9.2. *Let X_1 and X_2 be Banach spaces and $R \subseteq L_2([0,\tau];X_1) \oplus L_2([0,\tau];X_2)$ with $0 \in R$. Then, weak causality and strong causality are equivalent.*

Proof. For $R = \varnothing$ the assertion is trivial and, since strong causality implies weak causality, there is only one direction to show. Let $(u, f) \in R$. Then,

$$\forall (v, f_v) \in R: \ \inf \operatorname{spt}_0(u - v) \le \inf \operatorname{spt}_0(f - f_v)$$

implies

$$\inf \operatorname{spt}_0 u = \inf \operatorname{spt}_0(u - 0) \le \inf \operatorname{spt}_0(f - 0) = \inf \operatorname{spt}_0 f$$

because $0 \in R$.

□

Corollary 9.3. *Let X_1 and X_2 be Banach spaces and $R \subseteq L_2([0,\tau];X_1) \oplus L_2([0,\tau];X_2)$ linear. Then, weak causality, strong causality, and "linear" causality are equivalent.*

PROOF. For $R = \varnothing$ the assertion is trivial and, since linear relations contain zero and strong causality trivially implies "linear" causality, it suffices to show that "linear" causality implies weak causality. Let $(u, f_u), (v, f_v) \in R$. Then, $(u - v, f_u - f_v) \in R$ by linearity and, thus,

$$\inf \operatorname{spt}_0(u - v) \leq \inf \operatorname{spt}_0(f - f_v).$$

□

These are very neat properties as we only need to show weak causality of solutions for our construction to work whereas strong causality of solutions implies that the "vacuum solution" is zero, that is, a motionless fluid will remain at rest as long as no external force acts on it. This property does not hold if we have proper weak causality of solutions but is essential for the system to be physically senseful.

LEMMA 9.4. *Let X_1 and X_2 be Banach spaces, and $(R_t)_{t\in\mathbb{R}_{>0}}$ a family of left-unique*[1] *relations $R_t \subseteq L_2([0,t]; X_1) \oplus L_2([0,t]; X_2)$ such that*

$$\forall t \in \mathbb{R}_{>0}\ \forall s \in (0,t)\ \forall (u,f) \in R_t :\ \left(u|_{[0,s]}, f|_{[0,s]}\right) \in R_s \tag{6}$$

holds. Then, all R_t have weakly causal solutions.

PROOF. Suppose R_t does not have weakly causal solutions for some $t \in \mathbb{R}_{>0}$. Then there are $(u, f_u), (v, f_v) \in R_t$ with

$$\inf \sup_0 (u - v) < \inf \operatorname{spt}_0(f_u - f_v).$$

By left-uniqueness, this implies $u \neq v$ because f_u and f_v must be distinct. Choose $s \in (\inf \sup_0(u - v), \inf \operatorname{spt}_0(f_u - f_v))$. Then

$$u|_{[0,s]} \neq v_{[0,s]}$$

and

$$f_u|_{[0,s]} = f_v|_{[0,s]} \tag{$*$}$$

hold. But from $(*)$ and left-uniqueness of R_s we deduce

$$u|_{[0,s]} = v_{[0,s]}$$

which is a contradiction.

□

Since $\operatorname{grad}_{(0)}$, $\operatorname{div}_{(0)}$, ∂_0, $\otimes$, tr, and sym obviously are causal operators and C was defined to be causal, we conclude that all F_τ are weakly causal (which implies (6)) and, therefore, they all have strongly causal solutions. Choosing C to be local, as well, is not possible in our general setting because many non-Newtonian fluids have viscous memory, that is, C contains delay terms. However, it would break physics to assume the viscosity depended on the future. Hence, C ought to be causal. This was the last missing piece of our jigsaw and we can, now, state our main result.

THEOREM 9.5 (Well-posedness and Causality). *Let $\tau \in \mathbb{R}_{>0}$, $u_0 \in \mathfrak{TR}$, and $f \in L_2([0,\tau]; H)$.*

(i) *There exist $\tau' \in (0, \tau)$ and $u \in \mathfrak{MR}_\tau$ such that the Navier-Stokes equations*

$$\begin{aligned} \partial_0 u + Y^* Y u + B(u) =& f \quad \text{in } (0,\tau') \times M, \\ u(0) =& u_0 \text{ in } M \end{aligned}$$

are satisfied. Furthermore, solutions are strongly causal and unique in $\mathfrak{MR}_{\tau'}$.

[1] left-uniqueness resembles injectivity

(ii) *There exists an open neighborhood* $U \subseteq L_2([0,\tau];H) \times \mathfrak{TR}$ *of* (f,u_0) *and* $\tau' \in (0,\tau)$ *such that*

$$G\colon\ U \to \mathfrak{MR}_{\tau'};\ (g,v_0) \mapsto F_\tau^{-1}(g,v_0)|_{[0,\tau']}$$

is analytic and all $G(g,v_0)$ *solve the Navier-Stokes system in* $(0,\tau') \times M$ *with respect to the data* $(g,v_0) \in U$*, i.e., the solutions depend analytically on the data and* τ' *is locally constant.*

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