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ζ-functions of Fourier Integral Operators

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«Ει μη αληθες εστι, μη ειπης.»

Caesar Marcus Aurelius Antoninus Augustus

Abstract

Based on Guillemin's work on gauged Lagrangian distributions, we will introduce the notion of a gauged poly-log-homogeneous distribution as an approach to ζ-functions for a class of Fourier Integral Operators which includes cases of amplitudes with asymptotic expansion $\sum_{k\in\mathbb{N}} a_{m_k}$ where each a_{m_k} is log-homogeneous with degree of homogeneity m_k but violating $\Re(m_k) \to -\infty$. We will calculate the Laurent expansion for the ζ-function and give formulae for the coefficients in terms of the phase function and amplitude, as well as investigate generalizations to the Kontsevich-Vishik trace. Using stationary phase approximation, series representations for the Laurent coefficients and values of ζ-functions will be stated explicitly, and the kernel singularity structure will be studied. This will yield algebras of Fourier Integral Operators which purely consist of Hilbert-Schmidt operators and whose ζ-functions are entire, as well as algebras in which the generalized Kontsevich-Vishik trace is form-equivalent to the pseudo-differential operator case. Additionally, we will introduce an approximation method (mollification) for ζ-functions of Fourier Integral Operators whose amplitudes are poly-log-homogeneous at zero by ζ-functions of Fourier Integral Operators with "regular" amplitudes.

In part II, we will study Bochner-, Lebesgue-, and Pettis integration in algebras of Fourier Integral Operators. The integration theory will extend the notion of parameter dependent Fourier Integral Operators and is compatible with the Atiyah-Jänich index bundle as well as the ζ-function calculus developed in part I. Furthermore, it allows one to emulate calculations using holomorphic functional calculus in algebras without functional calculus, and to consider measurable families of Fourier Integral Operators as they appear, for instance, in heat- and wave-traces of manifolds whose metrics are subject to random (possibly singular) perturbations.

Contents

Notations

(a,b) open interval $\{x \in X;\ a < x\ \wedge\ x < b\}$ for any partially ordered set X; similarly, $[a,b]$ is the closed interval $\{x \in X;\ a \leq x\ \wedge\ x \leq b\}$, and $[a,b)$ and $(a,b]$ are $\{x \in X;\ a \leq x\ \wedge\ x < b\}$ and $\{x \in X;\ a < x\ \wedge\ x \leq b\}$ respectively

$(a_\iota)_{\iota \in I} \in X^I$ family notation of a map $I \ni \iota \mapsto a_\iota \in X$

$*$ Hodge-$*$-operator

$*$ convolution

0 zero-section, as in $T^*X \setminus 0$

$:$ definition as in $f(x) := 5$ or $a_n \to: a$ (defining the limit of a sequence $(a_n)_{n\in\mathbb{N}}$ to be called a)

$[A,B]$ commutator $AB - BA$

$[A]f$ pre-set/pre-image of the set A under the relation f, i.e. for $f \subseteq X \times Y$, $[A]f = \{x \in X;\ \exists y \in A:\ (x,y) \in f\}$

$\#S$ cardinality of the set S

$|\cdot|$ absolute value or modulus in $\mathbb{R}$ or $\mathbb{C}$, resp.

$\bigcap_{\iota \in I} A_\iota$ prefix notation of intersections ranging all $\iota \in I$

$\bigcup_{\iota \in I} A_\iota$ prefix notation of unions ranging all $\iota \in I$

$\mathcal{B}(\Omega)$ Borel σ-algebra of a topological space Ω

$\cap$ infix notation for intersections of sets

$\cdot$ place holder for the argument, as in $f(x,\cdot)$

$\circ$ composition of relations

$\mathbb{C}$ set of complex numbers

$\cong$ isomorphic

∂ Fréchet derivative

∂ boundary operators as in ∂A =closure of A minus the interior of A

∂^α multi-index notation

∂_j partial Fréchet derivative with respect to the j^{th} argument

∂_r radial derivative, that is $\partial_r f(x) = \left\langle \operatorname{grad} f(x), \frac{x}{\|x\|} \right\rangle$

$\partial_{\partial B}$ spherical derivative, that is, on ∂B_V

$\uplus$ disjoint union

$\Delta(X)$ see $\operatorname{diag}(X)$

δ_{diag} δ-distribution along the diagonal

δ_x Dirac δ-distribution centered at x

$\Delta_{\partial B}$ spherical Laplacian

det determinant

$\det_{\mathfrak{fp}}$ regularized generalized determinant

$\det_\zeta$ generalized ζ-determinant

$\operatorname{diag}(X \times X)$ diagonal in X^2, i.e. $\{(x,y) \in X^2;\ x = y\}$

dim dimension operator

div divergence

$\ell_p(I)$ set of absolutely p-summable families in $\mathbb{C}^I$ or $\mathbb{R}^I$

$\ell_p(I,X)$ set of absolutely p-summable families in X^I

$\emptyset$ empty set

$\equiv$ equality modulo some equivalence relation

$\exists$ “there exists”

exp exponential function with base e

$\forall$ “for all”

$\mathcal{F}$ Fourier transform - $(2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N}$ convention

$\mathfrak{fp}_z$	finite part at a point z
Γ	Γ-function
Γ	canonical relation
Γ'	twisted relation, for a relation Γ
γ_n	Stieltjes constants $\lim_{N\to\infty}\left(-\frac{(\ln N)^{n+1}}{n+1}+\sum_{k=1}^{N}\frac{(\ln k)^n}{k}\right)$
$\gamma_n(h)$	generalized Stieltjes constants $\lim_{N\to\infty}\left(-\frac{(\ln(N+h))^{n+1}}{n+1}+\sum_{k=1}^{N}\frac{(\ln(k+h))^n}{k+h}\right)$
Γ_{ui}	upper incomplete Γ-function
$\Leftrightarrow$	biconditional
ilc_z	initial Laurent coefficient at a point z
$\Im$	imaginary part of a complex number
$\in$	"is element of" as in $a \in A$
$\in$	function that maps singletons to their element, that is, $\in(\{x\}) = x$
inf	infimum
$\int_X d\mathrm{vol}_X$	integration with respect to vol_X
λ	Lebesgue measure
$\langle S\rangle$	algebra generated by elements of S
$\langle\cdot,\cdot\rangle$	dual pairing in $X \oplus X'$
$\langle\cdot,\cdot\rangle_X$	scalar product in X
$\langle\cdot,\cdot\rangle_{X\oplus X'}$	dual pairing in $X \oplus X'$
ln	logarithm with base e
$\mapsto$	"maps to" as in $x \mapsto f(x)$
max	maximum
min	minimum
$\mathcal{M}$	Mellin transform
$\mathcal{M}^T$	upper Mellin transform
$\mathcal{M}_T$	lower Mellin transform

$\nearrow$	convergence from below
$\mathbb{N}$	set of positive integers (without zero)
$\mathbb{N}_0$	set of non-negative integers (with zero)
$\Vert\cdot\Vert_X$	norm in a normed space X
$\Vert\cdot\Vert_{\mathrm{Lip}}$	(canonical) operator norm, i.e. the norm in $L(X,Y)$ for an operator in $L(X,Y)$
oilc_z	order of the initial Laurent coefficient at a point z
$\overline{A}^T$	closure/completion of A in the topology T or with respect to the topology induced by T
$\oplus$	topological direct sum with product topology, i.e. for topological vector spaces $X \oplus Y$ is $X \times Y$ and the semi-norms are generated by $p_{ij}(x,y) = \Vert(p_i^X(x), p_j^Y(y))\Vert$ with some norm $\Vert\cdot\Vert$ on $\mathbb{R}^2$
order	order of an operator
$\otimes$	tensor product
$\mathcal{P}$	power set
pr_j	projection to the j^{th} argument
pr_V	orthoprojection to the space V
$\prod_{\iota\in I} a_\iota$	prefix notation for products ranging over all ι in an ordered set I
$\psi DO(X)$	ring of pseudo-differential operators on X
$\Re$	real part of a complex number
$\mathrm{res}\,\alpha$	residue of a log-homogeneous distribution α
$\mathrm{res}_z\, f$	residue of the meromorphic function f at z
$\mathrm{tr}^{\mathrm{res}}$	residue trace
$\varrho(A)$	residue form
$\varrho(A)$	resolvent set of an operator A
$\mathbb{R}$	set of real numbers

$\searrow$	convergence from above
$\smallsetminus$	"set minus", as in $A \smallsetminus B = \{a \in A;\ a \notin B\}$
sgn	sign function
$\sigma(A)$	spectrum of an operator A
$\sigma(A)$	symbol of an operator A
$\sigma_d(A)$	discrete spectrum of an operator A
$\sigma_p(A)$	point spectrum of an operator A
$\sim$	asymptotic expansion
spt	support of a function
$\subseteq$	subset, as in $A \subseteq B$ - A is a subset of B
$\subsetneq$	proper subset, as in $A \subsetneq B \iff A \subseteq B \ \wedge\ A \neq B$
$\xrightarrow{s}$	strong convergence
$\sum_{\iota \in I} a_\iota$	prefix notation for sums ranging over all ι in an ordered set I
sup	supremum
sym	symmetrization operator, $\mathrm{sym} A = \frac{A+A^*}{2}$
$\Rightarrow$	implication
Θ_σ	spectral Θ-function
$\to$	convergence in standard topology
tr	trace
$\vee$	logical disjunction
vol_X	Riemannian volume measure on an orientable Riemannian manifold X
$\wedge$	logical conjunction
$\wedge$	wedge-product on the exterior algebra
$\overset{*}{\rightharpoonup}$	weak-$*$-convergence
$\rightharpoonup$	weak convergence
ζ	generic ζ-function

$\zeta_\sigma(A)$ spectral ζ-function of an operator A

ζ_H Riemann-Hurwitz-ζ-function

ζ_R Riemann-ζ-function

$\mathbb{Z}$ set of integers

$\flat$ musical isomorphism

$A/_B$ quotient space of A being factorized by B

A^* adjoint relation of a relation $A \subseteq X \oplus Y$, i.e. $A^* = (-A^\perp)^{-1}$ (functional minus, i.e. $(x,y) \in A \;\Leftrightarrow\; (x,-y) \in -A$)

$A^\perp$ orthogonal complement or annihilator of A

A^{-1} inverse of A

a_{d-j} log-homogeous amplitude with degree of homogeneity $d-j$

B^A set of all left-total functions $f:\ A \to B$

B_V $B_V[0,1]$

$B_V(a,r)$ open ball in V centered at a with radius r

$B_V[a,r]$ closed ball in V centerd at a with radius r

$C(A)$ $C(A,\mathbb{R})$ or $C(A,\mathbb{C})$ depending on the context

$C(A,B)$ set of continuous functions $f \in B^A$

$C^\infty(A,B)$ set of functions in $C(A,B)$ which can be differentiated arbitrarily often

$C^\omega(A,B)$ set of analytic functions in $C(A,B)$

$C^k(A,B)$ set of k-times differentiable functions in $C(A,B)$

$C_0(A,B)$ closure of $C_c(A,B)$ in $C(A,B)$

$C_c(A,B)$ set of compactly supported elements of $C(A,B)$

d exterior derivative

$d\mathrm{vol}_X$ Riemannian volume form on an orientable Riemannian manifold X

d^* co-derivative on exterior algebra

f' Fréchet derivative of the function f

$f:\ D(f) \subseteq A \to B;\ x \mapsto f(x)$ a function f defined on $D(f)$ interpreted as a subset of A mapping each $x \in D(f)$ to $f(x) \in B$

$f[A]$ post-set/image of the set A under the relation f, i.e. for $f \subseteq X \times Y$, $f[A] = \{y \in Y;\ \exists x \in A:\ (x,y) \in f\}$

H^k_{dR} k^{th} de Rham cohomology group

$I(X,\Lambda)$ set of Lagrangian distributions on X with respect to Λ

$I^m(X,\Lambda)$ set of Lagrangian distributions of order m on X with respect to Λ

$I_{\mathrm{compact}}(X,\Lambda)$ set of compactly supported Lagrangian distributions on X with respect to Λ

k_{KV} Kontsevich-Vishik regularized kernel

$L(V)$ set of bounded linear functionals on a topological vector space V

$L_p(X)$ Lebesgue space L_p on some measure space X

m multiplication operator with the argument

P^t transpose of a pseudo-differential operator P

S^m Hörmander class

S_p the set $\{s \in S;\ p(s)\}$ if S is a set and p a predicate

T^*X co-tangent bundle of a manifold X

TX tangent bundle of a manifold X

V' topological dual space of a topological vector space V

W^s_p Sobolev space of "s-times" weakly differentiable functions in L_p

z^* complex conjugate of z

$\binom{r}{k}$ binomial coefficient $\prod_{j=0}^{k-1} \frac{r-j}{j+1}$

Introduction

An important class of functionals on an algebra are traces, i.e. functionals that vanish on commutators. Traces not only give insight into the structure of a given algebra but also allow invariants of the algebra to be calculated and, hence, the objects the algebra is associated with. In particular, exotic traces (non-trivial traces which are not a multiple of the classical trace on trace-class operators) have many applications in the theory of ideals in $L(H)$ and non-commutative geometry. In geometric analysis, on the other hand, algebras often are modules of semi-group representations of some geometric or topological structure, e.g. a manifold, foliation, a fractal, or quantum field theory. As such, traces give rise to geometric, topological, spectral, or physical invariants which, in turn, can be used to classify and characterize those structures. A generic application would look like

$$\text{terms depending on an operator } A \quad = \quad \text{terms depending on a manifold } M.$$

The Atiyah-Singer Index Theorem, for instance, is of this form and states that the analytical index of an elliptic differential operator between smooth vector bundles on an finite-dimensional compact manifold coincides with its topological index.

A very interesting class of traces and trace-like functionals arise from the notion of (operator) ζ-functions which were introduced by Ray and Singer [**59**, **60**] using Seeley's work on complex powers of elliptic pseudo-differential operators [**68**]. In mathematical physics, Hawking [**37**] first used these ζ-functions as a tool of regularization for path integrals, very much in the light of regularizing divergent series

$$\sum_{n\in\mathbb{N}} 1 \quad \text{``=''} \quad \zeta_R(0) = -\frac{1}{2} \quad \text{and} \quad \sum_{n\in\mathbb{N}} n \quad \text{``=''} \quad \zeta_R(-1) = -\frac{1}{12}$$

where ζ_R denotes Riemann's ζ-function. Considering Lidskii's theorem

$$\operatorname{tr} A = \sum_{\lambda\in\sigma(A)\setminus\{0\}} \mu_\lambda \lambda$$

for a trace-class operator A (where $\sigma(A)$ denotes the spectrum of A and μ_λ the algebraic multiplicity of the eigenvalue λ), we obtain the spectral ζ-function on $\mathbb{C}$

$$\zeta_\sigma(A)(z) \coloneqq \sum_{\lambda\in\sigma(A)\setminus\{0\}} \mu_\lambda \lambda^{-z} = \operatorname{tr} A^{-z}$$

by meromorphic extension provided that A has purely discrete spectrum (in Hawking's case A is a differential operator), the series $\sum_{\lambda\in\sigma(A)\setminus\{0\}} \mu_\lambda \lambda^{-z}$ converges unconditionally in some open set $\Omega \subseteq \mathbb{C}$ (usually a half-space for $\Re(z)$ sufficiently large), and the resulting function extends meromorphically to $\mathbb{C}$.

Very closely related are regularizations of infinite products

$$\prod_{n\in\mathbb{N}} a_n = \prod_{n\in\mathbb{N}} \exp(\ln a_n) = \exp\left(\sum_{n\in\mathbb{N}} \ln a_n\right) = \exp\left(\operatorname{tr} \ln A\right)$$

if $(a_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of an operator A such that $\ln A$ is well-defined and of trace-class. This is why this product is also called the determinant $\det A$ of A. Using the spectral ζ-function, we observe

$$\begin{aligned}\det A &= \exp\left(\operatorname{tr} \ln A\right)\\ &= \exp\left(\operatorname{tr}\left(A^{-z} \ln A\right)\big|_{z=0}\right)\\ &= \exp\left(-\partial\left(z \mapsto \operatorname{tr} A^{-z}\right)(0)\right)\\ &= \exp\left(-\zeta_\sigma(A)'(0)\right).\end{aligned}$$

In other words, not just ζ-functions are important but also their derivatives. Such ζ-determinants were introduced by Ray and Singer in [**59**, **60**].

However, considering families $z \mapsto A^z$ is very restrictive (especially since for many algebras the term A^z is not well-defined and even if it is possible to define complex powers, it may not be possible for every A). It is common, therefore, to study more general families like $z \mapsto G(z)A$ with $G(0) = 1$; in particular, $G(z) = g^z$ for some suitable operator g is a viable choice in algebras that allow complex power for some elements. These have important applications in the theory of pseudo-differential operators and such ζ-functions have been widely studied (cf. e.g. [**67**]); in fact, the entire Laurent expansion is known for ζ-functions of families of the type $z \mapsto Ag^z$ (cf. [**56**]).

For pseudo-differential operators with polyhomogeneous amplitudes, the ζ-function is a meromorphic function with isolated simple poles only and its Laurent coefficients can be used to define traces. Of particular importance are the non-commutative residue (cf. [**33**, **77**, **78**]), which corresponds to the pole, and the Kontsevich-Vishik trace (cf. [**47**, **48**]) which corresponds to the constant Laurent coefficient. In order to obtain the Laurent expansion, it is necessary to take derivatives which produce logarithmic terms in the amplitude. ζ-functions for such operators are still meromorphic but may fail to have only simple poles. Generalizations to the non-commutative residue and the Kontsevich-Vishik trace for such operators with log-terms have also been studied (cf. e.g. [**51**]).

While the theory for pseudo-differential operators can solve many problems, there is still a need to replace them by Fourier Integral Operators. A prime example would be the case of wave trace invariants. Similarly, in the realm of mathematical physics, Radzikowski [**57**, **58**] realized the importance of the wave front set in quantum field theories on curved space-time which inherently means that Fourier Integral Operators take the role pseudo-differential operators played in more "classical" settings. Even though the theory for pseudo-differential operators is well-developed, for Fourier Integral Operators very little is known. Guillemin [**34**] showed that ζ-functions and the residue trace exist for gauged Lagrangian distributions with poly-homogeneous amplitudes and, thus, certain algebras of Fourier Integral Operators, Boutet de Monvel and Guillemin have considered the class of Toeplitz operators and generalized Szegő projectors (cf. [**7**, **8**]), and especially wave traces and related examples have been studied (cf. e.g. [**36**, **79**]). Whether or not there exists a suitable extension of the Kontsevich-Vishik trace, for instance, has been unknown.

Thus, one of the aims of this thesis is to study possible extensions of the Kontsevich-Vishik trace to Fourier Integral Operators. Since calculating the constant Laurent coefficient of a meromorphic function with simple poles requires us to calculate at least one derivative, it is necessary to consider log-terms in the amplitude. As to be expected, being able to handle one derivative will be sufficient to compute all derivatives and, thus, the entire Laurent expansion.

The thesis is structured in two parts. In part I, we will calculate the Laurent expansion and study generalizations of the Kontsevich-Vishik trace while part II will mostly focus on integration techniques in algebras of Fourier Integral Operators.

Chapter 1 contains a short overview of the most important definitions and theorems about Fourier Integral Operators and their algebras. In chapters 2 and 3 we will not see any Fourier Integral Operators directly, but define the notion of gauged poly-log-homogeneous distributions, their ζ-functions, and calculate the Laurent expansion. The definition in chapter 2 will seem rather restrictive since we will only allow affine-linear functions as degrees of homogeneity. However, we will see in chapter 3 that any meromorphic family of poly-log-homogeneous distributions has a ζ-function which is germ-equivalent to a ζ-function of a gauged poly-log-homogeneous distribution provided none of the degrees of homogeneity is germ-equivalent to a critical constant.

In chapter 4, we will return to Fourier Integral Operators. In fact, we will see that gauged poly-log-homogeneous distributions are a generalization of Guillemin's approach in [**34**]. Hence, Lagrangian distributions as considered in [**34**] and, in particular, pseudo-differential operators are covered. Furthermore, it includes the operators considered by Paycha and Scott [**56**], that is, those cases where the entire Laurent expansion for pseudo-differential ζ-functions is known, as well as generalized Toeplitz operators and Szegő projectors as studied by Boutet de Monvel and Guillemin [**7**,**8**]. In particular, we will obtain the Laurent expansion for ζ-functions of gauged Fourier Integral Operators which can be extended to the case of meromorphic germs of Fourier Integral Operators using the results of chapter 3.

Chapter 5 will be all about examples. Here, we will consider the heat trace

$$\operatorname{tr} e^{-t|\Delta|} = \frac{\operatorname{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi t)^{\frac{N}{2}}} \sum_{\gamma\in\Gamma} \exp\left(-\frac{\|\gamma\|_{\ell_2(N)}^2}{4t}\right)$$

on the flat torus $\mathbb{R}^N/\Gamma$ where Δ is the Dirichlet Laplacian, as well as calculate all the Laurent coefficients of ζ-functions of gauged fractional Laplacians

$$\zeta\left(s \mapsto \sqrt{|\Delta|}^{s+\alpha}\right)(z) = 2\zeta_R(-z-\alpha),$$

and gauged shifted fractional Laplacians

$$\zeta\left(s \mapsto \left(h + \sqrt{|\Delta|}\right)^{s+\alpha}\right)(z) = 2\zeta_H(-z-\alpha; h) - h^{z+\alpha}$$

on $\mathbb{R}/_{2\pi\mathbb{Z}}$ where ζ_R denotes the Riemann-ζ-function and ζ_H the Riemann-Hurwitz-ζ-function. In particular, the case of gauged fractional Laplacians is highly interesting since it violates the assumptions of our Laurent expansion quite strongly in the following sense. As of that point, we can only consider families of Fourier Integral Operators whose amplitudes $a(z)(x,y,\xi)$ satisfy $a(z)(x,y,\cdot)\in C^\infty(\mathbb{R}^N)$ for every $x,y\in X$ and $z\in\mathbb{C}$ where X is the underlying manifold. This is true for the gauged shifted fractional Laplacians but not in the non-shifted case.

However, it turns out that the "non-shifted" ζ-function is the compact limit of the "shifted" ζ-functions sending the shift to zero. This observation not only validates the example but is largely generalizable.[1] The generalization, which we will call mollification, will be discussed in chapter 6 and is essentially a procedure showing that any gauged poly-log-homogeneous distribution which is poly-log-homogeneous everywhere on $\mathbb{R}^N\setminus\{0\}$ can be written as a limit of gauged poly-log-homogeneous distribution with regular amplitudes such that the corresponding ζ-functions are compactly convergent. In other words, the Laurent expansion holds in that case, as well, and we have obtained a complete extension of the pseudo-differential case. In particular, we will now turn our focus to the Kontsevich-Vishik trace and other formulae related to the Laurent coefficients.

In chapter 7, we will study conditions to decide whether or not the ζ-function is holomorphic in a neighborhood of zero. In particular, this will yield a generalized Kontsevich-Vishik trace which is unique in the sense that any other extension of the Kontsevich-Vishik trace must coincide with this generalization modulo terms that vanish under ζ-regularization or cannot be given by a globally defined density (provided the kernel of the operator is defined as a globally defined density).

[1] It is also possible to use well-known facts about extensions of log-homogeneous distributions on $\mathbb{R}^N\setminus\{0\}$ to $\mathbb{R}^N$ if validating the Laurent expansion in this specific case were the only reason for these considerations. However, generalizing that approach would only yield the Laurent expansion for Fourier Integral Operators with log-homogeneous amplitudes up to a holomorphic function which has to be added. Furthermore, it is not directly applicable to gauged poly-log-homogeneous distributions in general.

Other than giving a positive answer to the question of a generalized Kontsevich-Vishik trace, the main consequence is that we can obtain Guillemin's results on the commutator structure **[34, 35]** from this generalized approach.

In order to actually be able to calculate the Laurent coefficients (and, thus, the generalized Kontsevich-Vishik trace) for a given gauged Fourier Integral Operator, chapter 8 focuses on the stationary phase approximation of the Laurent coefficients and the kernel singularity structure of Fourier Integral Operators. Here, we will calculate the kernel singularity structure explicitly and find two "polar opposites" in the set of Fourier Integral Operator algebras. One class of algebras, that also contains the Toeplitz operators and generalized Szegő projectors **[7]**, is closest to the pseudo-differential operator case, in the sense that the generalized Kontsevich-Vishik trace is form-equivalent to the Kontsevich-Vishik trace in the pseudo-differential case. In fact, we will obtain (3) and (4) in **[7]** and extend the results of **[7]** by calculating the Kontsevich-Vishik trace. For the other class of algebras, every term that appears in the generalized Kontsevich-Vishik trace but not in the pseudo-differential Kontsevich-Vishik trace is non-trivial. In particular, splitting off finitely many terms in the expansion is not possible since every single one of them will have a contribution, in general. This is closely related to the interesting fact that every operator in such an algebra is Hilbert-Schmidt and has continuous kernel; a property that is independent of the Hörmander class of the amplitude. In particular, ζ-functions of families of Fourier Integral Operators in such algebras have no poles.

At this point, we will have extended a number of pseudo-differential results[2] to Fourier Integral Operators. However, there are many others that cannot be

[2] e.g. (2.21) in **[47]**, (4.11), (5.19), Lemma 5.4, Proposition 5.5, and Theorem 5.6 (ii-v) in **[51]**, (9) in **[55]**, and (0.12), (0.14), (0.17), (0.18), and (2.20) in **[56]**

tackled. The major obstacle here is the fact that the algebra of pseudo-differential operators is closed with respect to holomorphic functional calculus whereas most algebras of Fourier Integral Operators are not. Hence, any result that requires the holomorphic functional calculus cannot be extended directly unless one finds an independent proof that does not make use of the functional calculus. Similarly, the mere question of replacing the phase function in an integral using holomorphic functional calculus for pseudo-differential operators means that we do not even know whether the new integral is well-defined in a suitable algebra of Fourier Integral Operators. Furthermore, if we consider variational formulae (e.g. the variational formula for the multiplicative anomaly of ζ-determinants), then we would like to be able to integrate a family f of gauged operators and their ζ-functions $\zeta \circ f$ and have the result be independent of the order of calculation, i.e. $\int \zeta \circ f = \zeta\left(\int f\right)$. In other words, we need to make sense of $\int f$ for operators, kernels, and ζ-functions such that all these notions can be used interchangeably. Hence, developing a suitable integration theory in algebras of Fourier Integral Operators would be highly useful and is the focus of part II of this thesis.

Another driving factor for considering integrals of measurable families of (gauged) Fourier Integral Operators are stochastic Fourier Integral Operators, that is, measurable functions of Fourier Integral Operators or, similarly, parameter dependent Fourier Integral Operators as they appear in the treatment of linear partial differential equations with discontinuous/stochastic coefficients [**28**]. Although the approach considered in part II is still very technically involved, it does not require Colombeau algebras [**13**, **26**, **27**] and is a natural extension of parameter dependent Fourier Integral Operators in the sense of chapters 2.1.2 and 2.2 of [**63**] as well as vertical Fourier Integral Operators associated with fibrations (cf. e.g. chapter 5 in [**67**]). In other words, it is a direct connection between stochastic Fourier Integral Operators and "standard" Fourier Integral Operator techniques of geometric analysis.

Part II starts with two chapters on various integrals in topological vector spaces. In chapter 9, we will consider Bochner- and Lebesgue-integrals, i.e. integrals in the strong topology of the algebra with respect to measurable functions (pre-sets of measurable sets are measurable) and strongly measurable functions (almost everywhere sequential limits of simple functions). Since the L_p-theory in locally convex topological vector spaces is notoriously filled with subtleties, an exhaustive account of the main theorems tailored to our applications is contained in chapter 9. However, these integrals have a major drawback: a priori, they take values in the completion of the algebra but there are canonical topologies on algebras of Fourier Integral Operators which are only quasi-complete and not complete. Luckily, with the notion of Pettis-integral, which is a weaker notion and the subject of chapter 10, quasi-completeness is sufficient and we can prove that the Bochner- and Lebesgue-integrals take values in the algebra.

Chapter 11 addresses an important side effect of having an integration theory; namely, we have a theory of measurable functions with values in an algebra of Fourier Integral Operators which extends the theory of continuous functions with values in an algebra of Fourier Integral Operators, i.e. parameter dependent Fourier Integral Operators [**63**] as well as the idea of families of operators such as they appear in the index theorem for families. There, we have a fibration $M \to B$ and an operator D_b on each fiber M_b such that $b \mapsto D_b$ is a continuous function. For pseudo-differential operators this is deeply connected with the family index and the Atiyah-Jänich index bundle. In chapter 11 we will, therefore, topologize the set of index bundles and show that the Atiyah-Jänich index bundle construction is a continuous map with respect to the gap-topology on the operator side. In other words, measurable families of Fourier Integral Operators in the sense of our integration theory yield measurable "index bundles" such that the restriction to continuous families is compatible with the Atiyah-Jänich case.

With this prelude, chapter 12 shows an example of how to emulate the holomorphic functional calculus in algebras in which holomorphic functional calculus is not defined. More precisely, we consider an example calculation that makes heavy use of the holomorphic functional calculus on the pseudo-differential operator side, replace the phase function, show that these new integrals are well-defined within the new algebra, and calculate them.

Finally, in chapter 13, we will return to ζ-functions of Fourier Integral Operators. In order for our integration theory to be applicable, we will need to show that the ζ-function as an operator from the space of gauged Fourier Integral Operators to the space of meromorphic functions, or a suitable other target space, has a quasi-complete extension. Unfortunately, a suitable topology on the space of meromorphic functions such that the ζ-function (as an operator from the space of gauged Fourier Integral Operators with wave front set in a given cone to the space of meromorphic functions) can be quasi-completed remains unknown. Instead, we can consider many subspaces of ζ and we will introduce the space of ζ-functions with a suitable topology that almost allows ζ to be quasi-completed. Though slightly unsatisfactory, these results still allow us to effectively use the integration theory in conjunction with the ζ-function calculus and prove results like "the integral of a Laurent coefficient of a ζ-function of a family of gauged Fourier Integral Operators is equal to the Laurent coefficient of the ζ-function of the integrated family of gauged Fourier Integral Operators", i.e.

$$\int k^{\text{th}}\text{–Laurent coefficient}\left(\zeta(f(x))\right)dx \ = \ k^{\text{th}}\text{–Laurent coefficient}\left(\zeta\left(\int f\right)\right).$$

This will yield the possibility of considering random manifolds, e.g. a manifold whose metric is subject to random perturbations (for instance, a stochastic process in the space of metrics). As such we have a measurable map $\Omega \ni \omega \mapsto \Delta(\omega)$ where each $\Delta(\omega)$ is a Laplacian on a manifold. Then, we will obtain cases in which the expected heat trace and wave trace coefficients of a random manifold can be expressed as coefficients of the trace of $\mathbb{E}T(t)$ where $\mathbb{E}$ denotes the expectation value (integration in Ω) and T the pointwise heat semi-group $\left(T(t)(\omega) = e^{-t|\Delta(\omega)|}\right)$ or wave group $\left(T(t)(\omega) = W(t)(\omega) = e^{it\sqrt{|\Delta(\omega)|}}\right)$, respectively. In other words, for the heat semi-group we find

$$\operatorname{tr}\mathbb{E}e^{-t|\Delta|} = \frac{\mathbb{E}\ \mathrm{vol}(M)}{(4\pi t)^{\frac{\dim M}{2}}} + \frac{\mathbb{E}\ \text{total curvature}(M)}{3(4\pi)^{\frac{\dim M}{2}}t^{\frac{\dim M}{2}-1}} + \text{ higher order terms}$$

under certain conditions on the random manifold. In particular, we can show that $\mathbb{E}e^{-t|\Delta|}$ is a smoothing operator for $t \in \mathbb{R}_{>0}$ and

$$\operatorname{tr}\mathbb{E}e^{-t|\Delta|} = \mathbb{E}\operatorname{tr}e^{-t|\Delta|}$$

holds, for instance, if $\omega \mapsto e^{-t|\Delta(\omega)|}$ is Pettis integrable. Similarly, we obtain

$$\mathbb{E}\left(\zeta(W(t)g)(0)\right) = \zeta\left(\mathbb{E}(W(t))g\right)(0)$$

where g is a gauge (the result is independent of the particular choice of g), though we will need stronger assumptions in this case.

Example Let $\Gamma(\omega) = \times_{j=1}^{N} f_j(\omega)\mathbb{Z} \subseteq \mathbb{R}^N$ (that is, $\mathbb{R}^N/_{\Gamma(\omega)}$ has fundamental domain $\times_{j=1}^{N}[0, f_j(\omega)]$) where the f_j are positive and bounded measurable functions on a probability space (not necessarily independent). Let $\Delta(\omega)$ be the Laplacian on

$\mathbb{R}^N/_{\Gamma(\omega)}$ and $(T(t)(\omega))_{t\in\mathbb{R}_{\geq 0}}$ the heat-semigroup. Then, it can be shown (cf. e.g. chapter 5)

$$\operatorname{tr} T(t)(\omega) = \frac{\operatorname{vol}\left(\mathbb{R}^N/_{\Gamma(\omega)}\right)}{(4\pi t)^{\frac{N}{2}}} \sum_{\gamma\in\Gamma(\omega)} \exp\left(-\frac{\|\gamma\|^2_{\ell_2(N)}}{4t}\right)$$

which can also be written as

$$\operatorname{tr} T(t)(\omega) = \frac{\operatorname{vol}\left(\mathbb{R}^N/_{\Gamma(\omega)}\right)}{(4\pi t)^{\frac{N}{2}}} \sum_{\nu\in\mathbb{Z}^N} \exp\left(-\frac{\|\gamma_\nu(\omega)\|^2_{\ell_2(N)}}{4t}\right)$$

with $\gamma_\nu(\omega) = \sum_{j=1}^N \nu_j f_j(\omega) e_j$ where $(e_j)_{j\in\mathbb{N}_{\leq N}}$ is the canonical basis of $\mathbb{R}^N$. In other words, $\mathbb{E}\operatorname{vol}\left(\mathbb{R}^N/_\Gamma\right)$ is given by the $\nu = 0$ term of the series on the right hand side.

Furthermore, the kernel $\kappa_{\mathbb{E}T(t)}$ of $\mathbb{E}T(t)$ parametrized over $[0,1]^N$ is

$$\begin{aligned}\kappa_{\mathbb{E}T(t)}(x,y) =& \mathbb{E}\left(\sum_{\nu\in\mathbb{Z}^N}\int_{\mathbb{R}^N} e^{i\langle x-y-\gamma_\nu,\xi\rangle_{\mathbb{R}^N}} (2\pi)^{-N} e^{-t\|\xi\|^2_{\ell_2(N)}} d\xi \prod_{j=1}^N f_j\right)\\ =& \sum_{\nu\in\mathbb{Z}^N}\int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}}\, \mathbb{E}\left(e^{-i\langle\gamma_\nu,\xi\rangle_{\mathbb{R}^N}} \prod_{j=1}^N f_j\right)(2\pi)^{-N} e^{-t\|\xi\|^2_{\ell_2(N)}} d\xi,\end{aligned}$$

i.e.

$$\begin{aligned}\operatorname{tr}\mathbb{E}T(t) &= \int_{[0,1]^N}\sum_{\nu\in\mathbb{Z}^N}\int_{\mathbb{R}^N}\mathbb{E}\left(e^{-i\langle\gamma_\nu,\xi\rangle_{\mathbb{R}^N}}\prod_{j=1}^N f_j\right)(2\pi)^{-N}e^{-t\|\xi\|^2_{\ell_2(N)}}d\xi dx\\ &= \sum_{\nu\in\mathbb{Z}^N}\int_{\mathbb{R}^N}\mathbb{E}\left(\prod_{j=1}^N f_j e^{-i\nu_j\xi_j f_j}\right)(2\pi)^{-N}e^{-t\|\xi\|^2_{\ell_2(N)}}d\xi.\end{aligned}$$

Here, the $\nu = 0$ term yields

$$\int_{\mathbb{R}^N}\mathbb{E}\left(\prod_{j=1}^N f_j\right)(2\pi)^{-N}e^{-t\|\xi\|^2_{\ell_2(N)}}d\xi = \frac{\mathbb{E}\left(\prod_{j=1}^N f_j\right)}{(4\pi t)^{\frac{N}{2}}}.$$

In other words,

$$\mathbb{E}\operatorname{vol}\left(\mathbb{R}^N/_\Gamma\right) = \mathbb{E}\left(\prod_{j=1}^N f_j\right)$$

which is fully consistent with the trivial calculation

$$\mathbb{E}\operatorname{vol}\left(\mathbb{R}^N/_\Gamma\right) = \mathbb{E}\operatorname{vol}\left(\bigtimes_{j=1}^N [0,f_j]\right) = \mathbb{E}\left(\prod_{j=1}^N f_j\right).$$

■

The thesis also contains three appendices. Appendices B and C are mainly background information. Since we will be using the gap-topology on multiple occasions, appendix B contains an overview of the gap-topology and results on the perturbation of the spectrum with respect to the gap-topology. In appendix C, we introduce and prove the necessary theorem to prove that perturbed eigenvalues of an operator with respect to the gap-topology can be written as a Puiseux series.

Appendix A, on the other hand, covers the basic theorems of classical probability theory in algebras of Fourier Integral Operators. This is particularly interesting since the integration theory developed in part II and its application to ζ-functions give rise to the idea of treating more geometrical stochastic Fourier Integral Operator questions in this formalism rather than introducing the entire machinery of Colombeau algebras. Hence, we would like to make sure that such a probability theory in algebras of Fourier Integral Operators is sufficiently rich. In fact, appendix A contains most major theorems one would expect to encounter in an introduction to stochastics, including versions of the strong and weak law of large numbers and a Lindeberg type central limit theorem.

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Part I

The Laurent expansion of Fourier Integral Operator ζ-functions and a generalized Kontsevich-Vishik trace

CHAPTER 1

Fourier Integral Operators of trace-class

We will begin this chapter with a short account on algebras of Fourier Integral Operators associated with canonical relations. For details and proofs, please, refer to chapter 25 in [**38**], chapters 2 and 4 in [**20**], as well as [**39**].

Unless explicitly stated otherwise, let X be an orientable, compact, connected, finite dimensional Riemannian manifold without boundary and $T_0^*X := T^*X \setminus 0$ the co-tangent bundle without the zero-section.

Definition 1.1. *Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a relation satisfying*

(i) *Γ is symmetric, i.e. $\forall (p,q) \in \Gamma: \ (q,p) \in \Gamma$,*

(ii) *Γ is transitive, i.e. $\forall (p,q),(q,r) \in \Gamma: \ (p,r) \in \Gamma$,*

We will call any such Γ a canonical relation. Furthermore, we will assume that all canonical relations satisfy

(iii) *the composition $\Gamma \circ \Gamma$ is clean, i.e. $\Gamma \times \Gamma$ intersects $T^*X \times \operatorname{diag}(T^*X \times T^*X) \times T^*X$ in a manifold whose tangent plane is precisely the intersection of the tangent planes of $\Gamma \times \Gamma$ and $T^*X \times \operatorname{diag}(T^*X \times T^*X) \times T^*X$ where $\operatorname{diag}(T^*X \times T^*X) := \{(x,y) \in T^*X \times T^*X; \ x = y\}$,*

(iv) *the projection $\mathrm{pr}_1 : \ \Gamma \to T^*X; \ (p,q) \mapsto p$ is proper, i.e. pre-sets of compacta are compact.*

We will call the set

$$\Gamma' := \{((x,\xi),(y,\eta)) \in T_0^*X \times T_0^*X; \ ((x,\xi),(y,-\eta)) \in \Gamma\}$$

a twisted canonical relation.

REMARK The properties (iii) and (iv) will imply that the set of Fourier Integral Operators we will associate with these canonical relations form an associative algebra.

■

DEFINITION 1.2. *Let $N \in \mathbb{N}$. A function*

$$\vartheta \in C\left(X \times X \times \mathbb{R}^N\right) \cap C^\infty\left(X \times X \times \left(\mathbb{R}^N \smallsetminus \{0\}\right)\right)$$

is called a phase function if and only if it is positively homogeneous of degree 1 in the third argument, i.e.

$$\forall x, y \in X \;\; \forall \xi \in \mathbb{R}^N \;\; \forall \lambda \in \mathbb{R}_{>0} : \;\; \vartheta(x, y, \lambda\xi) = \lambda\vartheta(x, y\xi).$$

DEFINITION 1.3. *Let $U \subseteq \mathbb{R}^n$ be open, $N \in \mathbb{N}$, and $m \in \mathbb{R}$. The Hörmander class $S^m(U \times U \times \mathbb{R}^N)$ is defined as the set of all $a \in C^\infty(U \times U \times \mathbb{R}^N)$ such that for every $K \subseteq_{\text{compact}} U^2$ and all multi-indices α, β, γ there exists a constant $c \in \mathbb{R}_{>0}$ such that*

$$\forall (x, y) \in K \;\; \forall \xi \in \mathbb{R}^N \smallsetminus B_{\mathbb{R}^N}(0, 1) : \;\; \left|\partial_1^\alpha \partial_2^\beta \partial_3^\gamma a(x, y, \xi)\right| \le c\left(1 + \|\xi\|_{\ell_2(N)}\right)^{m - \|\gamma\|_{\ell_1(N)}}$$

holds.

DEFINITION 1.4. *A Fourier Integral Operator on X is a linear operator*

$$A : \;\; C_c^\infty(X) \to C_c^\infty(X)'$$

whose Schwartz kernel $k \in C_c^\infty(X \times X)'$ is a locally finite sum of local representations of the form

$$k(x, y) = \int_{\mathbb{R}^N} e^{i\vartheta(x, y, \xi)} a(x, y, \xi) d\xi,$$

i.e.

$$\forall \varphi, \psi \in C_c^\infty(X): \; A(\varphi)\psi = \sum_{i=1}^{n} \int_{X^2} k_i(x,y)\varphi(y)\psi(x) d\mathrm{vol}_{X^2}(x,y),$$

where, for each localization $U \subseteq X$, ϑ is a phase function and a is an element of some Hörmander class $S^m(U \times U \times \mathbb{R}^N)$. a is also called an amplitude or symbol.

Definition 1.5. *A Fourier Integral Operator A whose Schwartz kernel $k \in C_c^\infty(X \times X)'$ can be written in the form*

$$k(x,y) = \int_{\mathbb{R}^{\dim X}} e^{i\langle x-y,\xi\rangle_{\ell_2(\dim X)}} a(x,y,\xi) d\xi$$

is called a pseudo-differential operator.

Remark It is also possible to consider (truly) globally defined Fourier Integral Operators (cf. e.g. [**49**, **50**, **62**]). However, we will not only want to work with Fourier Integral Operators, but specifically gauged[1] Fourier Integral Operators. While gauging locally is easy (by replacing the amplitude a with the family $\hat{a}(z)(x,y,\xi) = \|\xi\|_{\ell_2(N)}^z a(x,y,\xi)$, for instance) and can be very advantageous (cf. $\mathcal{M}$-gauges; Definition 2.10 and Corollary 2.11), finding and working with global gauges is much more difficult (though the rewards may be worth it). Hence, we will assume the more general stance and allow gauged Fourier Integral Operators to have kernels which are not given by globally defined densities.

Incidentally, this also implies that most of our calculations can be performed locally. In other words, all integrals over the underlying manifold X are to be understood as locally finite sums of integrals with respect to the respective charts.

[1] The notion of gauged Fourier Integral Operators will be defined via the notion of gauged poly-log-homogeneous distributions in chapter 2 and their application to gauged Lagrangian distributions in chapter 4. More precisely, a family of Fourier Integral Operators is gauged if and only if it corresponds to a gauged poly-log-homogeneous distribution.

In particular, since we will generally assume X to be compact, these locally finite sums are, in fact, finite.

■

DEFINITION 1.6. *Let* $\vartheta \in C(X \times X \times \mathbb{R}^N) \cap C^\infty(X \times X \times (\mathbb{R}^N \smallsetminus \{0\}))$ *be a phase function. Then, we call*

$$C(\vartheta) := \left\{(x,y,\xi) \in X \times X \times \left(\mathbb{R}^N \smallsetminus \{0\}\right);\ \partial_3\vartheta(x,y,\xi) = 0\right\}$$

the critical set of ϑ.

ϑ *is called non-degenerate if and only of the family of differentials*

$$(d\partial_{3,j}\vartheta(x,y,\xi))_{j\in\mathbb{N}_{\le N}}$$

is linearly independent for every $(x,y,\xi) \in C(\vartheta)$ *where* $\partial_{3,j}$ *denotes the derivative with respect to the* j^{th} *component of the third argument.*

REMARK Note that the singular support, that is, the complement of the largest open set on which a distribution is C^∞, of

$$C_c^\infty(X^2) \ni \varphi \mapsto \int_{X^2}\int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)\varphi(x,y)d\xi \mathrm{dvol}_{X^2}(x,y) \in \mathbb{C}$$

is contained in the image of $C(\vartheta) \ni (x,y,\xi) \mapsto (x,y)$ and the non-degeneracy condition implies that $C(\vartheta)$ is a manifold of dimension $2\dim X$ (cf. (2.3.11) in [**20**]).

■

A closely related concept of "nice points" is the notion of regular directed points (cf. page 92 in [**61**]) and the wave front set.

DEFINITION 1.7. (i) *Let* $u \in C_c^\infty(\mathbb{R}^n)'$. *A point* $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \smallsetminus \{0\})$ *is called a regular directed point for* u *if and only if there exist neighborhoods*

U of x and V of ξ, as well as $g \in C_c^\infty(\mathbb{R}^n)$ with $g|_U = 1$, such that

$$\forall m \in \mathbb{R}_{>0}\ \exists c \in \mathbb{R}_{>0}\ \forall p \in V\ \forall \lambda \in \mathbb{R}_{\geq 0}:\ |\mathcal{F}(gu)(\lambda p)| \leq c\left(1 + \|\lambda\|_{\ell_2(n)}\right)^{-m}$$

where $\mathcal{F}$ denotes the Fourier transform.

(ii) *Let $u \in C_c^\infty(X)'$. A point $(x,\xi) \in T_0^*X$ is called a regular directed point for u if and only if (x,ξ) is a regular directed point with respect to a chart.*

(iii) *Let $u \in C_c^\infty(X)'$. Then, we define the wave front set $WF(u)$ of u as*

$$WF(u) := \{(x,\xi) \in T_0^*X;\ (x,\xi) \text{ is not a regular directed point for } u\}.$$

(iv) *Let $\Gamma \subseteq T^*X \times T^*X$ be a closed cone. Then, we define the Hörmander space*

$$\mathcal{D}'_\Gamma := \{v \in C_c^\infty(X)';\ WF(v) \subseteq \Gamma\}.$$

Remark (i) In other words, a point (x,ξ) is a regular directed point if the localization of the distribution near x has a Fourier transform which approaches zero faster than any polynomial in an open cone containing ξ.

(ii) Hörmander defined the spaces $\mathcal{D}'_\Gamma$ with a pseudo-topology, that is, he defined what convergent sequences and their limits are in these spaces. In general, this does not imply that there is an actual topology consistent with a pseudo-topology. In this case, however, there are multiple "natural" topologies on the Hörmander spaces $\mathcal{D}'_\Gamma$, i.e. the pseudo-topology is generated by multiple different topologies. These have been studied in [**15**–**17**].

We will have a brief look at these topologies in chapter 12 and 13; though it should be noted that the "natural" topologies are at least quasi-complete (cf. Proposition 29 in [**17**]), i.e. they are sufficiently nice for us to talk about Pettis-integrability and we do not need to dive into their topological properties too deeply.

■

DEFINITION 1.8. *Let $\Lambda \subseteq T^*(X^2)\setminus 0$ be a Lagrangian manifold*[2] *and A a Fourier Integral Operator of the form $A = \sum_{j=1}^n A_j$ where A_j has the kernel*

$$\int_{\mathbb{R}^{N_j}} e^{i\vartheta_j(x,y,\xi)} a_j(x,y,\xi) d\xi$$

such that each phase function ϑ_j is non-degenerate and defined in an open, conic subset $U_j \subseteq_{\text{open}} X \times X \times (\mathbb{R}^{N_j} \setminus \{0\})$,

$$U_j \cap C(\vartheta) \ni (x,y,\xi) \mapsto (x,y,\partial_1\vartheta(x,y,\xi),\partial_2\vartheta(x,y,\xi))$$

is a diffeomorphism onto an open subset $U_j^\Lambda \subseteq_{\text{open}} \Lambda$,[3] *and $a_j \in S^{m+\frac{\dim X - N_j}{2}}(X \times X \times \mathbb{R}^{N_j})$ with*

$$\operatorname{spt} a_j \subseteq \{(x,y,t\xi) \in X \times X \times \mathbb{R}^{N_j};\ (x,y,\xi) \in K \ \wedge \ t \in \mathbb{R}_{>0}\}$$

for some $K \subseteq_{\text{compact}} U_j$. Then, we say A is an element of $I^m(X \times X, \Lambda)$ (or more precisely, A has a kernel in $I^m(X \times X, \Lambda)$).

Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a canonical relation such that Γ' is a Lagrangian manifold. Let σ be the canonical 2-form in T^*X, then, Γ' being a Lagrangian manifold in $T^*X \times T^*X$ with respect to $\sigma \otimes \sigma$ is equivalent to Γ being a Lagrangian manifold

[2]that is, Λ is a symplectic sub-manifold of dimension $\dim X$ which is, furthermore, isotropic, i.e. the symplectic form restricts to zero. A manifold is called symplectic if it is equipped with a closed non-degenerate 2-form. A bi-linear form w on a finite dimensional vector space V is called non-degenerate if and only if $V \ni y \mapsto (x \mapsto w(x,y)) \in V'$ is an isomorphism.

[3]Note that the image of $C(\vartheta) \ni (x,y,\xi) \mapsto (x,y,-\partial_1\vartheta(x,y,\xi),-\partial_2\vartheta(x,y,\xi))$ contains the wave front set of the kernel of A; cf. Theorem 24 in [**9**].

in $T^*X \times T^*X$ with respect to $\sigma \otimes (-\sigma)$. If Γ is (the graph of) a $C^\infty(T^*X, T^*X)$-function, Γ being conic means that Γ is homogeneous of degree one. (see also chapter 4.2 in [**20**])

Definition 1.9. *Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a canonical relation. Γ is called a homogeneous canonical relation if and only if Γ is a Lagrangian manifold with respect to $\sigma \otimes (-\sigma)$.*

Definition 1.10. *Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a homogeneous canonical relation. Then, we call*

$$\mathcal{A}_\Gamma := \bigcup_{m \in \mathbb{R}} I^m(X \times X, \Gamma')$$

the algebra of Fourier Integral Operators associated with Γ.

Remark Aside from the fact that one might relax the conditions from $a_j = 0$ outside of $\{(x, y, t\xi) \in X \times X \times \mathbb{R}^{N_j};\ (x, y, \xi) \in K \ \wedge\ t \in \mathbb{R}_{>0}\}$ in Definition 1.8 to $a_j \in S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m$, all the assumptions above are more or less necessary for $\mathcal{A}_\Gamma$ to form an associative algebra; cf. Theorem 2.4.1 in [**20**] and Example 1 in [**35**].

It should also be noted that $A \in \mathcal{A}_\Gamma$ implies $k_A \in \mathcal{D}'_\Gamma$ if k_A is the Schwartz kernel of A (cf. Theorem 2.4.1 in [**20**]).

■

Definition 1.11. *Let $\Gamma \subseteq T_0^*X \times T_0^*X$ be a homogeneous canonical relation. Then, we call Γ canonically idempotent if and only if $\mathrm{pr}_2 :\ \Gamma \to T^*X;\ (p, q) \mapsto q$ is proper (pre-sets of compacta are compact), $\mathrm{pr}_2[\Gamma] \subseteq T^*X$ is an embedded submanifold, and $\mathrm{pr}_2 :\ \Gamma \to T^*X$ is a fibration[4] of Γ over $\mathrm{pr}_2[\Gamma]$.*

[4]A fibration is a continuous map $\pi :\ X \to Y$ between topological spaces X and Y satisfying the homotopy lifting property for every topological space Z, i.e. for any homotopy $f :\ Z \times [0, 1] \to Y$ and $f_0 :\ Z \to X$ such that $f(\cdot, 0) = \pi \circ f_0$ there exists a homotopy $\tilde{f} :\ Z \times [0, 1] \to X$ such that $f = \pi \circ \tilde{f}$ and $f_0 = \tilde{f}(\cdot, 0)$.

REMARK Note that Γ being canonically idempotent implies that $\mathcal{A}_\Gamma$ is a $\star$-algebra; cf. Definition 3.1 in [**35**] and Theorem 4.2.1 in [**39**].

■

LEMMA 1.12. *Let A be a Fourier Integral Operator with kernel $k \in I^m(X \times X, \Lambda)$. If $m < -\dim X$, then A is of trace-class.*

PROOF. Theorem 1.1 in [**19**] states that $A \in L(L_2(X))$ is of trace-class if k is in the Sobolev space $W_2^s(X \times X)$ for some $s > \frac{\dim X}{2}$. Furthermore, Theorem 4.4.7 in [**20**] implies $I^m(X \times X, \Lambda) \subseteq W_2^s(X \times X)$ provided $m < -\frac{\dim X}{2} - s$ (we assume that X is compact). In other words, $m < -\dim X$ implies $I^m(X \times X, \Lambda) \subseteq W_2^s(X \times X)$ for some $s > \frac{\dim X}{2}$ and, hence, the assertion.

□

In terms of the amplitude $a \in S^m(X \times X \times \mathbb{R}^N)$, the value $m = -N$ is critical since for $m < -N$ the trace integral (cf. Lemma 1.15) is well-defined. This follows from the fact that the kernel k is in $C^l(X \times X)$ provided that $m < -N - l$; cf. equation (2.6) in [**34**]. However, we will only need continuity here since $k \in C(X \times X)$ implies $k \in L_2(X \times X)$ because X is compact. This is interesting in its own right because integral operators in $L(L_2(X))$ are Hilbert-Schmidt if and only if their kernels are in $L_2(X \times X)$; cf. e.g. Example 11.12 in [**18**].

LEMMA 1.13. *Let*

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi$$

be a localization of the Schwartz kernel of an $A \in \mathcal{A}_\Gamma$ with $a \in S^m(U \times \mathbb{R}^N)$ for some $m < -N$ and $U \subseteq_{\text{open}} X^2$. Then, $k \in C(U)$.

PROOF. Let $((x_j, y_j))_{j\in\mathbb{N}} \in U^{\mathbb{N}}$, $(x_j, y_j) \to: (x,y) \in U$, and $\forall j \in \mathbb{N}: \ a_j := e^{i\vartheta(x_j,y_j,\cdot)} a(x_j, y_j, \cdot)$. By compactness of X and definition of S^m, there exists a measurable $v:\ \mathbb{R}^N \to \mathbb{R}$ such that

$$\forall (x,y) \in U\ \ \forall \xi \in \mathbb{R}^N:\ |a(x,y,\xi)| \le v(\xi)$$

and

$$\exists c \in \mathbb{R}_{>0}\ \ \forall \xi \in \mathbb{R}^N \setminus B_{\mathbb{R}^N}(0,1):\ v(\xi) \le c\left(1 + \|\xi\|_{\ell_2(N)}\right)^m.$$

In other words, $v \in L_1(\mathbb{R}^N)$ and $(a_j)_{j\in\mathbb{N}}$ is bounded by v. Furthermore, $(a_j)_{j\in\mathbb{N}}$ converges pointwise to $e^{i\vartheta(x,y,\cdot)}a(x,y,\cdot)$ and (by L_p-dominated convergence; cf. e.g. Theorem 12.9 in [**65**]) in L_1, as well. Hence,

$$e^{i\vartheta}a \in C\left(U, L_1\left(\mathbb{R}^N\right)\right).$$

Using the Fourier transform $\mathcal{F}$ and the Dirac distribution δ_0 at zero, we obtain

$$\begin{aligned}
k(x,y) &= \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi \\
&=(2\pi)^{\frac{N}{2}}(2\pi)^{-\frac{N}{2}}\int_{\mathbb{R}^N} e^{-i\langle 0,\xi\rangle_{\mathbb{R}^N}}e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi \\
&=(2\pi)^{\frac{N}{2}}\mathcal{F}\left(e^{i\vartheta(x,y,\cdot)}a(x,y,\cdot)\right)(0) \\
&=(2\pi)^{\frac{N}{2}}\left(\delta_0 \circ \mathcal{F}\right)\left(e^{i\vartheta(x,y,\cdot)}a(x,y,\cdot)\right).
\end{aligned}$$

Since $(2\pi)^{\frac{N}{2}}(\delta_0 \circ \mathcal{F})$ is a continuous linear functional on $L_1(\mathbb{R}^N)$ and $e^{i\vartheta}a$ depends continuously on $(x,y) \in U$, we obtain the assertion.

□

Definition 1.14. *Let $\mathcal{A}$ be an algebra of Fourier Integral Operators on X. Then, we call the sub-algebra*

$$\mathcal{A}^{\text{classical}} := \{A \in \mathcal{A};\ A \textit{ is of trace-class and has continuous Schwartz kernel}\}$$

the classical sub-algebra of $\mathcal{A}$.

LEMMA 1.15. *Let $\mathcal{A}$ be an algebra of Fourier Integral Operators and $A \in \mathcal{A}^{\text{classical}}$. Then,*

$$\operatorname{tr} A = \int_X k_A(x,x) d\mathrm{vol}_X(x).$$

PROOF. The integral $\int_X k_A(x,x) d\mathrm{vol}_X(x)$ is well-defined since k_A is continuous and X compact. Let $(e_\iota)_{\iota\in I}$ an orthonormal basis of $L_2(X)$. Then,

$$\forall \iota, i \in I: \ \psi_{\iota,i}: \ X^2 \to \mathbb{C}, \ (x,y) \mapsto e_\iota(x) e_i(y)^* \qquad \mathrm{vol}_{X^2}\text{-almost everywhere}$$

defines an orthonormal basis $(\psi_{\iota,i})_{\iota,i\in I}$ in $L_2(X^2)$. In particular,

$$k_A = \sum_{\iota,i\in I} \alpha_{\iota,i} \psi_{\iota,i}$$

converges in $L_2(X^2)$ and, using a Friedrichs' mollifier[5] $\varphi_\varepsilon \to \delta_{\text{diag}}$ $(\varepsilon \searrow 0)$ on X^2 where

$$\forall \varphi \in C_c^\infty(X^2): \ \delta_{\text{diag}}(\varphi) = \int_X \varphi(x,x) d\mathrm{vol}_X(x),$$

we obtain

$$\begin{aligned}
\int_X k_A(x,x) d\mathrm{vol}_X(x) &= \lim_{\varepsilon\searrow 0} \int_{X^2} k_A \varphi_\varepsilon d\mathrm{vol}_{X^2} \\
&= \lim_{\varepsilon\searrow 0} \int_{X^2} \sum_{i,j\in I} \alpha_{i,j} \psi_{i,j} \varphi_\varepsilon d\mathrm{vol}_{X^2} \\
&= \lim_{\varepsilon\searrow 0} \sum_{i,j\in I} \alpha_{i,j} \int_{X^2} \psi_{i,j} \varphi_\varepsilon d\mathrm{vol}_{X^2} \\
&= \sum_{i,j\in I} \alpha_{i,j} \int_X e_i(x) e_j(x)^* d\mathrm{vol}_X(x) \\
&= \sum_{j\in I} \alpha_{j,j} \int_X e_j(x) e_j(x)^* d\mathrm{vol}_X(x) \\
&= \sum_{j\in I} \alpha_{j,j}
\end{aligned}$$

[5] $\varphi_\varepsilon = \chi_\varepsilon * \delta_{\text{diag}}$ in local trivializations for some family $(\chi_\varepsilon)_{\varepsilon\in(0,1)}$ satisfying that there exists a $\chi \in C_c^\infty(\mathbb{R}^{2\dim X})$ with $\int_{\mathbb{R}^{2\dim X}} \chi(x)dx = 1$, $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(\varepsilon^{-1}x)$ and $\lim_{\varepsilon\searrow 0} \chi_\varepsilon = \delta_0$ in the sense of distributions.

$$
\begin{aligned}
&= \sum_{\iota,i,j\in I} \alpha_{\iota,i} \int_X e_\iota(x) e_j(x)^* d\mathrm{vol}_X(x) \int_X e_j(y) e_i(y)^* d\mathrm{vol}_X(x) \\
&= \sum_{\iota,i,j\in I} \int_{X^2} \alpha_{\iota,i} e_\iota(x) e_i(y)^* e_j(y) e_j(x)^* d\mathrm{vol}_{X^2}(x,y) \\
&= \sum_{j\in I} \int_{X^2} k_A(x,y) e_j(y) e_j(x)^* d\mathrm{vol}_{X^2}(x,y) \\
&= \sum_{j\in I} \left\langle e_j, \int_X k_A(\cdot,y) e_j(y) d\mathrm{vol}_X(y) \right\rangle_{L_2(X)} \\
&= \sum_{j\in I} \langle e_j, A e_j \rangle_{L_2(X)} \\
&= \operatorname{tr} A
\end{aligned}
$$

since $\sum_{j\in I} \langle e_j, Ae_j\rangle_{L_2(X)}$ is absolutely convergent.

□

Hence, decreasing the order of a Fourier Integral Operator sufficiently yields a trace-class operator. Thus, the idea is to replace a Fourier Integral Operator by a holomorphic family of Fourier Integral Operators such that the family maps into the trace-class operators for some open subset of the domain of holomorphy (which is assumed to be connected). In chapter 2, however, we will consider a different class of families of distributions which will turn out to be suitable to treat certain algebras $\mathcal{A}_\Gamma$.

CHAPTER 2

Gauged poly-log-homogeneous distributions

In this chapter, we consider distributions of the form

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}(\xi)$$

where M is an orientable,[1] compact, finite dimensional manifold without boundary and α is a holomorphic family given by an expansion[2]

$$\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$$

where $I \subseteq \mathbb{N}$, $\alpha_0(z) \in L_1(\mathbb{R}_{\geq 1}\times M)$ in an open neighborhood of $\{z\in\mathbb{C};\ \Re(z)\leq 0\}$ and each of the $\alpha_\iota(z)$ is log-homogeneous with degree of homogeneity $d_\iota + z \in \mathbb{C}$ and logarithmic order $l_\iota \in \mathbb{N}_0$, that is,

$$\exists \tilde{\alpha}_\iota \in C^\infty(\mathbb{C},\mathbb{C}^M)\ \forall r\in\mathbb{R}_{\geq 1}\ \forall \nu\in M:\ \alpha_\iota(z)(r,\nu) = r^{d_\iota+z}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\nu).$$

We will furthermore assume the following.

[1] Replacing $\alpha(z)(r,\xi)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}(r,\xi)$ by some family $d\omega(z)(r,\xi)$ allows us to also treat non-orientable manifolds but we will not need this in the following and choose orientability for the sake of simplicity.

[2] This is not meant to be an asymptotic expansion but an actual identity. However, for a classical symbol a with asymptotic expansion $\sum_{j\in\mathbb{N}} a_j$ where a_j is homogeneous of degree $m-j$ for some $m\in\mathbb{C}$, it is possible to choose a finite set $I = \{0,1,\ldots,J\}$ and α_0 will correspond to $a - \sum_{j=0}^{J} a_{m-j}$.

This is completely analogous to the Kontsevich-Vishik trace, i.e. splitting off finitely many terms with large degrees of homogeneity while the rest is integrable. The only difference is that those terms (that have been split off) might not regularize to zero anymore.

- The family $(\Re(d_\iota))_{\iota\in I}$ is bounded from above. (Note, we do not require $\Re(d_\iota) \to -\infty$. $\forall \iota \in I:\ \Re(d_\iota) = 42$ is entirely possible.)
- The map $I \ni \iota \mapsto (d_\iota, l_\iota)$ is injective.
- There are only finitely many ι satisfying $d_\iota = d$ for any given $d \in \mathbb{C}$.
- The family $((d_\iota - \delta)^{-1})_{\iota\in I}$ is in $\ell_2(I)$ for any $\delta \in \mathbb{C} \setminus \{d_\iota;\ \iota \in I\}$.
- Each $\sum_{\iota\in I} \tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$.[3]

Any such family α will be called a gauged poly-log-homogeneous distribution. Note that the generic case (that is, applications to Fourier Integral Operators with amplitudes of the form $a \sim \sum_{j\in\mathbb{N}_0} a_{m-j}$) implies that I is a finite set and all these conditions are, therefore, satisfied.

Example Let $A(z)$ be a pseudo-differential operator on an N-dimensional manifold X whose amplitude has an asymptotic expansion $a(z) \sim \sum_{j\in\mathbb{N}} a_j(z)$ where each $a_j(z)$ is homogeneous of degree $m - j + z$. Then, we may want to evaluate the meromorphic extension of

$$\begin{aligned}
\operatorname{tr} A(z) &= \int_X \int_{\mathbb{R}^N} a(z)(x,x,\xi)\, d\xi\, d\mathrm{vol}_X(x) \\
&= \int_X \int_{\mathbb{R}_{\geq 1} \times \partial B_{\mathbb{R}^N}} a(z)(x,x,\xi)\, d\xi\, d\mathrm{vol}_X(x) \\
&\quad + \int_X \int_{B_{\mathbb{R}^N}(0,1)} a(z)(x,x,\xi)\, d\xi\, d\mathrm{vol}_X(x)
\end{aligned}$$

at zero. The poly-log-homogeneous distribution here is

$$(*) \qquad \int_X \int_{\mathbb{R}_{\geq 1} \times \partial B_{\mathbb{R}^N}} a(z)(x,x,\xi)\, d\xi\, d\mathrm{vol}_X(x).$$

[3]Unconditional convergence of $\sum_{\iota\in I} \tilde{\alpha}_\iota(z)$ in $L_1(M)$ may also be replaced by the slightly weaker, though more artificial, condition $\sum_{\iota\in I} \|\tilde{\alpha}_\iota(z)\|^2_{L_1(M)} < \infty$.

However, we need at least conditional convergence or $\sum_{\iota\in I} \alpha_\iota$ would not make sense, and having only conditional convergence (rather than unconditional convergence) would give rise to complications later on, as we will split off critical terms and treat them separately.

At this point, we have many possibilities to write it $(*)$ in the form

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}(\xi).$$

The easiest choice is $M := \partial B_{\mathbb{R}^N}$ and $I := \{j \in \mathbb{N};\ \Re(m) - j \geq -N\}$. This ensures that

$$\int_X a(z)(x,x,\xi) - \sum_{j\in I} a_j(z)(x,x,\xi) d\mathrm{vol}_X(x)$$

is integrable in $\mathbb{R}_{\geq 1} \times \partial B_{\mathbb{R}^N}$. Furthermore, having a finite I ensures that all of the conditions above are satisfied and α can be defined by

$$\alpha_0(z)(r,\nu) := \int_X a(z)(x,x,r\nu) - \sum_{j\in I} a_j(z)(x,x,r\nu) d\mathrm{vol}_X(x)$$

and

$$\alpha_j(z)(r,\nu) := \int_X a_j(z)(x,x,r\nu) d\mathrm{vol}_X(x) = r^{m-j+z} \underbrace{\int_X a_j(z)(x,x,\nu) d\mathrm{vol}_X(x)}_{=:\tilde{\alpha}_j(z)(\nu)}$$

for $j \in I$.

■

Remark Note that these distributions are strongly connected to traces of Fourier Integral Operators, as well. In fact, Guillemin's argument in [**34**] relies heavily on the fact that the inner products $\langle u(z), f\rangle$ at question are integrals of the form

$$\int_{\mathbb{R}_{\geq 1}\times \partial B_{\mathbb{R}^N}} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times \partial B_{\mathbb{R}^N}}(\xi)$$

where α is a gauged polyhomogeneous distribution; cf. equation (2.15) in [**34**].

■

If the conditions above are satisfied, we obtain formally

$$\begin{aligned}\int_{\mathbb{R}_{\geq 1}\times M}\alpha(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} =& \underbrace{\int_{\mathbb{R}_{\geq 1}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}} + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\ =&\tau_0(z) + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}}\int_M \alpha_\iota(z)(\varrho,\nu)\varrho^{\dim M}d\mathrm{vol}_M(\nu)d\varrho\\ =&\tau_0(z) + \sum_{\iota\in I}\underbrace{\int_{\mathbb{R}_{\geq 1}}\varrho^{\dim M+d_\iota+z}\left(\ln\varrho\right)^{l_\iota}d\varrho}_{=:c_\iota(z)}\underbrace{\int_M\tilde{\alpha}_\iota(z)d\mathrm{vol}_M}_{=:\operatorname{res}\alpha_\iota(z)\in\mathbb{C}}\\ =&\tau_0(z) + \sum_{\iota\in I}c_\iota(z)\operatorname{res}\alpha_\iota(z)\end{aligned}$$

which now needs to be justified.

LEMMA 2.1. $c_\iota(z) = (-1)^{l_\iota+1}l_\iota!\left(\dim M + d_\iota + z + 1\right)^{-(l_\iota+1)}$

PROOF. Let Γ_{ui} be the upper incomplete Γ-function given by the meromorphic extension of

$$\Gamma_{ui}(s,x) \coloneqq \int_x^\infty t^{s-1}e^{-t}dt \qquad (\Re(s)>0,\ x\in\mathbb{R}_{\geq 0}).$$

Γ_{ui} satisfies $\Gamma_{ui}(s,0) = \Gamma(s)$ where Γ denotes the (usual) Γ-function, $\Gamma(s,\infty) = 0$, and $\partial_2\Gamma_{ui}(s,x) = -x^{s-1}e^{-x}$. Then, we obtain

$$\begin{aligned}\left(\mathbb{R}_{>0}\ni y\mapsto\frac{-\Gamma_{ui}(l+1,-(d+1)\ln y)}{(-(d+1))^{l+1}}\right)'(x) =&\frac{-\partial_2\Gamma_{ui}(l+1,-(d+1)\ln x)\frac{-(d+1)}{x}}{(-(d+1))^{l+1}}\\ =&\frac{(-(d+1)\ln x)^l e^{(d+1)\ln x}}{(-(d+1))^l x}\\ =&\frac{(\ln x)^l x^{d+1}}{x}\\ =&x^d(\ln x)^l.\end{aligned}$$

Hence, for $d<-1$,

$$\int_{\mathbb{R}_{\geq 1}}x^d(\ln x)^l dx = \frac{(-1)^{l+1}l!}{(d+1)^{l+1}}$$

which yields

$$c_\iota(z) = \int_{\mathbb{R}_{\geq 1}} \varrho^{\dim M + d_\iota + z} (\ln \varrho)^{l_\iota} \, d\varrho = \frac{(-1)^{l_\iota + 1} l_\iota!}{(\dim M + d_\iota + z + 1)^{l_\iota + 1}}$$

in a neighborhood of $\mathbb{R}_{<-\dim M - d_\iota - 1}$ (because any real analytic function can be extended locally to a holomorphic function) and, thence, by meromorphic extension everywhere in $\mathbb{C} \setminus \{-\dim M - d_\iota - z - 1\}$.

□

Since the $\operatorname{res} \alpha_\iota$ are holomorphic functions, we now know that $\sum_{\iota \in I} c_\iota \operatorname{res} \alpha_\iota$ is a meromorphic function with isolated poles only (if it converges), because the assumption $((d_\iota + \delta)^{-1})_{\iota \in I} \in \ell_2(I)$ implies that there may be at most finitely many d_ι in any compact subset of $\mathbb{C}$.

LEMMA 2.2. *For every $z \in \mathbb{C} \setminus \{-\dim M - d_\iota - 1;\ \iota \in I\}$, $\sum_{\iota \in I} c_\iota(z) \operatorname{res} \alpha_\iota(z)$ converges absolutely.*

PROOF. By assumption, $(c_\iota(z))_{\iota \in I} \in \ell_2(I)$ and $\sum_{\iota \in I} \tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$. This allows us to utilize the following theorem.

THEOREM (THEOREM 4.2.1 IN [**43**]) *Let $p \in \mathbb{R}_{\geq 1}$, $q = \begin{cases} 2 & , \ p \in [1,2] \\ p & , \ p \in \mathbb{R}_{>2} \end{cases}$, and $\sum_{j \in \mathbb{N}} x_j$ converges unconditionally in L_p. Then, $\sum_{j \in \mathbb{N}} \|x_j\|_{L_p}^q$ converges.*

Hence,

$$\begin{aligned}
\sum_{\iota \in I} |c_\iota(z) \operatorname{res} \alpha_\iota(z)| &\leq \sum_{\iota \in I} |c_\iota(z)| \, \|\tilde{\alpha}_\iota(z)\|_{L_1(M)} \\
&= \left\| \left(|c_\iota(z)| \, \|\tilde{\alpha}_\iota(z)\|_{L_1(M)} \right)_{\iota \in I} \right\|_{\ell_1(I)} \\
&= \|(|c_\iota(z)|)_{\iota \in I}\|_{\ell_2(I)} \left\| \left(\|\tilde{\alpha}_\iota(z)\|_{L_1(M)} \right)_{\iota \in I} \right\|_{\ell_2(I)} \\
&= \|(c_\iota(z))_{\iota \in I}\|_{\ell_2(I)} \sqrt{\sum_{\iota \in I} \|\tilde{\alpha}_\iota(z)\|_{L_1(M)}^2} < \infty.
\end{aligned}$$

□

DEFINITION 2.3. *Let α be a gauged poly-*log*-homogeneous distribution. Then, we define the ζ-function of α to be the meromorphic extension of*

$$\zeta(\alpha)(z) := \int_{\mathbb{R}_{\geq 1}\times M} \alpha(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M},$$

i.e.

$$\zeta(\alpha)(z) = \tau_0(z) + \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{l_\iota+1}}.$$

Now, that we know $\zeta(\alpha)$ exists as a meromorphic function, we will calculate its Laurent expansion.

DEFINITION 2.4. *Let f be a meromorphic function defined by its Laurent expansion $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ at $z_0 \in \mathbb{C}$ without essential singularity at z_0, that is, $\exists N \in \mathbb{Z}\ \forall n \in \mathbb{Z}_{\leq N}:\ a_n = 0$. Then, we define the order of the initial Laurent coefficient $\mathrm{oilc}_{z_0}(f)$ of f at z_0 to be*

$$\mathrm{oilc}_{z_0}(f) := \min\{n \in \mathbb{Z};\ a_n \neq 0\}$$

and the initial Laurent coefficient $\mathrm{ilc}_{z_0}(f)$ of f at z_0

$$\mathrm{ilc}_{z_0}(f) := a_{\mathrm{oilc}_{z_0}(f)}.$$

LEMMA 2.5. *Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ and $\beta = \beta_0 + \sum_{\iota\in I'} \beta_\iota$ be two gauged poly-*log*-homogeneous distributions with $\alpha(0) = \beta(0)$ and $\operatorname{res}\alpha_j(0) \neq 0$ if l_j is the maximal logarithmic order with $d_j = -\dim M - 1$. Then, $\mathrm{oilc}_0(\zeta(\alpha)) = \mathrm{oilc}_0(\zeta(\beta))$ and $\mathrm{ilc}_0(\zeta(\alpha)) = \mathrm{ilc}_0(\zeta(\beta))$.*

In other words, $\mathrm{oilc}_0(\zeta(\alpha))$ and $\mathrm{ilc}_0(\zeta(\alpha))$ depend on $\alpha(0)$ only and are, thus, independent of the gauge.

PROOF. Since $\alpha(0) = \beta(0)$, we obtain that $z \mapsto \gamma(z) := \frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution again. Furthermore,

$$\operatorname{oilc}_0(\zeta(\gamma)) \geq \min\{\operatorname{oilc}_0(\zeta(\alpha)), \operatorname{oilc}_0(\zeta(\beta))\} =: -l = -l_j - 1$$

holds because each pair (d_ι, l_ι) in the expansion of γ appears in at least one of the expansions of α or β. This implies that $z \mapsto z^l \zeta(\gamma)(z) = z^{l-1}\left(\zeta(\alpha)(z) - \zeta(\beta)(z)\right)$ is holomorphic at zero (equality holds for $\Re(z)$ sufficiently small and, thence, in general by meromorphic extension). Hence, the highest order poles of $\zeta(\alpha)$ and $\zeta(\beta)$ at zero must cancel out which directly implies $\operatorname{oilc}_0(\zeta(\alpha)) = \operatorname{oilc}_0(\zeta(\beta))$ and $\operatorname{ilc}_0(\zeta(\alpha)) = \operatorname{ilc}_0(\zeta(\beta))$.

□

LEMMA 2.6. *Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ and $\beta = \beta_0 + \sum_{\iota\in I'} \beta_\iota$ be two gauged poly-*log*-homogeneous distributions with $\alpha(0) = \beta(0)$ and $\forall \iota \in I \cup I' :\ d_\iota \neq -\dim M - 1$. Then, $\zeta(\alpha)(0) = \zeta(\beta)(0)$.*

PROOF. Again, since $\alpha(0) = \beta(0)$, we obtain that $z \mapsto \gamma(z) := \frac{\alpha(z)-\beta(z)}{z}$ is a gauged poly-log-homogeneous distribution and $\operatorname{oilc}_0(\zeta(\gamma)) \geq 0$. Hence

$$\zeta(\alpha)(0) - \zeta(\beta)(0) = \operatorname{res}_0\left(z \mapsto \frac{\zeta(\alpha)(z) - \zeta(\beta)(z)}{z}\right) = \operatorname{res}_0 \zeta(\gamma) = 0$$

where res_0 denotes the residue of a meromorphic function at zero.

□

DEFINITION 2.7. *Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ be a gauged poly-*log*-homogeneous distribution and $I_{z_0} := \{\iota \in I;\ d_\iota = -\dim M - 1 - z_0\}$. Then, we define*

$$\mathfrak{fp}_{z_0}(\alpha) := \alpha - \sum_{\iota\in I_{z_0}} \alpha_\iota = \alpha_0 + \sum_{\iota\in I\setminus I_{z_0}} \alpha_\iota .$$

COROLLARY 2.8. *$\zeta(\mathfrak{fp}_0\alpha)(0)$ is independent of the chosen gauge.*

DEFINITION 2.9. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution and* $\operatorname{res}\alpha_\iota \neq 0$ *for some $\iota \in I_0$. Then, we say $\zeta(\alpha)$ has a structural singularity at zero.*

REMARK Note that the pole structure of $\zeta(\alpha)$ does not only depend on the $\operatorname{res}\alpha_\iota$ but also on derivatives of α. A structural singularity is a property of $\alpha(0)$ in the sense that it cannot be removed under change of gauge. More precisely, choosing β such that $\alpha(0) = \beta(0)$ does not imply that the principal part of the Laurent expansion of $\zeta(\alpha) - \zeta(\beta)$ vanishes. However, if all $\operatorname{res}\alpha_\iota$ vanish ($\iota \in I_0$), then there exists a β with $\alpha(0) = \beta(0)$ such that $\zeta(\beta)$ is holomorphic in a neighborhood of zero (e.g. β being $\mathcal{M}$-gauged; see below). Having a non-vanishing $\operatorname{res}\alpha_\iota$ for some $\iota \in I_0$, on the other hand, implies that every $\zeta(\beta)$ with $\alpha(0) = \beta(0)$ has a pole at zero.

■

DEFINITION 2.10. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution. If all $\tilde\alpha_\iota$ are independent of the complex argument, i.e. $\alpha_\iota(z)(r,\nu) = r^{d_\iota+z}(\ln r)^{l_\iota}\tilde\alpha_\iota(0)(\nu) = r^z\alpha_\iota(0)(r,\nu)$, then we call this choice of gauge an $\mathcal{M}$-gauge (or Mellin-gauge).*

REMARK The $\mathcal{M}$-gauge for Fourier Integral Operators can always be chosen locally.

■

COROLLARY 2.11. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution.*

(i) *If α is $\mathcal{M}$-gauged, then all* $\operatorname{res}\alpha_\iota$ *are constants.*

(ii) *If* $\operatorname{res}\alpha_\iota(0) = 0$ *for some $\iota \in I$, then the corresponding pole in $\zeta(\alpha)$ can be removed by re-gauging.*

(iii) *If* $\operatorname{res}\alpha_\iota(0) \neq 0$ *for some* $\iota \in I_0$*, then the corresponding pole in* $\zeta(\alpha)$ *in independent from the gauge. In particular,* $\operatorname{res}\alpha_\iota(0)$ *does not depend on the gauge.*

Proof. (i) trivial.

(ii) The corresponding pole contributes the term $\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{l_\iota+1}}$ to the expansion of $\zeta(\alpha)$. Choosing an $\mathcal{M}$-gauge yields

$$\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{l_\iota+1}} = \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(0)}{(\dim M + d_\iota + z + 1)^{l_\iota+1}} = 0$$

by holomorphic extension.

(iii) Lemma 2.5 shows that $\operatorname{oilc}_0\zeta(\alpha_\iota)$ and $\operatorname{ilc}_0(\zeta(\alpha_\iota))$ are independent of the gauge. Since, $\operatorname{res}\alpha_\iota(0) \neq 0$, we obtain $\operatorname{oilc}_0\zeta(\alpha_\iota) = -l_\iota - 1$ and

$$\operatorname{res}\alpha_\iota(0) = \frac{\operatorname{ilc}_0\zeta(\alpha_\iota)}{(-1)^{l_\iota+1} l_\iota!}.$$

□

Remark Suppose we have a gauged distribution α such that

$$\forall z \in \mathbb{C}\ \forall (r,\xi) \in \mathbb{R}_{\geq 1} \times M:\ \alpha(z)(r,\xi) = r^z \alpha(0)(r,\xi)$$

is satisfied and we artificially continue α by zero to $\mathbb{R}_{>0} \times M$. Then,

$$\begin{aligned}\int_{\mathbb{R}_{>0}\times M} \alpha(z)(r,\xi) d\mathrm{vol}_{\mathbb{R}_{>0}\times M}(r,\xi) &= \int_{\mathbb{R}_{>0}} r^{\dim M + z} \underbrace{\int_M \alpha(0)(r,\xi) d\mathrm{vol}_M(\xi)}_{=:A(r)} dr \\ &= \mathcal{M}(A)(\dim M + z + 1)\end{aligned}$$

holds where $\mathcal{M}f(z) = \int_{\mathbb{R}_{>0}} t^{z-1} f(t) dt$ for $f:\ \mathbb{R}_{>0} \to \mathbb{R}$ measurable, whenever the integral exists, denotes the Mellin transform. Hence, the name "$\mathcal{M}$-gauge".

■

PROPOSITION 2.12 (Laurent expansion of $\zeta(\mathfrak{fp}_0\alpha)$). *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution with $I_0 = \varnothing$. Then,*

$$\zeta(\alpha)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n\alpha)(0)}{n!} z^n$$

holds in a sufficiently small neighborhood of zero.

*Let $\beta = \beta_0 + \sum_{\iota\in I'}\beta_\iota$ be a gauged poly-*log*-homogeneous distribution without structural singularities at zero, i.e. $\forall\iota \in I_0'$: $\operatorname{res}\beta_\iota = 0$. Then, there exists a gauge $\hat\beta$ such that*

$$\zeta\left(\hat\beta\right)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n\mathfrak{fp}_0\beta)(0)}{n!} z^n$$

holds in a sufficiently small neighborhood of zero.

PROOF. The first assertion is a direct consequence of the facts that the n^{th} Laurent coefficient of a holomorphic function f is given by $\frac{\partial^n f(0)}{n!}$ and

$$\partial^n\zeta(\alpha) = \partial^n \int_{\mathbb{R}_{\geq 1}\times M} \alpha \, d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} = \int_{\mathbb{R}_{\geq 1}\times M} \partial^n\alpha \, d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} = \zeta(\partial^n\alpha).$$

Now,

$$\zeta\left(\hat\beta\right)(z) = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n\mathfrak{fp}_0\beta)(0)}{n!} z^n$$

follows from the fact that we may choose an $\mathcal{M}$-gauge for β_ι with $\iota \in I_0'$ which yields $\zeta\left(\hat\beta\right) = \zeta(\mathfrak{fp}_0\beta)$.

∎

$\mathcal{M}$-gauging will, furthermore, yield the following theorem which can be very handy with respect to actual computations. In particular, the fact that we can remove the influence of higher order derivatives of α_ι with critical degree of homogeneity will imply that the generalized Kontsevich-Vishik density (which we will define in chapter 7) is globally defined, i.e. for $\mathcal{M}$-gauged families with polyhomogeneous amplitudes the residue trace density and the generalized Kontsevich-Vishik density both exist globally (provided the kernel patches together).

THEOREM 2.13. *Let* $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ *be a gauged poly*-log-*homogeneous distribution. Then, there exists a gauge* $\hat{\alpha}$ *such that*

$$\zeta\left(\hat{\alpha}\right)(z) = \sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(0)}{z^{l_\iota+1}} + \sum_{n\in\mathbb{N}_0}\frac{\zeta(\partial^n\mathfrak{fp}_0\alpha)(0)}{n!}z^n$$

holds in a sufficiently small neighborhood of zero.

PROOF. This follows directly from Proposition 2.12 using an $\mathcal{M}$-gauge for α_ι with $\iota\in I_0$.

□

REMARK In general, there will be correction terms arising from the Laurent expansion of $\operatorname{res}\alpha_\iota$. Incorporating these yields

$$\begin{aligned}\zeta(\alpha)(z) = &\sum_{\iota\in I_0}\left(\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(0)}{z^{l_\iota+1}} + \sum_{n=1}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\partial^n\operatorname{res}\alpha_\iota(0)}{n!}z^{n-l_\iota-1}\right)\\ &+\sum_{n\in\mathbb{N}_0}\left(\frac{\zeta(\partial^n\mathfrak{fp}_0\alpha)(0)}{n!} + \sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\partial^{n+l_\iota+1}\operatorname{res}\alpha_\iota(0)}{(n+l_\iota+1)!}\right)z^n.\end{aligned}$$

■

COROLLARY 2.14. *Let* $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ *and* $\beta = \beta_0 + \sum_{\iota\in I}\beta_\iota$ *be two gauged poly*-log-*homogeneous distributions with* $\alpha(0) = \beta(0)$ *and such that the degrees of homogeneity and logarithmic orders of* α_ι *and* β_ι *coincide. Then,*

$$\begin{aligned}\zeta(\alpha)(z)-\zeta(\beta)(z) = &\sum_{\iota\in I_0}\sum_{n=1}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\partial^n\operatorname{res}\left(\alpha_\iota-\beta_\iota\right)(0)}{n!}z^{n-l_\iota-1}\\ &+\sum_{n\in\mathbb{N}_0}\frac{\zeta(\partial^n\mathfrak{fp}_0\left(\alpha-\beta\right))(0)}{n!}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\partial^{n+l_\iota+1}\operatorname{res}\left(\alpha_\iota-\beta_\iota\right)(0)}{(n+l_\iota+1)!}z^n\end{aligned}$$

holds in a sufficiently small neighborhood of zero.

In chapter 4, we will see that Corollary 2.14 applied to pseudo-differential operators implies many well-known formulae, e.g. (2.21) in [**47**], (9) in [**55**], and (2.20) in [**56**].

EXAMPLE Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ and $\beta = \beta_0 + \sum_{\iota\in I} \beta_\iota$ be two gauged polyhomogeneous distributions with $\alpha(0) = \beta(0)$ and such that the degrees of homogeneity of α_ι and β_ι coincide. Then, $\#I_0 \le 1$ and (because) all l_ι are zero. Hence,

$$\zeta(\alpha)(z) = \sum_{\iota\in I_0} \frac{-\operatorname{res}\alpha_\iota(0)}{z} + \sum_{n\in\mathbb{N}_0} \left(\frac{\zeta(\partial^n \mathfrak{fp}_0 \alpha)(0)}{n!} - \sum_{\iota\in I_0} \frac{\partial^{n+1}\operatorname{res}\alpha_\iota(0)}{(n+1)!} \right) z^n$$

and

$$\zeta(\alpha)(z) - \zeta(\beta)(z) = \sum_{n\in\mathbb{N}_0} \left(\frac{\zeta(\partial^n \mathfrak{fp}_0 (\alpha-\beta))(0)}{n!} - \sum_{\iota\in I_0} \frac{\partial^{n+1}\operatorname{res}(\alpha_\iota - \beta_\iota)(0)}{(n+1)!} \right) z^n$$

holds in a sufficiently small neighborhood of zero. This shows that the residue trace $-\sum_{\iota\in I_0} \operatorname{res}\alpha_\iota(0)$ is well-defined and independent of the gauge for polyhomogeneous distributions. Higher orders of the Laurent expansion depend on the gauge.

Furthermore, $\zeta(\alpha) - \zeta(\beta)$ is holomorphic in a neighborhood of zero and

$$\begin{aligned}
(\zeta(\alpha) - \zeta(\beta))(0) &= \zeta(\mathfrak{fp}_0(\alpha-\beta))(0) - \sum_{\iota\in I_0} \partial \operatorname{res}(\alpha_\iota - \beta_\iota)(0) \\
&= \underbrace{\zeta(\mathfrak{fp}_0\alpha)(0) - \zeta(\mathfrak{fp}_0\beta)(0)}_{=0} - \sum_{\iota\in I_0} \partial \operatorname{res}(\alpha_\iota - \beta_\iota)(0) \\
&= -\sum_{\iota\in I_0} \partial \operatorname{res}(\alpha_\iota - \beta_\iota)(0).
\end{aligned}$$

Defining $\gamma_\iota(z) := \frac{\alpha_\iota(z)-\beta_\iota(z)}{z}$ and $\gamma(z) := \frac{\alpha(z)-\beta(z)}{z}$ we, thus, obtain

$$(\zeta(\alpha) - \zeta(\beta))(0) = -\sum_{\iota\in I_0} \partial \operatorname{res}(\alpha_\iota - \beta_\iota)(0) = -\sum_{\iota\in I_0} \operatorname{res}\gamma_\iota(0) = \operatorname{res}_0 \zeta(\gamma).$$

Since $\operatorname{res}\gamma_\iota(0) \neq 0$ implies that it is independent of gauge, we obtain that $\operatorname{res}_0 \zeta(\gamma)$ is independent of gauge which directly yields

$$(\zeta(\alpha) - \zeta(\beta))(0) = \operatorname{res}_0 \zeta(\gamma) = \operatorname{res}_0 \zeta(\partial(\alpha-\beta)).$$

In other words, $(\zeta(\alpha)-\zeta(\beta))(0)$ is a trace residue.

■

THEOREM 2.15 (Laurent expansion of $\zeta(\alpha)$). *Let $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution. Then,*

$$\begin{aligned}\zeta(\alpha)(z)=&\sum_{\iota\in I_0}\sum_{n=0}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^n\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}}\\&+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq1}\times M}}{n!}z^n\\&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{n!(\dim M+d_\iota+1)^{l_\iota+j+1}}z^n\\&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde{\alpha}_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}z^n\end{aligned}$$

holds in a sufficiently small neighborhood of zero.

In particular, if α is polyhomogeneous, we obtain

$$\begin{aligned}\zeta(\alpha)(z)=&\sum_{\iota\in I_0}\frac{-\int_M\alpha_\iota(0)d\mathrm{vol}_M}{z}+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq1}\times M}}{n!}z^n\\&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{j+1}j!\int_M\partial^{n-j}\alpha_\iota(0)d\mathrm{vol}_M}{n!(\dim M+d_\iota+1)^{j+1}}z^n\\&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{-\int_M\partial^{n+1}\alpha_\iota(0)d\mathrm{vol}_M}{(n+1)!}z^n\end{aligned}$$

in a sufficiently small neighborhood of zero.

PROOF. Note that having a gauged log-homogeneous distribution

$$\beta(z)(r,\xi)=r^{d+z}(\ln r)^l\tilde{\beta}(z)(\xi)$$

the residue $\operatorname{res}\beta=\int_M\tilde{\beta}\ d\mathrm{vol}_M$ does not depend on the logarithmic order. Hence, we may assume without loss of generality that $l=0$ and we had a gauged homogeneous distribution in the first place, i.e. replace β by

$$\hat{\beta}(z)(r,\xi)=r^{d+z}\tilde{\beta}(z)(\xi)$$

Then, we observe

$$\partial^n\beta(z)(r,\xi)=\sum_{j=0}^n\binom{n}{j}r^{d+z}(\ln r)^{l+j}\partial^{n-j}\tilde\beta(z)(\xi)$$

and

$$\partial^n\tilde\beta(z)(\xi)=\partial^n\left(x\mapsto r^{-d-x}\hat\beta(x)(\xi)\right)(z)=\sum_{j=0}^n\binom{n}{j}r^{-d-z}(-\ln r)^j\partial^{n-j}\hat\beta(z)(r,\xi)$$

for every $n\in\mathbb{N}_0$, $r\in\mathbb{R}_{\geq1}$, and $\xi\in M$. In particular, for $r=1$, we deduce

$$\partial^n\tilde\beta(z)=\partial^n\hat\beta(z)|_M,$$

i.e.

$$\partial^n\operatorname{res}\beta=\partial^n\int_M\tilde\beta\ d\mathrm{vol}_M=\int_M\partial^n\tilde\beta\ d\mathrm{vol}_M=\int_M\partial^n\hat\beta\ d\mathrm{vol}_M.$$

Especially, for β homogeneous, we have $\hat\beta=\beta$ and, therefore,

$$\partial^n\operatorname{res}\beta=\int_M\partial^n\tilde\beta\ d\mathrm{vol}_M=\int_M\partial^n\hat\beta\ d\mathrm{vol}_M=\int_M\partial^n\beta\ d\mathrm{vol}_M.$$

Hence,

$$\begin{aligned}\zeta(\partial^n\mathfrak{fp}_0\alpha)(z)=&\int_{\mathbb{R}_{\geq1}\times M}\partial^n\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{\geq1}\times M}\\&+\sum_{\iota\in I\setminus I_0}\sum_{j=0}^n\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde\alpha_\iota(z)d\mathrm{vol}_M}{(\dim M+d_\iota+z+1)^{l_\iota+j+1}}.\end{aligned}$$

This directly yields

$$\begin{aligned}\zeta(\alpha)(z)=&\sum_{\iota\in I_0}\left(\frac{(-1)^{l_\iota+1}l_\iota!\int_M\tilde\alpha_\iota(0)d\mathrm{vol}_M}{z^{l_\iota+1}}+\sum_{n=1}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^n\tilde\alpha_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}}\right)\\&+\sum_{n\in\mathbb{N}_0}\left(\frac{\zeta(\partial^n\mathfrak{fp}_0\alpha)(0)}{n!}+\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde\alpha_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}\right)z^n\\=&\sum_{\iota\in I_0}\sum_{n=0}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^n\tilde\alpha_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}}+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq1}\times M}}{n!}z^n\\&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^n\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde\alpha_\iota(0)d\mathrm{vol}_M}{n!(\dim M+d_\iota+1)^{l_\iota+j+1}}z^n\\&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde\alpha_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}z^n\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\iota\in I_0}\sum_{n=0}^{l_\iota} \frac{(-1)^{l_\iota+1} l_\iota! \int_M \partial^n \tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}} + \sum_{n\in\mathbb{N}_0} \frac{\int_{\mathbb{R}_{\geq 1}\times M} \partial^n \alpha_0(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}{n!} z^n \\
&+ \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_M \partial^{n-j}\tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{n!(\dim M + d_\iota + 1)^{l_\iota+j+1}} z^n \\
&+ \sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_M \partial^{n+l_\iota+1}\tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{(n+l_\iota+1)!} z^n.
\end{aligned}
$$

□

Definition 2.16. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution such that $\zeta(\alpha)$ is holomorphic in a neighborhood of zero. Then, we define the generalized ζ-determinant*

$$
\det_\zeta(\alpha) := \exp\left(\zeta(\alpha)'(0)\right).
$$

Remark This generalized ζ-determinant reduces to the ζ-determinants as studied by Kontsevich and Vishik in [**47**, **48**]. In other words, we do not expect it to be multiplicative if α corresponds to a general Fourier Integral Operator. Though an interesting question, we will not study classes of families of Fourier Integral Operators satisfying the multiplicative property, here.

■

Knowing the Laurent expansion of $\zeta(\alpha)$ we know that

$$
\begin{aligned}
\zeta(\alpha)'(0) &= \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} \\
&+ \sum_{\iota\in I\setminus I_0}\sum_{j=0}^{1} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_M \partial^{1-j}\tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{l_\iota+j+1}} \\
&+ \sum_{\iota\in I_0} \frac{(-1)^{l_\iota+1} l_\iota! \int_M \partial^{l_\iota+2}\tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{(l_\iota+1)!}
\end{aligned}
$$

holds. In particular, if $I_0 = \emptyset$,

$$
\begin{aligned}
\zeta(\alpha)'(0) &= \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} \\
&+ \sum_{\iota\in I}\sum_{j=0}^{1} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_M \partial^{1-j}\tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{l_\iota+j+1}}.
\end{aligned}
$$

If α were polyhomogeneous we obtained

$$\begin{aligned}\zeta(\alpha)'(0) = &\int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} \\ &+ \sum_{\iota\in I\setminus I_0}\sum_{j=0}^{1} \frac{(-1)^{j+1}\int_M \partial^{1-j}\alpha_\iota(0) d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{j+1}} - \sum_{\iota\in I_0}\int_M \alpha_\iota''(0) d\mathrm{vol}_M \\ = &\int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I\setminus I_0} \frac{-\operatorname{res}(\alpha_\iota')(0)}{\dim M + d_\iota + 1} \\ &+ \sum_{\iota\in I\setminus I_0} \frac{\operatorname{res}\alpha_\iota(0)}{(\dim M + d_\iota + 1)^2} - \sum_{\iota\in I_0}\operatorname{res}(\alpha_\iota'')(0)\end{aligned}$$

If we were to choose an $\mathcal{M}$-gauge, we would find $\partial\tilde{\alpha}_\iota = 0$ and may assume $I_0 = \varnothing$ ($\zeta(\alpha)$ cannot have a structural singularity and non-structural singularities do not appear within the ζ-function of an $\mathcal{M}$-gauged poly-log-homogeneous distribution), i.e.

$$\begin{aligned}\zeta(\alpha)'(0) = &\int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I\setminus I_0} \frac{(-1)^{l_\iota}(l_\iota+1)!\int_M \tilde{\alpha}_\iota(0) d\mathrm{vol}_M}{(\dim M + d_\iota + 1)^{l_\iota+2}} \\ = &\int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I} \frac{(-1)^{l_\iota}(l_\iota+1)!\operatorname{res}\alpha_\iota(0)}{(\dim M + d_\iota + 1)^{l_\iota+2}}\end{aligned}$$

and, for α additionally polyhomogeneous,

$$\zeta(\alpha)'(0) = \int_{\mathbb{R}_{\geq 1}\times M} \alpha_0'(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I} \frac{\operatorname{res}\alpha_\iota(0)}{(\dim M + d_\iota + 1)^2}.$$

Remark Note that $\zeta(\alpha)'(0)$ depends on the first $1 + \max(\{l_\iota + 1;\ \iota\in I_0\}\cup\{0\})$ derivatives of α. Hence, the generalized ζ-determinant does so, too, and is, thus, not independent of the gauge.

∎

CHAPTER 3

Remarks on more general gauged poly-log-homogeneous distributions

The results obtained for gauged poly-log-homogeneous distributions can largely be generalized. In fact, the degree of homogeneity $d_\iota(z)$ can be chosen arbitrarily as long as it is not germ equivalent to a critical constant. In this chapter, we will investigate these direct generalizations and consider distributions of the form

$$\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}(\xi)$$

where M is an orientable, compact, finite dimensional manifold without boundary and the holomorphic family α is given by an expansion

$$\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$$

where $I \subseteq \mathbb{N}$, $\alpha_0(z) \in L_1(\mathbb{R}_{\geq 1}\times M)$ in an open neighborhood of $\{z \in \mathbb{C};\ \Re(z) \leq 0\}$ and each of the $\alpha_\iota(z)$ is log-homogeneous with degree of homogeneity $d_\iota(z) \in \mathbb{C}$ and logarithmic order $l_\iota \in \mathbb{N}_0$, that is,

$$\exists \tilde{\alpha}_\iota \in \mathbb{C}^M\ \ \forall r \in \mathbb{R}_{\geq 1}\ \ \forall \nu \in M:\ \ \alpha_\iota(z)(r,\nu) = r^{d_\iota(z)}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\nu).$$

We will furthermore assume (for now) that every d_ι is an entire function,

$$\forall z \in [-\dim M - 1]d_\iota:\ \ d_\iota'(z) \neq 0,$$

the family $(\Re(d_\iota(z)))_{\iota\in I}$ is bounded from above for every $z \in \mathbb{C}$, $\sup_{\iota\in I}\Re(d_\iota(z)) \to -\infty$ $(\Re(z) \to -\infty)$, the maps $I \ni \iota \mapsto (d_\iota(z), l_\iota)$ are injective, there are only finitely many ι satisfying $d_\iota(z) = d$ for any given $d, z \in \mathbb{C}$, the families $((d_\iota(z)+\delta)^{-1})_{\iota\in I}$

are in $\ell_2(I)$ for any $z \in \mathbb{C}$ and $\delta \in \mathbb{C} \setminus \{d_\iota(z);\ \iota \in I\}$, and each $\sum_{\iota\in I} \tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$. Any such family α will be called a gauged poly-log-homogeneous distribution with holomorphic order.

If the conditions above are satisfied, we obtain

$$\begin{aligned}\int_{\mathbb{R}_{\geq 1}\times M} \alpha(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} =& \underbrace{\int_{\mathbb{R}_{\geq 1}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}_{=:\tau_0(z)\in\mathbb{C}} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} \\ =& \tau_0(z) + \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}}\end{aligned}$$

which converges absolutely. For $d_\iota(0) \neq -\dim M - 1$, we observe

$$\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}} = \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\left(\frac{(\dim M+1+d_\iota(0)+z)^{l_\iota+1}}{(\dim M+1+d_\iota(z))^{l_\iota+1}}\alpha_\iota\right)(z)}{(\dim M + 1 + d_\iota(0) + z)^{l_\iota+1}}$$

in a neighborhood of zero. Hence, let

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z} (\ln r)^{l_\iota} \underbrace{\frac{(\dim M + 1 + d_\iota(0) + z)^{l_\iota+1}}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}} \tilde{\alpha}_\iota(z)(\xi)}_{=:\tilde{\beta}_\iota(z)(\xi)}.$$

For $d_\iota(0) = -\dim M - 1$, there exists an entire function δ_ι such that

$$\dim M + 1 + d_\iota(z) = d'_\iota(0)z + \delta_\iota(z)z^2$$

and, since $d'_\iota(0) \neq 0$, we obtain that $z \mapsto d'_\iota(0) + \delta_\iota(z)z$ has no zeros in a neighborhood of zero. Then, we observe

$$\begin{aligned}\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}} =& \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{(d'_\iota(0)z + \delta_\iota(z)z^2)^{l_\iota+1}} \\ =& \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\alpha_\iota(z)}{z^{l_\iota+1}\left(d'_\iota(0) + \delta_\iota(z)z\right)^{l_\iota+1}} \\ =& \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\left(\frac{\alpha_\iota(z)}{(d'_\iota(0)+\delta_\iota(z)z)^{l_\iota+1}}\right)}{z^{l_\iota+1}} \\ =& \frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res}\left(\frac{\alpha_\iota(z)}{(d'_\iota(0)+\delta_\iota(z)z)^{l_\iota+1}}\right)}{(\dim M + 1 + d_\iota(0) + z)^{l_\iota+1}}\end{aligned}$$

and define

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}(\ln r)^{l_\iota}\underbrace{\frac{\tilde{\alpha}_\iota(z)(\xi)}{(d_\iota'(0)+\delta_\iota(z)z)^{l_\iota+1}}}_{=:\tilde{\beta}_\iota(z)(\xi)}.$$

Thus, we obtain the following observation.

Observation 3.1. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution with holomorphic order. Then, the ζ-function $\zeta(\alpha)$ is germ equivalent to $\zeta(\beta)$ with β as defined above. Thus, $\zeta(\alpha)$ inherits all local properties from $\zeta(\beta)$, i.e. all local properties of ζ-functions associated with gauged poly-*log*-homogeneous distributions.*

In particular, if $\operatorname{res}\alpha_\iota(0) \neq 0$ with $d_\iota(0) = -\dim M - 1$ and l_ι maximal, then the initial Laurent coefficient of $\zeta(\alpha)$ is

$$\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota(0)}{d_\iota'(0)^{l_\iota+1}}$$

and the $\zeta(\alpha)$ has the Laurent expansion

$$\begin{aligned}\zeta(\alpha)(z) = &\sum_{\iota\in I_0}\sum_{n=0}^{l_\iota}\frac{(-1)^{l_\iota+1}l_\iota!\int_M \partial^n\tilde{\beta}_\iota(0)d\mathrm{vol}_M}{n!\ z^{l_\iota+1-n}}\\ &+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq 1}\times M}\partial^n\alpha_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}}{n!}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_M\partial^{n-j}\tilde{\beta}_\iota(0)d\mathrm{vol}_M}{n!(\dim M + d_\iota + 1)^{l_\iota+j+1}}z^n\\ &+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_M\partial^{n+l_\iota+1}\tilde{\beta}_\iota(0)d\mathrm{vol}_M}{(n+l_\iota+1)!}z^n\end{aligned}$$

in a sufficiently small neighborhood of zero.

Proof. Note that zero is either a pole of $\zeta(\alpha)$ or a regular value, that is, we can choose a neighborhood uniformly for all ι with $d_\iota(0) \neq -\dim M - 1$. Since there are only finitely many ι with $d_\iota(0) = -\dim M - 1$, we obtain germ equivalence of the series representations and, since the Laurent expansion was solely determined from the series representation, the observation follows.

□

We may generalize this even further. Suppose α is meromorphic in $\mathbb{C}$, that is, holomorphic in $\Omega \subseteq_{\mathrm{open}} \mathbb{C}$ such that $\mathbb{C} \setminus \Omega$ is a set of isolated points in $\mathbb{C}$. Let $0 \in \Omega$ and let α satisfy all properties of being a gauged poly-log-homogeneous distribution with holomorphic order but on Ω instead of $\mathbb{C}$. Then, we call α a meromorphic gauged poly-log-homogeneous distribution with respect to zero. Since $0 \in \Omega$, we directly obtain that α is locally a gauged poly-log-homogeneous distribution and still all local properties are preserved just as they are in Observation 3.1.

Now, we can even drop the assumption

$$\forall z \in [-\dim M - 1]d_\iota : \ d_\iota'(z) \neq 0$$

in the definition of a meromorphic gauged poly-log-homogeneous distribution with respect to zero (in exchange for an increased logarithmic order). Instead, let

$$d_\iota(z) = -\dim M - 1 + \delta_\iota(z) z^{m_\iota}$$

with $\delta_\iota(0) \neq 0$ and call any such α a generalized meromorphic gauged poly-log-homogeneous distribution with respect to zero. Then,

$$\begin{aligned}
&\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res} \alpha_\iota(z)}{(\dim M + 1 + d_\iota(z))^{l_\iota+1}} \\
=&\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res} \alpha_\iota(z)}{(\delta_\iota(z) z^{m_\iota})^{l_\iota+1}} \\
=&\frac{(-1)^{l_\iota+1} l_\iota! \operatorname{res} \left(\delta_\iota^{-l_\iota-1} \alpha_\iota\right)(z)}{z^{m_\iota(l_\iota+1)}} \\
=&\frac{(-1)^{m_\iota(l_\iota+1)} (m_\iota(l_\iota+1)-1)! \operatorname{res} \left((-1)^{m_\iota(l_\iota+1)+l_\iota+1} \frac{l_\iota!}{(m_\iota(l_\iota+1)-1)!} \delta_\iota^{-l_\iota-1} \alpha_\iota\right)(z)}{z^{m_\iota(l_\iota+1)}} \\
=&\frac{(-1)^{m_\iota(l_\iota+1)} (m_\iota(l_\iota+1)-1)! \operatorname{res} \left((-1)^{m_\iota(l_\iota+1)+l_\iota+1} \frac{l_\iota!}{(m_\iota(l_\iota+1)-1)!} \delta_\iota^{-l_\iota-1} \alpha_\iota\right)(z)}{(\dim M + 1 + d_\iota(0) + z)^{m_\iota(l_\iota+1)}}
\end{aligned}$$

shows that choosing

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}\,(\ln r)^{m_\iota(l_\iota+1)-1}\underbrace{\frac{(-1)^{m_\iota(l_\iota+1)+l_\iota+1}l_\iota!}{(m_\iota(l_\iota+1)-1)!}\delta_\iota(z)^{-l_\iota-1}\tilde{\alpha}_\iota(z)(\xi)}_{=:\tilde{\beta}_\iota(z)(\xi)}$$

for $\iota \in I$ with $d_\iota(0) = -\dim M - 1$ also yields germ equivalence and, again, all local properties are preserved.

Hence, we can state the following Definition and Theorem.

Definition 3.2. *Let $\Omega \subseteq_{\text{open}} \mathbb{C}$, $\Omega_0 \subseteq_{\text{open}} \Omega$, $0 \in \Omega$, and $\alpha = (\alpha(z))_{z\in\Omega}$ a holomorphic family of the form*

$$\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$$

where

- *$I \subseteq \mathbb{N}$,*
- *$\forall z \in \Omega:\ \alpha_0(z) \in L_1(\mathbb{R}_{\geq 1} \times M)$,*
- *$\forall z \in \Omega_0:\ \alpha(z) \in L_1(\mathbb{R}_{\geq 1} \times M)$,*
- *each of the $\alpha_\iota(z)$ is* log-*homogeneous with degree of homogeneity $d_\iota(z) \in \mathbb{C}$ and logarithmic order $l_\iota \in \mathbb{N}_0$, that is,*

  $$\exists \tilde{\alpha}_\iota \in \mathbb{C}^M\ \ \forall r \in \mathbb{R}_{\geq 1}\ \ \forall \nu \in M:\ \alpha_\iota(z)(r,\nu) = r^{d_\iota(z)}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\nu),$$

- *each d_ι is holomorphic in Ω,*
- *none of the d_ι is germ equivalent to $-\dim M - 1$ at zero (i.e. none of the d_ι is the constant $-\dim M - 1$),*
- *the maps $I \ni \iota \mapsto (d_\iota(z), l_\iota)$ are injective,*
- *there are only finitely many ι satisfying $d_\iota(z) = d$ for any given $d \in \mathbb{C}$ and $z \in \Omega$,*
- *the families $((d_\iota(z) + \delta)^{-1})_{\iota\in I}$ are in $\ell_2(I)$ for any $z \in \Omega$ and $\delta \in \mathbb{C} \setminus \{d_\iota(z);\ \iota \in I\}$,*

- *and each* $\sum_{\iota\in I}\tilde\alpha_\iota(z)$ *converges unconditionally in* $L_1(M)$.

If every connected component of Ω *has non-empty intersection with* Ω_0*, then we call* α *a generalized gauged poly-*log*-homogeneous distribution and*

$$\zeta(\alpha) := \int_{\mathbb{R}_{\geq 1}\times M}\alpha_0 d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota}{(\dim M+1+d_\iota)^{l_\iota+1}}$$

the associated ζ*-function of* α.

Otherwise (in particular, if $\Omega_0=\varnothing$*), we call* α *an abstract generalized gauged poly-*log*-homogeneous distribution and*

$$\zeta(\alpha) := \int_{\mathbb{R}_{\geq 1}\times M}\alpha_0 d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M} + \sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha_\iota}{(\dim M+1+d_\iota)^{l_\iota+1}}$$

the associated ζ*-function of* α.

Remark Because abstract generalized gauged poly-log-homogeneous distributions have empty Ω_0 on some connected component of Ω, we will still obtain the Laurent expansion and all other local properties derived from the series expansion we used to define the ζ-function here but applications to Fourier Integral Operators might lose all properties that are obtained from meromorphic extension of the classical trace, e.g. traciality.

■

Theorem 3.3. *Let* $\alpha=\alpha_0+\sum_{\iota\in I}\alpha_\iota$ *and* $\beta=\beta_0+\sum_{\iota\in I}\beta_\iota$ *be (abstract) generalized gauged poly-*log*-homogeneous distributions with* $\beta_0=\alpha_0$,

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}(\ln r)^{l_\iota}\underbrace{\frac{(\dim M+1+d_\iota(0)+z)^{l_\iota+1}}{(\dim M+1+d_\iota(z))^{l_\iota+1}}\tilde\alpha_\iota(z)(\xi)}_{=\tilde\beta_\iota(z)(\xi)}$$

for $\iota\in I$ *with* $d_\iota(0)\neq-\dim M-1$*, and*

$$\beta_\iota(z)(r,\xi) := r^{d_\iota(0)+z}(\ln r)^{m_\iota(l_\iota+1)-1}\underbrace{\frac{(-1)^{m_\iota(l_\iota+1)+l_\iota+1}l_\iota!}{(m_\iota(l_\iota+1)-1)!}\delta_\iota(z)^{-l_\iota-1}\tilde\alpha_\iota(z)(\xi)}_{=\tilde\beta_\iota(z)(\xi)}$$

for $\iota \in I$ *with* $d_\iota(z) = -\dim M - 1 + \delta_\iota(z) z^{m_\iota}$ *in a neighborhood of zero and* δ_ι *holomorphic such that* $\delta_\iota(0) \neq 0$.

Then, the ζ*-function* $\zeta(\alpha)$ *is germ equivalent to* $\zeta(\beta)$ *at zero. In particular,* $\zeta(\alpha)$ *has the Laurent expansion*

$$\begin{aligned}
\zeta(\alpha)(z) = & \sum_{\iota \in I_0} \sum_{n=0}^{m_\iota(l_\iota+1)-1} \frac{(-1)^{m_\iota(l_\iota+1)}(m_\iota(l_\iota+1)-1)! \int_M \partial^n \tilde{\beta}_\iota(0) d\mathrm{vol}_M}{n!\ z^{m_\iota(l_\iota+1)-n}} \\
& + \sum_{n \in \mathbb{N}_0} \frac{\int_{\mathbb{R}_{\geq 1} \times M} \partial^n \alpha_0(0) d\mathrm{vol}_{\mathbb{R}_{\geq 1} \times M}}{n!} z^n \\
& + \sum_{n \in \mathbb{N}_0} \sum_{\iota \in I \setminus I_0} \sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_M \partial^{n-j} \tilde{\beta}_\iota(0) d\mathrm{vol}_M}{n!(\dim M + d_\iota + 1)^{l_\iota+j+1}} z^n \\
& + \sum_{n \in \mathbb{N}_0} \sum_{\iota \in I_0} \frac{(-1)^{m_\iota(l_\iota+1)}(m_\iota(l_\iota+1)-1)! \int_M \partial^{n+m_\iota(l_\iota+1)} \tilde{\beta}_\iota(0) d\mathrm{vol}_M}{(n+m_\iota(l_\iota+1))!} z^n
\end{aligned}$$

in a sufficiently small neighborhood of zero.

CHAPTER 4

Application to gauged Lagrangian distributions

If we consider a dual pair $\langle u(z), f\rangle$ where $u:\ \mathbb{C}\to I(X,\Lambda)$ is a gauged Lagrangian distribution ($I(X,\Lambda)$ is the space of lagrangian distributions with micro-support in the closed conic Lagrangian sub-manifold Λ of $T^*X\smallsetminus\{X\times\{0\}\}$; cf. [**34**] and chapter 25 in [**38**]), then we obtain integrals of the form

$$\langle u(z), f\rangle = \int_X\int_{\mathbb{R}^N} e^{i\vartheta(x,\xi)}a(z)(x,\xi)\ d\xi\ d\mathrm{vol}_X(x).$$

In particular, we are considering distributions of the form

$$u_1(x) = \int_{\mathbb{R}^N} e^{i\vartheta(x,\xi)}a(x,\xi)d\xi.$$

If ϑ is non-degenerate, then Theorem 25.1.3 in [**38**] shows that the Fourier transform of u_1 satisfies (in local coordinates)

$$\forall y\in\mathbb{R}^{\dim X}\smallsetminus B_{\mathbb{R}^{\dim X}}:\ \mathcal{F}u_1(y) = e^{-iH(y)}v(y)$$

with $\Lambda = \{(H'(y),y);\ y\in\mathbb{R}^{\dim X}\smallsetminus\{0\}\}$ where $v\in S^{m-\frac{\dim X}{4}}\left(\mathbb{R}^{\dim X}\right)$ if $u_1\in I^m_{\text{compact}}\left(\mathbb{R}^{\dim X},\Lambda\right)$ and $B_{\mathbb{R}^{\dim X}}$ is the closed unit ball in $\mathbb{R}^{\dim X}$. Furthermore, Theorem 25.1.5' in [**38**] shows that v is poly-log-homogeneous if and only if a is and we obtain

$$\begin{aligned}
u_1(x) =& \underbrace{\int_{B_{\mathbb{R}^{\dim X}}} e^{i\langle x,y\rangle}\mathcal{F}u_1(y)dy}_{=:\hat\tau_0(x)} + \int_{\mathbb{R}^{\dim X}\smallsetminus B_{\mathbb{R}^{\dim X}}} e^{i\langle x,y\rangle-iH(y)}v(y)dy\\
=&\hat\tau_0(x) + \int_{\mathbb{R}^{\dim X}\smallsetminus B_{\mathbb{R}^{\dim X}}} e^{i\langle x,y\rangle-i\langle H'(y),y\rangle}v(y)dy\\
=&\hat\tau_0(x) + \int_{\mathbb{R}^{\dim X}\smallsetminus B_{\mathbb{R}^{\dim X}}} e^{i\langle x-H'(y),y\rangle}v(y)dy.
\end{aligned}$$

Changing coordinates locally from x to $x - H'(y)$ yields an integral of the form

$$\int_{\mathbb{R}^{\dim X} \setminus B_{\mathbb{R}^{\dim X}}} e^{i\langle x,y\rangle} v(y)\ dy$$

which is paired with another Lagrangian distribution f. In particular, extending v by zero on $B_{\mathbb{R}^{\dim X}}$ yields the inverse Fourier transform $\mathcal{F}^{-1}(v)(x)$ since, by Theorem 21.2.10 in [**38**], we may assume that $X = \mathbb{R}^{\dim X}$ (cf. also the proof of Theorem 2.1 in [**34**]).

Returning to

$$\langle u(z), f\rangle = \int_X \int_{\mathbb{R}^N} e^{i\vartheta(x,\xi)} a(z)(x,\xi)\ d\xi\ d\mathrm{vol}_X$$

we will split off the integral

$$\tau_0(z) := \int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,\xi)} a(z)(x,\xi)\ d\xi\ d\mathrm{vol}_X(x)$$

which defines a holomorphic function and we are left with

$$\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,\xi)} a(z)(x,\xi)\ d\mathrm{vol}_X(x)\ d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)$$

which can be re-parametrized (choosing suitable coordinates in a conic neighborhood of Λ) into the form

$$\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi).$$

For $f = P^t\delta_0$ with some pseudo-differential operator P whose symbol p is poly-log-homogeneous, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}} \alpha(z)(\xi) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)\\
=&\langle u(z), f\rangle - \tau_0(z)\\
=&\left\langle x \mapsto \int_{\mathbb{R}^N\setminus B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a(z)(x,\xi)\ d\mathrm{vol}_{\mathbb{R}^N\setminus B_{\mathbb{R}^N}}(\xi), P^t\delta_0\right\rangle\\
=&\left\langle \mathcal{F}^{-1}(v(z)), P^t\delta_0\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle P\mathcal{F}^{-1}(v(z)), \delta_0 \right\rangle \\
&= \left\langle x \mapsto \int_{\mathbb{R}^{\dim X}} e^{i\langle x,\xi\rangle} p(x,\xi) \mathcal{F}\left(\mathcal{F}^{-1}(v(z))\right)(\xi) d\xi, \delta_0 \right\rangle \\
&= \left\langle x \mapsto \int_{\mathbb{R}^{\dim X}} e^{i\langle x,\xi\rangle} p(x,\xi) v(z)(\xi) d\xi, \delta_0 \right\rangle \\
&= \int_{\mathbb{R}^{\dim X} \setminus B_{\mathbb{R}^{\dim X}}} p(0,\xi) v(z)(\xi) \ d\mathrm{vol}_{\mathbb{R}^{\dim X} \setminus B_{\mathbb{R}^{\dim X}}}(\xi)
\end{aligned}
$$

which is a distribution as considered in chapter 2.[1] In other words, if A is a gauged Fourier Integral Operator with phase function ϑ and amplitude a on X, then

$$
\begin{aligned}
\zeta(A)(z) = &\underbrace{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} a(z)(x,x,\xi) \ d\xi \ d\mathrm{vol}_X(x)}_{=:\tau_0(A)(z)} \\
&+ \int_{\mathbb{R}_{\geq 1} \times \partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} a(z)(x,x,\xi) \ d\mathrm{vol}_X(x) \ d\mathrm{vol}_{\mathbb{R}_{\geq 1} \times \partial B_{\mathbb{R}^N}}(\xi)
\end{aligned}
$$

exists and inherits all properties described in chapter 2 because δ_{diag} is of the form $P^t \delta_0$ for some pseudo-differential operator P with polyhomogeneous symbol.

THEOREM 4.1. *If $a = a_0 + \sum_{\iota \in I} a_\iota$ is the amplitude of a poly-*log*-homogeneous Fourier Integral Operator A with phase function ϑ and A_ι the gauged Fourier Integral Operator with phase function ϑ and amplitude a_ι, then*

$$
\operatorname{res} A_\iota(z) := \int_{\partial B_{\mathbb{R}^N}} \int_X e^{i\vartheta(x,x,\xi)} \tilde{a}_\iota(z)(x,x,\xi) \ d\mathrm{vol}_X(x) \ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)
$$

and

$$
\begin{aligned}
&\zeta(A)(z) \\
&= \sum_{n \in \mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n a(0)(x,x,\xi) \ d\xi \ d\mathrm{vol}_X(x)}{n!} z^n \\
&\quad + \sum_{\iota \in I_0} \sum_{n=0}^{l_\iota} \frac{(-1)^{l_\iota+1} l_\iota! \int_{\Delta(X) \times \partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^n \tilde{a}_\iota(0) \ d\mathrm{vol}_{\Delta(X) \times \partial B_{\mathbb{R}^N}}}{n! \ z^{l_\iota+1-n}}
\end{aligned}
$$

[1]This parametrization was already observed by Duistermaat and Hörmander in the proof of Theorem 5.4.1 in [**21**]. Furthermore, it is crucial for Guillemin's work on the residue trace; cf. (2.15) in [**34**].

$$+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}\int_X e^{i\vartheta(x,x,\xi)}\partial^n a_0(0)(x,x,\xi)\ \mathrm{dvol}_X(x)\ \mathrm{dvol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)}{n!}z^n$$

$$+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}\partial^{n-j}\tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}}z^n$$

$$+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}\partial^{n+l_\iota+1}\tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(n+l_\iota+1)!}z^n$$

holds in a neighborhood of zero where $\Delta(X) := \{(x,y)\in X^2;\ x=y\}$.

For a polyhomogeneous a this reduces to

$$\begin{aligned}\zeta(A)(z) = &\sum_{n\in\mathbb{N}_0}\frac{\int_X\int_{B_{\mathbb{R}^N}(0,1)}e^{i\vartheta(x,x,\xi)}\partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!}z^n\\
&-\sum_{\iota\in I_0}\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}z^{-1}\\
&+\sum_{n\in\mathbb{N}_0}\frac{\int_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}e^{i\vartheta}\partial^n a_0(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}}{n!}z^n\\
&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{j+1}j!\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}\partial^{n-j}a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{j+1}}z^n\\
&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{-\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}\partial^{n+1}a_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(n+1)!}z^n,\end{aligned}$$

i.e.

$$\begin{aligned}\zeta(A)(z) = &-\sum_{\iota\in I_0}\operatorname{res}A_\iota(0)z^{-1}-\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I_0}\frac{\operatorname{res}\partial^{n+1}A_\iota(0)}{(n+1)!}z^n\\
&+\sum_{n\in\mathbb{N}_0}\frac{\int_X\int_{B_{\mathbb{R}^N}(0,1)}e^{i\vartheta(x,x,\xi)}\partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!}z^n\\
&+\sum_{n\in\mathbb{N}_0}\frac{\int_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}e^{i\vartheta}\partial^n a_0(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}}{n!}z^n\\
&+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{j+1}j!\operatorname{res}\partial^{n-j}A_\iota(0)}{n!(N+d_\iota)^{j+1}}z^n\end{aligned}$$

where $\partial^n A_\iota$ is the gauged Fourier Integral Operator with phase ϑ and amplitude $\partial^n a_\iota$.

From this last formula, and the knowledge that $\operatorname{res} A_\iota(0)$ is independent of the gauge, we obtain the following well-known result (cf. [**34**]).

THEOREM 4.2. *Let A and B be polyhomogeneous Fourier Integral Operators. Let G_1 and G_2 be gauged Fourier Integral Operators with $G_1(0) = AB$ and $G_2(0) = BA$. Then,*

$$\operatorname{res}_0 \zeta(G_1) = \operatorname{res}_0 \zeta(G_2),$$

i.e. the residue of the ζ-function is tracial and $A \mapsto \operatorname{res}_0 \zeta\left(\hat{A}\right)$ is a well-defined trace where $\hat{A}$ is any choice of gauge for A.

PROOF. This is a direct consequence of the following two facts.

(i) $\operatorname{res}_0 \zeta(G_j) = -\sum_{\iota\in I_0} \operatorname{res}(G_j)_\iota(0)$ is independent of the gauge ($j \in \{1,2\}$).

(ii) $\zeta\left(\hat{A}B\right) = \zeta\left(B\hat{A}\right)$ holds for any gauge $\hat{A}$ of A because it is true for $\Re(z)$ sufficiently small.

Hence, $\operatorname{res}_0 \zeta(G_1) = \operatorname{res}_0 \zeta(\hat{A}B) = \operatorname{res}_0 \zeta(B\hat{A}) = \operatorname{res}_0 \zeta(G_2)$.

□

Similarly, for $I_0(AB) = \varnothing$, $G_1(0) = AB$, and $G_2(0) = BA$, we obtain that $\zeta(G_1)(0) = \zeta(G_2)(0)$ where we used that $\zeta(\mathfrak{fp}_0\alpha)(0)$ is independent of gauge. In other words, we may also generalize the Kontsevich-Vishik trace to $\zeta(\mathfrak{fp}_0 A)(0)$ where $\mathfrak{fp}_0 A$ is the gauged Fourier Integral Operator with phase ϑ and amplitude $a - \sum_{\iota\in I_0} a_\iota$.

DEFINITION 4.3. *Let A be a Fourier Integral Operator with phase function ϑ and poly-*log*-homogeneous amplitude $a = a_0 + \sum_{\iota\in I} a_\iota$. Let $\hat{A}$ be a gauged poly-*log*-homogeneous Fourier Integral Operator with $\hat{A}(0) = A$ with phase function ϑ and amplitude $\hat{a} = \hat{a}_0 + \sum_{\iota\in I} \hat{a}_\iota$, and $\mathfrak{fp}_0\hat{A}$ the part of $\hat{A}$ corresponding to the amplitude*

$a-\sum_{\iota\in I_0}a_\iota$, *that is, all but the terms with critical degree of homogeneity. Then, we call*

$$\operatorname{tr}_{KV}A := \zeta(\mathfrak{fp}_0\hat{A})(0)$$

the generalized Kontsevich-Vishik trace of A*.*

In particular, we may also consider the regularized generalized determinant

$$\det_{\mathfrak{fp}}(A) := \exp\left(\zeta(\mathfrak{fp}_0 A)'(0)\right)$$

where

$$\begin{aligned}
&\zeta(\mathfrak{fp}_0 A)(z)\\
&=\sum_{n\in\mathbb{N}_0}\frac{\int_X\int_{B_{\mathbb{R}^N}(0,1)}e^{i\vartheta(x,x,\xi)}\partial^n a(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)}{n!}z^n\\
&\quad+\sum_{n\in\mathbb{N}_0}\frac{\int_{\Delta(X)\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}^N})}e^{i\vartheta}\partial^n a_0(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}^N})}}{n!}z^n\\
&\quad+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I\setminus I_0}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}\partial^{n-j}\tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}}z^n,
\end{aligned}$$

i.e.

$$\begin{aligned}
\zeta(\mathfrak{fp}_0 A)'(0)=&\int_X\int_{B_{\mathbb{R}^N}(0,1)}e^{i\vartheta(x,x,\xi)}a'(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)\\
&+\int_{\Delta(X)\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}^N})}e^{i\vartheta}a_0'(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}^N})}\\
&+\sum_{\iota\in I\setminus I_0}\frac{(-1)^{l_\iota+1}l_\iota!\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}\tilde{a}_\iota'(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(N+d_\iota)^{l_\iota+1}}\\
&+\sum_{\iota\in I\setminus I_0}\frac{(-1)^{l_\iota+2}(l_\iota+1)!\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}}e^{i\vartheta}\tilde{a}_\iota(0)\ \mathrm{dvol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(N+d_\iota)^{l_\iota+2}}
\end{aligned}$$

which reduces to

$$\begin{aligned}
\zeta(\mathfrak{fp}_0 A)'(0)=&\int_X\int_{B_{\mathbb{R}^N}(0,1)}e^{i\vartheta(x,x,\xi)}a'(0)(x,x,\xi)\ d\xi\ \mathrm{dvol}_X(x)\\
&+\int_{\Delta(X)\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}^N})}e^{i\vartheta}a_0'(0)\ \mathrm{dvol}_{\Delta(X)\times(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}^N})}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\iota\in I\setminus I_0} \frac{\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\tilde{a}_\iota'(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{N+d_\iota}\\
&+ \sum_{\iota\in I\setminus I_0} \frac{\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta}\tilde{a}_\iota(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{(N+d_\iota)^2}\\
=&\tau_0(\partial A)(0) + \int_{\Delta(X)\times\left(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}\right)} e^{i\vartheta}a_0'(0)\ d\mathrm{vol}_{\Delta(X)\times\left(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}\right)}\\
&- \sum_{\iota\in I\setminus I_0} \frac{\mathrm{res}(\partial A_\iota)(0)}{N+d_\iota} + \sum_{\iota\in I\setminus I_0} \frac{\mathrm{res}\, A_\iota(0)}{(N+d_\iota)^2}
\end{aligned}
$$

for polyhomogeneous A. This will further reduce nicely if we choose an $\mathcal{M}$-gauge for the A_ι on $X\times\left(\mathbb{R}^N\setminus B_{\mathbb{R}^N}(0,1)\right)$ and constant "gauge" (i.e. no gauge) for a_0 on $X\times\mathbb{R}^N$ and for a on $X\times B_{\mathbb{R}^N}(0,1)$. In that case, we obtain

$$\zeta(\mathfrak{fp}_0 A)'(0) = \sum_{\iota\in I\setminus I_0} \frac{\mathrm{res}\, A_\iota(0)}{(N+d_\iota)^2}.$$

To be fair, this would be a gauge in a generalized sense for Fourier Integral Operators because such a gauge may not yield $C^\infty\left(X\times X\times\mathbb{R}^N\right)$-amplitudes $a(z)$ though the set of exceptions is the null set $X\times\partial B_{\mathbb{R}^N}$. If we wanted to avoid that, we would have to gauge the $X\times B_{\mathbb{R}^N}(0,1)$ part, as well, and the correction term can easily be estimated by

$$
\begin{aligned}
|\tau_0(A)'(0)| =& \left|\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)}a'(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)\right|\\
\leq& \mathrm{vol}_X(X)\ \mathrm{vol}_{\mathbb{R}^N}\left(B_{\mathbb{R}^N}(0,1)\right)\ \|a'(0)\|_{L_\infty(\Delta(X)\times B_{\mathbb{R}^N}(0,1))}\\
\leq& \mathrm{vol}_X(X)\ \mathrm{vol}_{\mathbb{R}^N}\left(B_{\mathbb{R}^N}(0,1)\right)\ \|a'(0)\|_{L_\infty(X\times X\times B_{\mathbb{R}^N})}.
\end{aligned}
$$

An important class of gauges (since they can be constructed fairly easily) are multiplicative gauges.

Definition 4.4. *Let A be a Fourier Integral Operator and G a gauged Fourier Integral Operator with $G(0)=1$ such that each $G(z)$ and all derivatives $\partial^n G(z)$ are composable with A. Then, we call $AG(\cdot)$ a multiplicative gauge of A.*

*A multiplicative gauge G is called exponential if and only if there exists a poly-*log*-homogeneous Fourier Integral Operator operator G_0 such that the derivative G' of G satisfies*

$$\forall z \in \mathbb{C}: \; G'(z) = G(z)G_0.$$

Note that the name "multiplicative" just means that we gauge the operator by multiplication with a previously chosen family. This is analogous to "Q-weighted" generalized ζ-functions $\zeta(A,Q,z) := \zeta(s \mapsto AQ^s)(z)$ for pseudo-differential operators, i.e. $G = (z \mapsto Q^z)$.

Remark If we consider a canonical relation Γ and the corresponding algebra of Fourier Integral Operators $\mathcal{A}_\Gamma$, then we may be inclined to search for multiplicative gauges in $\mathcal{A}_\Gamma$. Unfortunately, the identity will not be an element of $\mathcal{A}_\Gamma$, in general. An appropriate candidate of an algebra to consider if looking for a multiplicative gauge, therefore, should be the unitalization $\mathcal{A}_\Gamma \oplus \mathbb{C}$ of $\mathcal{A}_\Gamma$. If $\mathcal{A}_\Gamma$ is unital already, taking the direct sum with $\mathbb{C}$ will not change anything at all. Note that we interpret the element $(a,\lambda) \in \mathcal{A}_\Gamma \oplus \mathbb{C}$ to be $a+\lambda$ which directly yields the following structure.

- $(a,0) = a \in \mathcal{A}_\Gamma$, $(0,1) = 1$
- $\forall \lambda \in \mathbb{C}: \; \lambda(a,\mu) + (b,\nu) = (\lambda a, \lambda\mu) + (b,\nu) = (\lambda a + b, \lambda\mu + \nu)$
- $(a,\lambda)(b,\mu) = (a+\lambda)(b+\mu) = ab + a\mu + \lambda b + \lambda\mu = (ab + \mu a + \lambda b, \lambda\mu)$

Since derivatives should exists within the algebra and we might be interested in using a functional calculus, it may be necessary to also include an $L(L_2(X))$-closure of $\mathcal{A}_\Gamma \oplus \mathbb{C}$.

However, keeping the search for multiplicative gauges simple, we may gauge with properly supported pseudo-differential operators $G(z)$ (cf. section 18.4 in [**69**]) at the cost of potentially leaving the algebra even further, that is, $AG(z)$ should not be expected to be in $\overline{\mathcal{A}_\Gamma \oplus \mathbb{C}}^{L(L_2(X))}$ anymore. In other words, it is easy to find gauges for $A \in \mathcal{A}_\Gamma$ but the gauged operators may be "very far away from" $\mathcal{A}_\Gamma$.

■

Let P be a gauged pseudo-differential operator. Then, we may also consider

$$\langle P(z)u, f\rangle$$

as a gauge. This is due to Theorems 18.2.7 and 18.2.8 in [**38**]. In particular, if f is a Lagrangian distribution, then it can be represented in the form $\int e^{i\langle x,\xi\rangle} a_f(x,\xi)d\xi$ which is nothing other than $P_f\delta_0$ where P_f is the pseudo-differential operator with amplitude a_f. Hence,

$$\langle P(z)u, f\rangle = \langle P_f' P(z)u, \delta_0\rangle.$$

For traces, though, a multiplicative gauge yields

$$\zeta(A)(z) = \langle \mathfrak{g}(z) \circ k_A, \delta_{\mathrm{diag}}\rangle$$

where $\mathfrak{g}(z) \circ k_A$ is the kernel of $G(z)A$ and $\forall \varphi \in C(X): \ \delta_{\mathrm{diag}}(\varphi) = \int_X \varphi(x,x)dx$ (i.e. δ_{diag} is the kernel of the identity).

Example Suppose u is an $\mathcal{M}$-gauged log-homogeneous distribution. We, thus, obtain

$$u(0)(x) = \tau_0(u(0))(x) + \int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle} v(0)(\xi) = \tilde{\tau}_0(u(0))(x) + (P_u\delta_0)(x)$$

where P_u is a pseudo-differential operator with amplitude $p_u(x,\xi) = v(\xi)$ for $\xi \in \mathbb{R}^N \setminus B_{\mathbb{R}^N}$. Furthermore, the complex power H^z with $H := \sqrt{|\Delta|}$ has the amplitude $p_z(x,\xi) = (2\pi)^{-N} \|\xi\|_{\ell_2(N)}^z$ where $|\Delta|$ is the (non-negative) Dirichlet Laplacian. This follows from $|\Delta|^{-1} = \mathcal{F}^{-1} \|m\|_{\ell_2(N)}^{-2} \mathcal{F}$ where m is the maximal multiplication operator with the argument on $L_2(\mathbb{R}^N)$

$$D(m) := \left\{ f \in L_2(\mathbb{R}^N);\ \left(\mathbb{R}^n \ni \xi \mapsto \xi f(\xi) \in \mathbb{C}^N\right) \in L_2(\mathbb{R}^N;\mathbb{C}^N) \right\},$$

$$m : D(m) \subseteq L_2(\mathbb{R}^N) \to L_2(\mathbb{R}^N; \mathbb{C}^N);\ f \mapsto (\xi \mapsto \xi f(\xi))\,.$$

$(-\Delta)^{-1}$ is well-known to be a compact operator. Hence, let $r-1$ be its spectral radius. Then, the holomorphic functional calculus yields

$$\begin{aligned}
H^z =& \left(|\Delta|^{-1}\right)^{-\frac{z}{2}} \\
=& \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \left(\lambda - (-\Delta)^{-1}\right)^{-1} d\lambda \\
=& \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \left((-\Delta)^{-1}\right)^{j} d\lambda \\
=& \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \left(\mathcal{F}^{-1} \|m\|_{\ell_2(N)}^{-2} \mathcal{F}\right)^{j} d\lambda \\
=& \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \mathcal{F}^{-1} \left(\|m\|_{\ell_2(N)}^{-2}\right)^{j} \mathcal{F} d\lambda \\
=& \mathcal{F}^{-1} \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \sum_{j\in\mathbb{N}_0} \lambda^{-(j+1)} \left(\|m\|_{\ell_2(N)}^{-2}\right)^{j} d\lambda \mathcal{F} \\
=& \mathcal{F}^{-1} \frac{1}{2\pi i}\int_{r\partial B_{\mathbb{C}}} \lambda^{-\frac{z}{2}} \left(\lambda - \|m\|_{\ell_2(N)}^{-2}\right)^{-1} d\lambda \mathcal{F} \\
=& \mathcal{F}^{-1} \left(\|m\|_{\ell_2(N)}^{-2}\right)^{-\frac{z}{2}} \mathcal{F} \\
=& \mathcal{F}^{-1} \|m\|_{\ell_2(N)}^{z} \mathcal{F}.
\end{aligned}$$

Using the composition formula for pseudo-differential operators, we obtain that $(2\pi)^N H^z P_u$ has the amplitude (for $\|\xi\|_{\ell_2(N)} \geq 1$)

$$\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} \partial_2^\alpha \left((2\pi)^N p_z\right)(x,\xi) \underbrace{(-i\partial_1)^\alpha p_u(x,\xi)}_{=0\ \Leftarrow\ \alpha\neq 0} = \|\xi\|_{\ell_2(N)}^{z} v(0)(\xi) = v(z)(\xi).$$

In other words,

$$u(z) \equiv (2\pi)^N H^z u(0)$$

modulo whatever happens on $B_{\mathbb{R}^N}$.

■

Example Let A be a poly-log-homogeneous Fourier Integral Operator and u a poly-log-homogeneous distribution with $I_0(A) = I_0(u) = \varnothing$. Suppose G and P are exponential multiplicative gauges, that is,

$$G'(z) = G(z)G_0 \qquad \text{and} \qquad P'(z) = P(z)P_0,$$

for A and u, respectively. Then

$$\zeta(GA)(z) = \sum_{n\in\mathbb{N}_0} \frac{\partial^n \zeta(GA)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n GA)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(GG_0^n A)(0)}{n!} z^n$$

and

$$\zeta(Pu)(z) = \sum_{n\in\mathbb{N}_0} \frac{\partial^n \zeta(Pu)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(\partial^n Pu)(0)}{n!} z^n = \sum_{n\in\mathbb{N}_0} \frac{\zeta(PP_0^n u)(0)}{n!} z^n$$

hold in sufficiently small neighborhoods of zero. Using

$$\begin{aligned}
&\zeta(GG_0^k A)(z) \\
&= \sum_{n\in\mathbb{N}_0} \frac{\int_X \int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} \partial^n \sigma(GG_0^k A)(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)}{n!} z^n \\
&+ \sum_{n\in\mathbb{N}_0} \frac{\int_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta} \partial^n \sigma(GG_0^k A)_0(0)\ d\mathrm{vol}_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}}{n!} z^n \\
&+ \sum_{n\in\mathbb{N}_0} \sum_{\iota\in I} \sum_{j=0}^{n} \frac{(-1)^{l_\iota+j+1}(l_\iota+j)! \int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta} \partial^{n-j} \tilde{\sigma}(GG_0^k A)_\iota(0)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}} z^n,
\end{aligned}$$

where $\sigma(G_0^k A)$ denotes the amplitude of $G_0^k A$, we obtain

$$\begin{aligned}
\zeta(GA)(z) &= \sum_{k\in\mathbb{N}_0} \frac{\zeta(GG_0^k A)(0)}{k!} z^k \\
&= \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \Bigg(\int_{\Delta(X)\times B_{\mathbb{R}^N}} e^{i\vartheta} \sigma(G_0^k A)\ d\mathrm{vol}_{\Delta(X)\times B_{\mathbb{R}^N}} \\
&\qquad + \int_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})} e^{i\vartheta} \sigma(G_0^k A)_0\ d\mathrm{vol}_{\Delta(X)\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})} \\
&\qquad + \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota!\,\mathrm{res}(G_0^k A)_\iota}{(N+d_\iota)^{l_\iota+1}} \Bigg) z^k
\end{aligned}$$

in a sufficiently small neighborhood of zero. For $\zeta(PP_0^n u)(0)$, we will denote the gauged poly-log-homogeneous distribution associated with $PP_0^k u$ by $\alpha\left(PP_0^k u\right)$ and use

$$\begin{aligned}\zeta(PP_0^k u)(z) &= \sum_{n\in\mathbb{N}_0}\frac{1}{n!}\int_{B_{\mathbb{R}^N}(0,1)}\partial^n\alpha\left(PP_0^k u\right)(0)d\mathrm{vol}_{B_{\mathbb{R}^N}(0,1)}z^n\\ &\quad+\sum_{n\in\mathbb{N}_0}\frac{\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}\partial^n\alpha\left(PP_0^k u\right)_0(0)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}}{n!}z^n\\ &\quad+\sum_{n\in\mathbb{N}_0}\sum_{\iota\in I}\sum_{j=0}^{n}\frac{(-1)^{l_\iota+j+1}(l_\iota+j)!\int_{\partial B_{\mathbb{R}^N}}\partial^{n-j}\tilde{\alpha}\left(PP_0^k u\right)_\iota(0)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}}{n!(N+d_\iota)^{l_\iota+j+1}}z^n\end{aligned}$$

to obtain

$$\begin{aligned}\zeta(Pu)(z) &= \sum_{k\in\mathbb{N}_0}\frac{\zeta(PP_0^k u)(0)}{k!}z^k\\ &= \sum_{k\in\mathbb{N}_0}\frac{1}{k!}\Bigg(\int_{B_{\mathbb{R}^N}(0,1)}\alpha\left(P_0^k u\right)d\mathrm{vol}_{B_{\mathbb{R}^N}(0,1)}\\ &\qquad+\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}\alpha\left(P_0^k u\right)_0 d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}\\ &\qquad+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\alpha\left(P_0^k u\right)_\iota}{(N+d_\iota)^{l_\iota+1}}\Bigg)z^k.\end{aligned}$$

■

Example If we consider a multiplicatively gauged $A(z) = BQ^z$ where Q may be non-invertible but is an element of an admissible algebra of Fourier Integral Operators with holomorphic functional calculus, e.g. a pseudo-differential operator of order 1 (order $q > 0$ can be obtained using the results of chapter 3) and spectral cut (the following is to be interpreted in this setting), then $Q^0 = 1 - 1_{\{0\}}(Q)$ where

$$1_{\{0\}}(Q) := \frac{1}{2\pi i}\int_{\partial B(0,\varepsilon)}(\lambda - Q)^{-1}\,d\lambda$$

with ε sufficiently small such that $B(0,\varepsilon)\cap\sigma(Q) = \{0\}$. In other words, $1_{\{0\}}(Q)$ is the projector onto the null space of Q. Thus, assuming $I_0 = \varnothing$ (that is, the Kontsevich-Vishik trace $\mathrm{tr}_{KV}(A(0))$ is well-defined and coincides with $\zeta(A)(0)$), we obtain

$$\zeta(A)(0) = \operatorname{tr}_{KV}\left(BQ^0\right) = \operatorname{tr}_{KV}(B) - \operatorname{tr}_{KV}\left(B1_{\{0\}}(Q)\right)$$

and

$$\begin{aligned}\forall k \in \mathbb{N}:\ \zeta(\partial^k A)(0) &= \operatorname{tr}_{KV}\left(B(\ln Q)^k Q^0\right)\\ &= \operatorname{tr}_{KV}\left(B(\ln Q)^k\right) - \operatorname{tr}_{KV}\left(B(\ln Q)^k 1_{\{0\}}(Q)\right)\end{aligned}$$

where we note that there still is a dependence on the spectral cut used to define the operators Q^z and $\ln Q$. These generalize the formulae (0.17) and (0.18) in **[56]** (note that the factors $(-1)^k$ are due to sign convention Q^z vs. Q^{-z}).

∎

PROPOSITION 4.5. *Let $A(z) = BQ^z$ be polyhomogeneous with Q as above, $\mathfrak{fp}\zeta$ the finite part of ζ, and $\operatorname{tr}_{\mathfrak{fp}}$ the finite part of the trace integral (that is, removing the principal part from the Laurent expansion $\zeta(A)$ and evaluating at zero; cf.* ***[47]****,* ***[48]****,* ***[51]****, and* ***[56]****). Furthermore, let c_k be the coefficient of $\frac{z^k}{k!}$ in the Laurent expansion of $\zeta(A)$ with $k \in \mathbb{N}_0$.*

Then, we obtain

$$\begin{aligned}c_k =&\zeta\left(\partial^k \mathfrak{fp}_0 A\right)(0) + \sum_{\iota\in I_0}\int_X\int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)}\partial^k a_\iota(0)(x,x,\xi)\ d\xi\ \operatorname{dvol}_X(x)\\ &- \sum_{\iota\in I_0}\frac{1}{k+1}\operatorname{res}\left(\partial^{k+1}A_\iota\right)(0)\\ =&\mathfrak{fp}\zeta\left(\partial^k A\right)(0) - \frac{1}{k+1}\operatorname{res}\left(\partial^{k+1}A\right)(0)\\ =&\operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k Q^0\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1}Q^0\right).\end{aligned}$$

In particular,

$$c_0 = \operatorname{tr}_{\mathfrak{fp}}(B) - \operatorname{res}(B\ln Q) - \operatorname{tr}_{\mathfrak{fp}}\left(B1_{\{0\}}(Q)\right)$$

and

$$\forall k \in \mathbb{N}: \; c_k = \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k\right) - \frac{1}{k+1}\operatorname{res}\left(B(\ln Q)^{k+1}\right) - \operatorname{tr}_{\mathfrak{fp}}\left(B(\ln Q)^k 1_{\{0\}}(Q)\right)$$

*generalize equations (0.12) and (0.14) in [**56**] (keeping in mind the factors* $(-1)^k$ *due to sign convention).*

If Q *is invertible, then* $1_{\{0\}}(Q) = 0$*, and for another admissible and invertible operator* Q'*, we obtain*

$$c_0(Q) - c_0(Q') = -\operatorname{res}\left(B\left(\ln Q - \ln Q'\right)\right) \tag{4.1}$$

*which is a generalization of equation (2.21) in [**47**] and (9) in [**55**].*

Furthermore, for $A(z) = [B, CQ^z]$ *with invertible* Q *(that is* $c_0 = 0$ *since* $\zeta(A) = 0$*), we obtain*

$$\operatorname{tr}_{\mathfrak{fp}}([B,C]) = \operatorname{res}([B, C\ln Q])$$

*a generalization of (2.20) in [**56**].*

Example Applying our ζ-calculus and the considerations above to complex powers also allows us to reproduce the variation formula for the multiplicative anomaly (2.18) in [**47**] using effectively the same proof. However, it should be noted that this approach now also works in algebras of Fourier Integral Operators provided they contain complex powers (or, at least, such that the ζ-functions are still defined).

$$\begin{aligned}&\partial_t\partial_s\left(\zeta\left(z \mapsto (A_tB)^z\right)(s) - \zeta\left(z \mapsto A_t^z\right)(s) - \zeta\left(z \mapsto B^z\right)(s)\right)\\ =&\partial_s\left(\partial_t\zeta\left(z \mapsto (A_tB)^z\right)(s) - \partial_t\zeta\left(z \mapsto A_t^z\right)(s)\right)\end{aligned}$$

can be evaluated using a suitable contour Γ and $C \in \{B, 1\}$ which yields

$$\partial_t\zeta\left(z \mapsto (A_tC)^z\right) = \zeta\left(z \mapsto \partial_t\frac{1}{2\pi i}\int_\Gamma \lambda^z(\lambda - A_tC)^{-1}d\lambda\right)$$

$$
\begin{aligned}
&=\zeta\left(z \mapsto \frac{1}{2\pi i}\int_\Gamma \lambda^z (A_t'C)(\lambda - A_tC)^{-2} d\lambda\right)\\
&=\zeta\left(z \mapsto (A_t'C)\frac{1}{2\pi i}\int_\Gamma \lambda^z \left(-\partial_\lambda(\lambda - A_tC)^{-1}\right) d\lambda\right)\\
&=\zeta\left(z \mapsto (A_t'C)\frac{1}{2\pi i}\int_\Gamma (\partial_\lambda\lambda^z)(\lambda - A_tC)^{-1} d\lambda\right)\\
&=\zeta\left(z \mapsto (A_t'C)\frac{1}{2\pi i}\int_\Gamma z\lambda^{z-1}(\lambda - A_tC)^{-1} d\lambda\right)\\
&=\zeta\left(z \mapsto z(A_t'C)(A_tC)^{-1}(A_tC)^z\right)\\
&=\zeta\left(z \mapsto zA_t'A_t^{-1}(A_tC)^z\right).
\end{aligned}
$$

Taking the other derivative, we obtain

$$
\begin{aligned}
\partial_s\partial_t\zeta\left(z \mapsto (A_tC)^z\right)(s) =&\partial_s\zeta\left(z \mapsto zA_t'A_t^{-1}(A_tC)^z\right)(s)\\
=&\zeta\left(z \mapsto \partial_z\left(zA_t'A_t^{-1}(A_tC)^z\right)\right)(s)\\
=&\zeta\left(z \mapsto A_t'A_t^{-1}(A_tC)^z + z\partial_z\left(A_t'A_t^{-1}(A_tC)^z\right)\right)(s)\\
=&(1+s\partial_s)\zeta\left(z \mapsto A_t'A_t^{-1}(A_tC)^z\right)(s).
\end{aligned}
$$

However, by assumption $\zeta\left(z \mapsto A_t'A_t^{-1}(A_tC)^z\right)$ is holomorphic near zero, i.e. its derivative $\zeta\left(z \mapsto A_t'A_t^{-1}(A_tC)^z\right)'$ is holomorphic near zero; hence,

$$
s\partial_s\zeta\left(z \mapsto A_t'A_t^{-1}(A_tC)^z\right)(s) \to 0 \qquad (s \to 0).
$$

In other words,

$$
\begin{aligned}
&\partial_t\partial_s\left(\zeta\left(z \mapsto (A_tB)^z\right)(s) - \zeta\left(z \mapsto A_t^z\right)(s) - \zeta\left(z \mapsto B^z\right)(s)\right)\\
=&\zeta\left(z \mapsto A_t'A_t^{-1}(A_tB)^z\right)(s) - \zeta\left(z \mapsto A_t'A_t^{-1}A_t^z\right)(s)
\end{aligned}
$$

which, according to equation (4.1), yields

$$
\begin{aligned}
\partial_t \ln F(A_t,B) =&\partial_t\partial_s\left(\zeta\left(z \mapsto (A_tB)^z\right)(s) - \zeta\left(z \mapsto A_t^z\right)(s) - \zeta\left(z \mapsto B^z\right)(s)\right)\\
=&\zeta\left(z \mapsto A_t'A_t^{-1}(A_tB)^z\right)(s) - \zeta\left(z \mapsto A_t'A_t^{-1}A_t^z\right)(s)
\end{aligned}
$$

$$= -\operatorname{res}\left(A_t' A_t^{-1}\left(\frac{\ln(A_t B)}{\operatorname{order} A_t B} - \frac{\ln A_t}{\operatorname{order} A_t}\right)\right)$$

with the multiplicative anomaly

$$F(A,B) := \frac{\exp\left(\zeta\left(z\mapsto (AB)^z\right)'(0)\right)}{\exp\left(\zeta\left(z\mapsto A^z\right)'(0)\right)\exp\left(\zeta\left(z\mapsto B^z\right)\right)'(0)}.$$

Choosing a multiplicative gauge G with $G' = GG_0$, we obtain a different variation formula of the multiplicative anomaly; namely,

$$\begin{aligned}\partial_t\left(\zeta(A_tB_tG)' - \zeta(A_tG)' - \zeta(B_tG)'\right) =& \zeta(A_t'B_tG)' + \zeta(A_tB_t'G)' - \zeta(A_t'G)' - \zeta(B_t'G)'\\ =& \zeta(A_t'(B_t-1)G)' + \zeta((A_t-1)B_t'G)'\\ =& \zeta(A_t'(B_t-1)G') + \zeta((A_t-1)B_t'G')\\ =& \zeta(A_t'(B_t-1)GG_0) + \zeta((A_t-1)B_t'GG_0).\end{aligned}$$

■

Remark Note that the mechanism explored in this chapter works whenever there is a representation $\int_{\mathbb{R}^N}\underbrace{\int_X e^{i\vartheta(x,x,\xi)}a(x,x,\xi)d\mathrm{vol}_X(x)}_{=:\alpha(\xi)}d\xi$ with poly-log-homogeneous α. In particular, we may consider algebras that do not have the form $\mathcal{A}_\Gamma$ where Γ intersects the co-normal bundle of the identity cleanly. Above, we used that $\langle k, \delta_{\mathrm{diag}}\rangle$ can be written as $\langle Pk, \delta_0\rangle$ for some pseudo-differential operator P, i.e. we used the clean intersection property to obtain the poly-log-homogeneous distribution form. However, for $\Re(z)$ sufficiently small, the gauged $k(z)$ is continuous, that is, $\langle k(z), \delta_{\mathrm{diag}}\rangle$ is well-defined and if we can show it extends meromorphically, the clean intersection property won't be necessary.

■

CHAPTER 5

The heat trace, fractional, and shifted fractional Laplacians on flat tori

In this chapter, we will apply Theorem 4.1 to some examples which are well-known or can be easily checked through spectral considerations.

Example (the Heat Trace on the flat torus $\mathbb{R}^N/_\Gamma$) Let $\Gamma \subseteq \mathbb{R}^N$ be a discrete group generated by a basis of $\mathbb{R}^N$, $|\Delta|$ the Dirichlet Laplacian on $\mathbb{R}^N$, δ the Dirichlet Laplacian on $\mathbb{R}^N/_\Gamma$, and T the semi-group generated by $-\delta$ on $\mathbb{R}^N/_\Gamma$. It is well-known that

$$\operatorname{tr} T(t) = \frac{\operatorname{vol}_{\mathbb{R}^N/_\Gamma}\left(\mathbb{R}^N/_\Gamma\right)}{(4\pi t)^{\frac{N}{2}}} \sum_{\gamma\in\Gamma} \exp\left(-\frac{\|\gamma\|_{\ell_2(N)}^2}{4t}\right)$$

holds; cf. e.g. equation 3.2.3.28 in [**67**]. Furthermore, the kernel κ_δ of δ is given by the kernel $\kappa_{|\Delta|}$ via $\kappa_\delta(x,y) = \sum_{\gamma\in\Gamma} \kappa_{|\Delta|}(x, y+\gamma)$; cf. e.g. section 3.2.2 in [**67**]. In other words,

$$\kappa_\delta(x,y) = \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^N} e^{i\langle x-y-\gamma,\xi\rangle} (2\pi)^{-N} \|\xi\|_{\ell_2(N)}^2 \, d\xi.$$

Hence, using functional calculus, we obtain

$$\kappa_{T(t)}(x,y) = \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^N} e^{i\langle x-y-\gamma,\xi\rangle} (2\pi)^{-N} e^{-t\|\xi\|_{\ell_2(N)}^2} d\xi.$$

Considering some gauge of $T(t)$ we obtain from the Laurent expansion (Theorem 4.1)

$$\begin{aligned}&\zeta(T(t))(0)\\ &= \int_{\mathbb{R}^N/_\Gamma\times B_{\mathbb{R}^N}} \sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle} (2\pi)^{-N} e^{-t\|\xi\|_{\ell_2(N)}^2} \, d\operatorname{vol}_{\mathbb{R}^N/_\Gamma\times B_{\mathbb{R}^N}}(x,\xi)\end{aligned}$$

$$+\int_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}\sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle}(2\pi)^{-N}\left(e^{-t\|\cdot\|^2_{\ell_2(N)}}\right)_0(\xi)\ d\mathrm{vol}_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}(x,\xi)$$
$$+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\,\mathrm{res}(T(t))_\iota}{(N+d_\iota)^{l_\iota+1}}.$$

Since $\left(\xi\mapsto e^{-t\|\xi\|^2_{\ell_2(N)}}\right)\in\mathcal{S}(\mathbb{R}^N)$, we can choose $I=\varnothing$ and $\left(e^{-t\|\cdot\|^2_{\ell_2(N)}}\right)_0 = e^{-t\|\cdot\|^2_{\ell_2(N)}}$ which yields

$$\begin{aligned}
&\zeta(T(t))(0)\\
=&\int_{\mathbb{R}^N/\Gamma\times B_{\mathbb{R}^N}}\sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle}(2\pi)^{-N}e^{-t\|\xi\|^2_{\ell_2(N)}}\ d\mathrm{vol}_{\mathbb{R}^N/\Gamma\times B_{\mathbb{R}^N}}(x,\xi)\\
&+\int_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}\sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle}(2\pi)^{-N}e^{-t\|\xi\|^2_{\ell_2(N)}}\ d\mathrm{vol}_{\mathbb{R}^N/\Gamma\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N})}(x,\xi)\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(2\pi)^N}\int_{B_{\mathbb{R}^N}}\sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle}e^{-t\|\xi\|^2_{\ell_2(N)}}\ d\mathrm{vol}_{B_{\mathbb{R}^N}}(\xi)\\
&+\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(2\pi)^N}\int_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}\sum_{\gamma\in\Gamma} e^{-i\langle\gamma,\xi\rangle}e^{-t\|\xi\|^2_{\ell_2(N)}}\ d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}^N}}(\xi)\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(2\pi)^N}\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^N} e^{-i\langle\gamma,\xi\rangle}e^{-t\|\xi\|^2_{\ell_2(N)}}\ d\xi\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi^2)^{\frac{N}{2}}}\sum_{\gamma\in\Gamma}\pi^{\frac{N}{2}}t^{-\frac{N}{2}}e^{-\frac{\|\gamma\|^2_{\ell_2(N)}}{4t}}\\
=&\frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi t)^{\frac{N}{2}}}\sum_{\gamma\in\Gamma}\exp\left(-\frac{\|\gamma\|^2_{\ell_2(N)}}{4t}\right),
\end{aligned}$$

i.e. precisely what we wanted to obtain.

■

Please note that the following example of fractional Laplacians exceeds the applicability of the ζ-function Laurent expansion as it is for now. However, we will consider shifted versions of the fractional Laplacian afterwards (there applicability is given) and show in chapter 6 that the Laurent expansion still holds in the non-shifted version (the relationship between the shifted and non-shifted fractional Laplacian are, in fact, the basis of the idea leading to the notion of mollification which will allow us to extend the ζ-function calculus to amplitudes that are poly-log-homogeneous everywhere on $\mathbb{R}^N \setminus \{0\}$).

Example (fractional Laplacian on the flat torus $\mathbb{R}/_{2\pi\mathbb{Z}}$) Let $H := \sqrt{|\Delta|}$ on $\mathbb{T} := \mathbb{R}/_{2\pi\mathbb{Z}}$ where $|\Delta|$ denotes the (non-negative) Laplacian. It is well-known that the spectrum $\sigma(H)$ of H is discrete, satisfies $\sigma(H) = \mathbb{N}_0$, and each non-zero eigenvalue has multiplicity 2. Furthermore, the symbol of H^z has the kernel

$$\kappa_{H^z}(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{|\xi|^z}{2\pi} d\xi.$$

The singular part is given for $n = 0$ and $\sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{|\xi|^z}{2\pi} d\xi$ is regular.[1]

Let $\alpha \in (-1,0)$. Since ζ is the spectral ζ-function, we obtain (μ_λ denoting the multiplicity of λ and $\Re(z) < -1$)

$$\zeta\left(s \mapsto H^s H^\alpha\right)(z) = \sum_{\lambda\in\sigma(H)\setminus\{0\}} \mu_\lambda \lambda^{z+\alpha} = 2\sum_{n\in\mathbb{N}} n^{z+\alpha} = 2\zeta_R(-z-\alpha)$$

where ζ_R denotes Riemann's ζ-function. In particular,

$$\zeta\left(s \mapsto H^s H^\alpha\right)(0) = 2\zeta_R(-\alpha).$$

On the other hand, we have the Laurent expansion (Theorem 4.1)

$$\begin{aligned}\zeta\left(s \mapsto H^s H^\alpha\right)(z) = \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \Bigg(&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}} e^{i\vartheta} \sigma\left((\ln H)^k H^\alpha\right) \,\mathrm{dvol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}} \\ &+ \int_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}})} e^{i\vartheta} \sigma\left((\ln H)^k H^\alpha\right)_0 \,\mathrm{dvol}_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}})} \\ &+ \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota!\,\mathrm{res}\left(\left((\ln H)^k H^\alpha\right)_\iota\right)}{(1+d_\iota)^{l_\iota+1}} \Bigg) z^k,\end{aligned}$$

i.e.

$$\zeta\left(s \mapsto H^s H^\alpha\right)(0) = \int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}} e^{i\vartheta} \sigma\left(H^\alpha\right) \,\mathrm{dvol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}}$$

[1] Here, we will assume this is well-known. However, it would also follow from the fact that the kernel is C^∞ in a neighborhood of the diagonal which we will prove independently from any results of this chapter (beginning of chapter 8).

$$+\int_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}})} e^{i\vartheta}\sigma\left(H^{\alpha}\right)_0 \ \mathrm{dvol}_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}})}$$
$$+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(H^{\alpha}\right)_\iota}{(1+d_\iota)^{l_\iota+1}}.$$

Plugging in our kernel yields

$$\begin{aligned}\zeta\left(s\mapsto H^sH^{\alpha}\right)(0) =& \sum_{n\in\mathbb{Z}}\int_0^{2\pi}\int_{-1}^{1} e^{-2\pi in\xi}\frac{|\xi|^{\alpha}}{2\pi}\ d\xi\ dx\\ &+\sum_{n\in\mathbb{Z}\smallsetminus\{0\}}\int_0^{2\pi}\int_{\mathbb{R}_{\leq 1}\cup\mathbb{R}_{\geq 1}} e^{-2\pi in\xi}\frac{|\xi|^{\alpha}}{2\pi}\ d\xi\ dx\\ &-\frac{1}{1+\alpha}\int_0^{2\pi}\int_{\partial B_{\mathbb{R}}}\frac{|\xi|^{\alpha}}{2\pi}\ \mathrm{dvol}_{\partial B_{\mathbb{R}}}(\xi)\ dx\\ =&\int_{-1}^{1}|\xi|^{\alpha}\,d\xi+\sum_{n\in\mathbb{Z}\smallsetminus\{0\}}\int_{\mathbb{R}} e^{-2\pi in\xi}\,|\xi|^{\alpha}\,d\xi\\ &-\frac{1}{1+\alpha}\int_{\partial B_{\mathbb{R}}}|\xi|^{\alpha}\,\mathrm{dvol}_{\partial B_{\mathbb{R}}}(\xi).\end{aligned}$$

Since $\alpha\in(-1,0)$ and $\mathrm{vol}_{\partial B_{\mathbb{R}}}$ is the sum of point measures $\delta_{-1}+\delta_1$, we obtain

$$\int_{-1}^{1}|\xi|^{\alpha}\,d\xi = 2\int_0^1\xi^{\alpha}d\xi=\frac{2}{\alpha+1}=\frac{1}{1+\alpha}\int_{\partial B_{\mathbb{R}}}|\xi|^{\alpha}\,\mathrm{dvol}_{\partial B_{\mathbb{R}}}(\xi),$$

i.e.

$$\zeta\left(s\mapsto H^sH^{\alpha}\right)(0)=\sum_{n\in\mathbb{Z}\smallsetminus\{0\}}\int_{\mathbb{R}} e^{-2\pi in\xi}\,|\xi|^{\alpha}\,d\xi.$$

Using that the Fourier transform of $\xi\mapsto|\xi|^{\alpha}$ is

$$\int_{\mathbb{R}} e^{-2\pi ix\xi}\,|\xi|^{\alpha}\,d\xi=\frac{2\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(\alpha+1)}{|2\pi x|^{\alpha+1}}$$

and Riemann's functional equation

$$\zeta_R(z)=2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta_R(1-z),$$

we obtain (in the sense of meromorphic extensions)

$$\begin{aligned}\zeta\left(s\mapsto H^sH^{\alpha}\right)(0) &= \sum_{n\in\mathbb{Z}\smallsetminus\{0\}}\int_{\mathbb{R}} e^{-2\pi in\xi}\,|\xi|^{\alpha}\,d\xi\\ &=\sum_{n\in\mathbb{Z}\smallsetminus\{0\}}\frac{2\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(\alpha+1)}{|2\pi n|^{\alpha+1}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}} \cdot 2\sum_{n\in\mathbb{N}} \frac{1}{n^{\alpha+1}} \\
&= 2\underbrace{2(2\pi)^{(-\alpha)-1}\sin\left(\frac{-\alpha\pi}{2}\right)\Gamma(1-(-\alpha))\zeta_R(1-(-\alpha))}_{=\zeta_R(-\alpha)}.
\end{aligned}
$$

■

Remark Using identification via meromorphic extension of

$$
\zeta_R(z) = \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \frac{\sin\left(\frac{-z\pi}{2}\right)\Gamma(z+1)}{|2\pi n|^{z+1}}
$$

and, therefore,

$$
\forall z \in \mathbb{C}\smallsetminus\{-1\}: \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i n\xi} |\xi|^z \, d\xi = 2\zeta_R(-z)
$$

as well as

$$
\int_{-1}^{1} |\xi|^z \, d\xi = \frac{1}{1+z} \int_{\partial B_{\mathbb{R}}} |\xi|^z \, d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi),
$$

the example above extends to all $\alpha \in \mathbb{C}\smallsetminus\{-1\}$, i.e.

$$
\zeta_R = \left(\alpha \mapsto \frac{1}{2}\zeta\left(s \mapsto H^s H^{-\alpha}\right)(0)\right).
$$

■

Example (the generalized ζ-determinant of $s \mapsto H^{s+\alpha}$) Let $\alpha \in \mathbb{C}\smallsetminus\{-1\}$. In order to calculate $\det_\zeta\left(s \mapsto H^s H^\alpha\right) = \exp\left(\zeta\left(s \mapsto H^s H^\alpha\right)'(0)\right)$, it suffices to know the derivative $\zeta\left(s \mapsto H^s H^\alpha\right)'(0)$. From the spectral ζ-function we directly obtain

$$
\zeta\left(s \mapsto H^s H^\alpha\right)'(0) = \partial\left(z \mapsto 2\zeta_R(-z)\right)(\alpha) = -2\zeta_R'(-\alpha).
$$

On the other hand, we may invest

$$
\begin{aligned}
\zeta\left(s \mapsto H^s H^\alpha\right)(z) = \sum_{k\in\mathbb{N}_0} \frac{1}{k!}\Bigg(&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}} e^{i\vartheta}\sigma\left((\ln H)^k H^\alpha\right) \, d\mathrm{vol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}} \\
&+ \int_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}})} e^{i\vartheta}\sigma\left((\ln H)^k H^\alpha\right)_0 \, d\mathrm{vol}_{\Delta(\mathbb{T})\times(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}})}
\end{aligned}
$$

$$+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(\left(\ln H\right)^k H^\alpha\right)_\iota}{(1+d_\iota)^{l_\iota+1}}\Bigg)z^k,$$

(Theorem 4.1) again, and find

$$\begin{aligned}\zeta\left(s\mapsto H^sH^\alpha\right)'(0)=&\int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}}e^{i\vartheta}\sigma\left(\ln HH^\alpha\right)\ d\mathrm{vol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}}\\&+\int_{\Delta(\mathbb{T})\times\left(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}}\right)}e^{i\vartheta}\sigma\left(\ln HH^\alpha\right)_0\ d\mathrm{vol}_{\Delta(\mathbb{T})\times\left(\mathbb{R}_{\geq1}\times\partial B_{\mathbb{R}}\right)}\\&+\sum_{\iota\in I}\frac{(-1)^{l_\iota+1}l_\iota!\operatorname{res}\left(\ln HH^\alpha\right)_\iota}{(1+d_\iota)^{l_\iota+1}}.\end{aligned}$$

Using the amplitude $\frac{\ln|\xi|}{2\pi}$ of $\ln H$ on $\mathbb{R}$, yields that

$$\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{i(x-y-2\pi n)\xi}\frac{|\xi|^\alpha\ln|\xi|}{2\pi}\ d\xi$$

is the kernel of $\ln HH^\alpha$ on $\mathbb{T}$. Again, the singular part is given for $n=0$ yielding $\#I=1$, $d_\iota=\alpha$, and $l_\iota=1$, as well as

$$\begin{aligned}\zeta\left(s\mapsto H^sH^\alpha\right)'(0)=&\int_0^{2\pi}\int_{-1}^1\sum_{n\in\mathbb{Z}}e^{-2\pi in\xi}\frac{|\xi|^\alpha\ln|\xi|}{2\pi}\ d\xi\ dx\\&+\int_0^{2\pi}\int_{\mathbb{R}_{<-1}\cup\mathbb{R}_{>1}}\sum_{n\in\mathbb{Z}\setminus\{0\}}e^{-2\pi in\xi}\frac{|\xi|^\alpha\ln|\xi|}{2\pi}\ d\xi\ dx\\&+\frac{1}{(1+\alpha)^2}\int_0^{2\pi}\int_{\partial B_{\mathbb{R}}}\frac{|\xi|^\alpha\ln|\xi|}{2\pi}\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)\ dx\\=&\int_{-1}^1|\xi|^\alpha\ln|\xi|\ d\xi+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\alpha\ln|\xi|\ d\xi+\frac{2}{(1+\alpha)^2}.\end{aligned}$$

Note that

$$\int_{-1}^1|\xi|^\alpha\ln|\xi|\ d\xi=2\int_0^1\xi^\alpha\ln\xi\ d\xi=-\frac{2}{(\alpha+1)^2}$$

holds for $\Re(\alpha)>-1$ and, hence, by meromorphic extension

$$\begin{aligned}\zeta\left(s\mapsto H^sH^\alpha\right)'(0)=&\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\alpha\ln|\xi|\ d\xi\\=&\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}\partial\left(\beta\mapsto|\xi|^\beta\right)(\alpha)\ d\xi\\=&\partial\left(\beta\mapsto\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}|\xi|^\beta\ d\xi\right)(\alpha)\end{aligned}$$

$$
\begin{aligned}
&= \partial\left(\beta \mapsto 2\zeta_R(-\beta)\right)(\alpha) \\
&= -2\zeta_R'(-\alpha).
\end{aligned}
$$

■

Similarly, we can take higher order derivatives.

EXAMPLE ($\partial^k \zeta\left(s \mapsto H^s H^\alpha\right)(0)$ ON $\mathbb{R}/_{2\pi\mathbb{Z}}$) Regarding higher order derivatives the spectral ζ-function yields

$$\partial^k \zeta\left(s \mapsto H^s H^\alpha\right)(0) = \partial^k\left(z \mapsto 2\zeta_R(-z)\right)(\alpha) = (-1)^k \cdot 2\partial^k \zeta_R(-\alpha).$$

From

$$
\begin{aligned}
\zeta\left(s \mapsto H^s H^\alpha\right)(z) = \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \Bigg(& \int_{\Delta(\mathbb{T})\times B_{\mathbb{R}}} e^{i\vartheta} \sigma\left((\ln H)^k H^\alpha\right)\, d\mathrm{vol}_{\Delta(\mathbb{T})\times\partial B_{\mathbb{R}}} \\
& + \int_{\Delta(\mathbb{T})\times\left(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}}\right)} e^{i\vartheta} \sigma\left((\ln H)^k H^\alpha\right)_0\, d\mathrm{vol}_{\Delta(\mathbb{T})\times\left(\mathbb{R}_{\geq 1}\times\partial B_{\mathbb{R}}\right)} \\
& + \sum_{\iota\in I} \frac{(-1)^{l_\iota+1} l_\iota!\, \mathrm{res}\left((\ln H)^k H^\alpha\right)_\iota}{(1+d_\iota)^{l_\iota+1}} \Bigg) z^k
\end{aligned}
$$

(Theorem 4.1) we obtain

$$
\begin{aligned}
\partial^k \zeta\left(s \mapsto H^s H^\alpha\right)(0) = & \int_0^{2\pi} \int_{-1}^1 \sum_{n\in\mathbb{Z}} e^{-2\pi i n\xi} \frac{|\xi|^\alpha (\ln|\xi|)^k}{2\pi}\, d\xi\, dx \\
& + \int_0^{2\pi} \int_{\mathbb{R}\setminus B_{\mathbb{R}}} \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-2\pi i n\xi} \frac{|\xi|^\alpha (\ln|\xi|)^k}{2\pi}\, d\xi\, dx \\
& + \frac{(-1)^{k+1} k!}{(1+\alpha)^{k+1}} \int_0^{2\pi} \int_{\partial B_{\mathbb{R}}} \frac{|\xi|^\alpha}{2\pi}\, d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)\, dx \\
= & 2\int_0^1 \xi^\alpha (\ln\xi)^k\, d\xi + \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i n\xi} |\xi|^\alpha (\ln|\xi|)^k\, d\xi \\
& - \frac{2\cdot(-1)^k k!}{(1+\alpha)^{k+1}} \\
= & 2\partial^k\left(\beta \mapsto \int_0^1 \xi^\beta d\xi\right)(\alpha) - \frac{2\cdot(-1)^k k!}{(1+\alpha)^{k+1}} \\
& + \partial^k\left(\beta \mapsto \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i n\xi} |\xi|^\beta\, d\xi\right)(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\underbrace{\partial^k \left(\beta \mapsto (1+\beta)^{-1}\right)(\alpha)}_{=(-1)^k k!(1+\alpha)^{-(k+1)}} - \frac{2\cdot(-1)^k k!}{(1+\alpha)^{k+1}} + \partial^k \left(\beta \mapsto 2\zeta_R(-\beta)\right)(\alpha) \\
&= (-1)^k \cdot 2\partial^k \zeta_R(-\alpha).
\end{aligned}
$$

■

Finally, let us calculate the residue of $\zeta\left(s \mapsto H^s H^{-1}\right)$.

Example ($\operatorname{res}_0 \zeta\left(s \mapsto H^s H^{-1}\right)$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$) $\zeta\left(s \mapsto H^s H^{-1}\right)(z) = 2\zeta_R(1-z)$ shows that $\operatorname{res}_0 \zeta\left(s \mapsto H^s H^{-1}\right) = -2\operatorname{res}_1 \zeta_R = -2$. Also, using the Laurent expansion (Theorem 4.1) of $\zeta(A)$ for $A = \left(s \mapsto H^s H^{-1}\right)$, we obtain

$$
\operatorname{res}_0 \zeta\left(s \mapsto H^s H^{-1}\right) = -\int_0^{2\pi} \int_{\partial B_{\mathbb{R}}} \frac{|\xi|^{-1}}{2\pi} \, d\mathrm{vol}_{\partial B_{\mathbb{R}}} \, dx = -2.
$$

■

Furthermore, we can consider shifted fractional Laplacians which do not have singular amplitudes, that is, these are actually covered by the theory we have developed so far. They will also lead to the crucial observation that will help incorporate the case of singular amplitudes and, thus, justify the example of fractional Laplacians.

Example (shifted fractional Laplacians on $\mathbb{R}/_{2\pi\mathbb{Z}}$) Again, let $H := \sqrt{|\Delta|}$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$. Furthermore, let $h \in (0,1]$ and $G := h + H$. Then,

$$
\zeta\left(s \mapsto G^{s+\alpha}\right)(z) = \sum_{n\in\mathbb{Z}} (h+|n|)^{z+\alpha} = 2\sum_{n\in\mathbb{N}_0} (h+n)^{z+\alpha} - h^{z+\alpha} = 2\zeta_H(-z-\alpha;h) - h^{z+\alpha}
$$

where $\zeta_H(z;h)$ denotes the Riemann-Hurwitz-ζ-function. In order to use our formalism above (Theorem 4.1), we will need to write $\xi \mapsto (h+|\xi|)^\alpha$ as a series of poly-log-homogeneous functions. Using Newton's binomial theorem

$$
\forall x,y \in \mathbb{R}\ \forall r \in \mathbb{C} : \left(|x| > |y| \ \Rightarrow\ (x+y)^r = \sum_{k\in\mathbb{N}_0} \binom{r}{k} x^{r-k} y^k\right)
$$

where

$$
\binom{r}{k} := \frac{1}{k!}\prod_{j=0}^{k-1}(r-j) = \frac{r(r-1)\cdots(r-k+1)}{k!},
$$

we obtain

$$(h+|\xi|)^{\alpha} = \sum_{k\in\mathbb{N}_0} \binom{\alpha}{k} |\xi|^{\alpha-k} h^k$$

for $|\xi| \geq 1$, i.e. the kernel

$$k_{G^{z+\alpha}}(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x-y-2\pi n)\xi} \frac{1}{2\pi} (h+|\xi|)^{z+\alpha} d\xi$$

of $G^{z+\alpha}$ is, in fact, given by a poly-log-homogeneous amplitude. For $\alpha = -1$, the critical term in zero is given by the $k = 0$ term of $\sum_{k\in\mathbb{N}_0} \binom{\alpha}{k} |\xi|^{\alpha-k} h^k$, i.e.

$$\operatorname{res}_0 \zeta\left(s \mapsto G^{s-1}\right) = -\int_{\partial B_{\mathbb{R}}} |\xi|^{-1} d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi) = -2.$$

On the other hand, the spectral calculation yields

$$\begin{aligned}\operatorname{res}_0 \zeta\left(s \mapsto G^{s-1}\right) &= \operatorname{res}_0\left(z \mapsto 2\zeta_H(-z+1;h) - h^{z+\alpha}\right) = 2\operatorname{res}_0\left(z \mapsto \zeta_H(-z+1;h)\right) \\ &= -2\operatorname{res}_0\left(z \mapsto \zeta_H(z-1;h)\right) = -2\operatorname{res}_1 \zeta_H(\cdot;h) = -2.\end{aligned}$$

For $\alpha \neq -1$ and $|\xi| \geq 1$,

$$(h+|\xi|)^{\alpha} = \sum_{k\in\mathbb{N}_0} \binom{\alpha}{k} h^k |\xi|^{\alpha-k}$$

implies $\alpha - k = -1$ if and only if $k = \alpha + 1 \in \mathbb{N}_0$. However, since $\binom{\alpha}{\alpha+1} = 0$ for $\alpha \in \mathbb{N}_0$, we obtain $I_0 = \varnothing$ and

$$\begin{aligned}\zeta\left(s \mapsto G^{s+\alpha}\right)(0) &= \sum_{n\in\mathbb{Z}} \int_{-1}^{1} e^{-2\pi i n\xi} (h+|\xi|)^{\alpha} d\xi \\ &\quad + \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}\setminus[-1,1]} e^{-2\pi i n\xi} (h+|\xi|)^{\alpha} d\xi \\ &\quad + \sum_{k\in\mathbb{N}_0} \frac{-1}{1+\alpha-k} \int_{\partial B_{\mathbb{R}}} \binom{\alpha}{k} h^k |\xi|^{\alpha-k} d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi) \\ &= \int_{-1}^{1} (h+|\xi|)^{\alpha} d\xi - \sum_{k\in\mathbb{N}_0} \frac{2}{1+\alpha-k} \binom{\alpha}{k} h^k \\ &\quad + \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i n\xi} (h+|\xi|)^{\alpha} d\xi.\end{aligned}$$

Observing

$$\begin{aligned}\int_{-1}^{1}(h+|\xi|)^\alpha d\xi =&2\int_0^1 (h+\xi)^\alpha d\xi\\ =&2\int_h^{1+h}\xi^\alpha d\xi\\ =&\frac{2}{\alpha+1}\left((1+h)^{\alpha+1}-h^{\alpha+1}\right)\\ =&\frac{-2h^{\alpha+1}}{\alpha+1}+\frac{2}{\alpha+1}\sum_{k\in\mathbb{N}_0}\binom{\alpha+1}{k}h^k\\ =&\frac{-2h^{\alpha+1}}{\alpha+1}+2\sum_{k\in\mathbb{N}_0}\frac{1}{\alpha-k+1}\binom{\alpha}{k}h^k\end{aligned}$$

leaves us with

$$\zeta\left(s\mapsto G^{s+\alpha}\right)(0) =\frac{-2h^{\alpha+1}}{\alpha+1}+\underbrace{\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}(h+|\xi|)^\alpha d\xi}_{\text{non-singular}}.$$

This is precisely what we expect since the principal part of $\zeta_H(z;h)$ near 1 is $\frac{h^{1-z}}{z-1}$ (cf. equation 3.1.1.10 in [**67**]), i.e.

$$\zeta\left(s\mapsto G^{s+\alpha}\right)(0) = 2\zeta_H(-z-\alpha;h)-h^{z+\alpha}$$

has principal part $2\frac{h^{1+\alpha}}{-\alpha-1}$.

Unfortunately, evaluating $\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}e^{-2\pi in\xi}(h+|\xi|)^\alpha d\xi$ is a wee tricky. We will use that

$$\int_{\mathbb{R}}(h+|\xi|)^\alpha d\xi =2\int_{\mathbb{R}_{\geq 0}}(h+\xi)^\alpha d\xi = 2\int_{\mathbb{R}_{\geq h}}\xi^\alpha d\xi = -\frac{2h^{\alpha+1}}{\alpha+1}$$

holds for $\Re(\alpha)<-1$ and note

$$\zeta\left(s\mapsto G^{s+\alpha}\right)(0) = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2\pi in\xi}(h+|\xi|)^\alpha d\xi$$

by meromorphic extension. Furthermore, we obtain

$$\zeta\left(s\mapsto G^{s+\alpha}\right)(0) = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2\pi in\xi}(h+|\xi|)^\alpha d\xi$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}_{\geq 0}} e^{-2\pi i n\xi}(h+|\xi|)^{\alpha} d\xi + \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}_{<0}} e^{-2\pi i n\xi}(h+|\xi|)^{\alpha} d\xi \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \int_{\mathbb{R}_{\geq h}} e^{-2\pi i n\xi}\xi^{\alpha} d\xi + \sum_{n\in\mathbb{Z}} \int_{-\infty}^{0-} e^{-2\pi i n\xi}(h-\xi)^{\alpha} d\xi \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \int_{\mathbb{R}_{\geq h}} e^{-2\pi i n\xi}\xi^{\alpha} d\xi + \sum_{n\in\mathbb{Z}} \left(-\int_{\infty}^{h+} e^{-2\pi i n(h-\xi)}\xi^{\alpha} d\xi \right) \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \int_{\mathbb{R}_{\geq h}} e^{-2\pi i n\xi}\xi^{\alpha} d\xi + \sum_{n\in\mathbb{Z}} e^{-2\pi i n h} \int_{\mathbb{R}_{>h}} e^{2\pi i n\xi}\xi^{\alpha} d\xi \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \int_{\mathbb{R}_{\geq h}} e^{-2\pi i n\xi}\xi^{\alpha} d\xi + \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \int_{\mathbb{R}_{>h}} e^{-2\pi i n\xi}\xi^{\alpha} d\xi \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \left(\int_{\mathbb{R}} e^{-2\pi i n\xi} 1_{\mathbb{R}_{\geq h}}(\xi)\xi^{\alpha} d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi} 1_{\mathbb{R}_{>h}}(\xi)\xi^{\alpha} d\xi \right).
\end{aligned}
$$

For $\varepsilon \in (0,1)$ let

$$
\varphi_{\varepsilon}(x) := \begin{cases} 0 & , \ x \in \mathbb{R}_{\leq h-\varepsilon} \\ \varepsilon^{-1}(x-h+\varepsilon) & , x \in (h-\varepsilon, h) \\ 1 & , \ x \in \mathbb{R}_{\geq h} \end{cases}
$$

and

$$
\psi_{\varepsilon}(x) := \begin{cases} 0 & , \ x \in \mathbb{R}_{\leq h} \\ \varepsilon^{-1}(x-h) & , x \in (h, h+\varepsilon) \\ 1 & , \ x \in \mathbb{R}_{\geq h+\varepsilon} \end{cases} .
$$

Then,

$$
\begin{aligned}
\zeta\left(s \mapsto G^{s+\alpha}\right)(0) &= \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \left(\int_{\mathbb{R}} e^{-2\pi i n\xi} 1_{\mathbb{R}_{\geq h}}(\xi)\xi^{\alpha} d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi} 1_{\mathbb{R}_{>h}}(\xi)\xi^{\alpha} d\xi \right) \\
&= \sum_{n\in\mathbb{Z}} e^{2\pi i n h} \lim_{\varepsilon\searrow 0} \left(\int_{\mathbb{R}} e^{-2\pi i n\xi} \varphi_{\varepsilon}(\xi)\xi^{\alpha} d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi} \psi_{\varepsilon}(\xi)\xi^{\alpha} d\xi \right)
\end{aligned}
$$

can be evaluated using the Poisson summation formula on a lattice Λ (cf. Chapter VII.2 Theorem 2.4 in [**71**])

$$
\sum_{\lambda\in\Lambda} f(x+\lambda) = \sum_{\lambda\in\Lambda} \mathcal{F}f(\lambda)e^{2\pi i\lambda x}
$$

which yields (we can move $\lim_{\varepsilon\searrow 0}$ freely in and out of integrals and series due to meromorphic extension, dominated convergence, and the series converging absolutely for $\Re(\alpha) < -1$)

$$\begin{aligned}
\zeta\left(s\mapsto G^{s+\alpha}\right)(0) =& \lim_{\varepsilon\searrow 0}\sum_{n\in\mathbb{Z}} e^{2\pi i n h}\left(\int_{\mathbb{R}} e^{-2\pi i n\xi}\varphi_\varepsilon(\xi)\xi^\alpha d\xi + \int_{\mathbb{R}} e^{-2\pi i n\xi}\psi_\varepsilon(\xi)\xi^\alpha d\xi\right)\\
=& \lim_{\varepsilon\searrow 0}\sum_{n\in\mathbb{Z}}\left(\varphi_\varepsilon(h+n)(h+n)^\alpha + \psi_\varepsilon(h+n)(h+n)^\alpha\right)\\
=& \lim_{\varepsilon\searrow 0}\left(\sum_{n\in\mathbb{N}_0}\varphi_\varepsilon(h+n)(h+n)^\alpha + \sum_{n\in\mathbb{N}}\psi_\varepsilon(h+n)(h+n)^\alpha\right)\\
=& \sum_{n\in\mathbb{N}_0}(h+n)^\alpha + \sum_{n\in\mathbb{N}}(h+n)^\alpha\\
=& 2\zeta_H(-\alpha;h) - h^\alpha.
\end{aligned}$$

Considering derivatives, we obtain

$$\partial^m\zeta\left(s\mapsto G^s G^\alpha\right)(0) = 2(-1)^m\partial^m\zeta_H(-\alpha;h) - h^\alpha(\ln h)^m$$

from the spectral ζ-function while the Laurent expansion (Theorem 4.1) yields

$$\begin{aligned}
&\partial^m\zeta\left(s\mapsto G^s G^\alpha\right)(0)\\
=&\sum_{n\in\mathbb{Z}}\int_{-1}^{1} e^{-2\pi i n\xi}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&+\sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}\setminus[-1,1]} e^{-2\pi i n\xi}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&+\sum_{k\in\mathbb{N}_0}\sum_{j=0}^{m}\frac{(-1)^{j+1}j!\int_{\partial B_{\mathbb{R}}}\partial^{m-j}\left(\beta\mapsto\binom{\beta}{k}h^k|\xi|^{\beta-k}\right)(\alpha)\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)}{(\alpha-k+1)^{j+1}}\\
=&\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2\pi i n\xi}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&-2\int_{\mathbb{R}_{\geq 1}}(h+|\xi|)^\alpha\left(\ln(h+|\xi|)\right)^m\ d\xi\\
&+\partial^m\left(\beta\mapsto\sum_{k\in\mathbb{N}_0}\frac{-\int_{\partial B_{\mathbb{R}}}\binom{\beta}{k}h^k|\xi|^{\beta-k}\ d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi)}{\beta-k+1}\right)(\alpha)\\
=&\partial^m\left(\beta\mapsto\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2\pi i n\xi}(h+|\xi|)^\beta\ d\xi\right)(\alpha)
\end{aligned}$$

$$
\begin{aligned}
&-2\partial^m\left(\beta\mapsto\int_{\mathbb{R}_{\geq 1}}(h+|\xi|)^\beta\ d\xi\right)(\alpha)+\partial^m\left(\beta\mapsto\sum_{k\in\mathbb{N}_0}\frac{-2\binom{\beta}{k}h^k}{\beta-k+1}\right)(\alpha)\\
=&\partial^m\left(\beta\mapsto 2\zeta_H(-\beta;h)-h^\beta\right)(\alpha)\\
&-2\partial^m\left(\beta\mapsto-\frac{(1+h)^{\beta+1}}{\beta+1}\right)(\alpha)-2\partial^m\left(\beta\mapsto\frac{(1+h)^{\beta+1}}{\beta+1}\right)(\alpha)\\
=&2(-1)^m\partial^m\zeta_H(-\alpha;h)-h^\alpha(\ln h)^m.
\end{aligned}
$$

■

CHAPTER 6

Mollification of singular amplitudes

In this chapter we will address the fact that many applications consider amplitudes which are homogeneous on $\mathbb{R}^N \setminus \{0\}$ rather than just $\mathbb{R}^N \setminus B_{\mathbb{R}^N}(0,1)$. In particular for pseudo-differential operators, this is the classical case. However, it does not add too many problems because we can use a cut-off function near zero and extend the symbol as a distribution to $\mathbb{R}^N$ (which is uniquely possible up to the critical degrees of homogeneity). Then, we are left with a Fourier transform of a compactly supported distribution, i.e. the corresponding kernel is continuous and we can take the trace. In the general Fourier Integral Operator case, on the other hand, the situation is more complicated. Hence, in this chapter, we will show that the Laurent expansion holds for such amplitudes, as well, and not just modulo trace-class operators. We will prove this result by showing that we can always find a sequence of "nice" families of operators (that is, the amplitudes are C^∞ in $B_{\mathbb{R}^N}(0,1)$) such that their ζ-functions converge compactly (this process is called "mollification"). Once compact convergence is shown, we know that all local properties (in particular the Laurent expansion) are preserved taking the limit. In other words, by the end of this chapter, the ζ-function calculus considered above will fully contain the pseudo-differential case.

The idea of mollification is strongly intertwined with the examples of the shifted and non-shifted fractional Laplacians in the previous chapter. Our calculations of $\zeta\left(s \mapsto H^s H^\alpha\right)$ have been pushing the boundaries of our formulae in the sense that the Laurent expansion of Fourier Integral Operators assumes integrability of all amplitudes $a(z)$ on $B_{\mathbb{R}^N}$. This is obviously not true for $a(z)(x,y,\xi) = |\xi|^{z+\alpha}$ if $\Re(z) < -1 - \Re(\alpha)$ (recall $H := \sqrt{|\Delta|}$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$ where $|\Delta|$ is the non-negative Laplacian on $\mathbb{R}/_{2\pi\mathbb{Z}}$). Hence, we would have to consider the Laurent expansion in a more general version where we also allowed

$$z \mapsto \int_X \int_{B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} a(z)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)$$

to have a non-vanishing principal part.

However, we may use $\zeta\left(s \mapsto G^s G^\alpha\right)$ to justify the calculations as they are by taking the limit $h \searrow 0$ in $\zeta\left(s \mapsto G^s G^\alpha\right)$ (recall $G := h + H$ with $h \in (0,1]$). Note that

$$\zeta_H(z;h) = \frac{1}{z-1} + \sum_{n\in\mathbb{N}_0} \frac{(-1)^n}{n!} \gamma_n(h)(z-1)^n$$

and

$$\zeta_R(z) = \frac{1}{z-1} + \sum_{n\in\mathbb{N}_0} \frac{(-1)^n}{n!} \gamma_n (z-1)^n$$

hold with infinite radius of convergence where the Stieltjes constants γ_n and generalized Stieltjes constants $\gamma_n(h)$ are given by

$$\gamma_n := \lim_{N\to\infty} \left(-\frac{(\ln N)^{n+1}}{n+1} + \sum_{k=1}^{N} \frac{(\ln k)^n}{k} \right),$$

$$\gamma_n(h) := \lim_{N\to\infty} \left(-\frac{(\ln(N+h))^{n+1}}{n+1} + \sum_{k=1}^{N} \frac{(\ln(k+h))^n}{k+h} \right).$$

These imply $\gamma_n(h) \to \gamma_n$ $(h \searrow 0)$ and, hence,

$$\lim_{h\searrow 0} \zeta_H(-z-\alpha;h) = \zeta_R(-z-\alpha) \qquad \text{compactly.}$$

On the other hand,

$$\begin{aligned} \zeta_H(z;h) - h^{-z} &= \sum_{n\in\mathbb{N}} (h+n)^{-z} \\ &= \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}_0} \binom{-z}{k} h^k n^{-z-k} \end{aligned}$$

$$= \sum_{k\in\mathbb{N}_0} \binom{-z}{k} h^k \zeta_R(z+k)$$

holds by meromorphic extension and, thus,

$$\lim_{h\searrow 0} \zeta_H(z;h) - h^{-z} = \zeta_R(z) \qquad \text{compactly.}$$

Finally, we obtain

$$\begin{aligned}\lim_{h\searrow 0} \zeta\left(s\mapsto G^s G^\alpha\right)(z) =& \lim_{h\searrow 0}\left(\zeta_H(-z-\alpha;h)+\zeta_H(-z-\alpha;h)-h^{z+\alpha}\right)\\ =& 2\zeta_R(-z-\alpha)\\ =& \zeta\left(s\mapsto H^s H^\alpha\right)(z)\end{aligned}$$

compactly. In fact, knowing a bit more about ζ_H we can get the result from the fact that

$$\forall n\in\mathbb{Z}:\ \zeta_H(s;n) = \begin{cases}\zeta_R(s)+\sum_{k=1}^{n} k^{-s} &, n\le 0\\ \zeta_R(s)-\sum_{k=1}^{n-1} k^{-s} &, n>0\end{cases}$$

which directly implies $\zeta_H(s;1)=\zeta_H(s;0)=\zeta_R(s)$ and, hence,

$$\begin{aligned}2\zeta_H(-z-\alpha;h)-h^{z+\alpha} =& \zeta_H(-z-\alpha;h)+\zeta_H(-z-\alpha;1+h)\\ \to& \zeta_H(-z-\alpha;0)+\zeta_H(-z-\alpha;1) \qquad (h\to 0)\\ =& 2\zeta_R(-z-\alpha)\end{aligned}$$

where the limit is compact again using Vitali's theorem (cf. Theorem 6.1 below). In any case, the important observation is

$$\lim_{h\searrow 0} \zeta\left(s\mapsto G^s G^\alpha\right) = \zeta\left(s\mapsto H^s H^\alpha\right) \qquad \text{compactly.}$$

Let us have a closer look at what happens with respect to the amplitude when we replace H by G. Here, we regularized the kernel $a(z)(x,y,\xi) = |\xi|^z$ by adding an $h \in (0,1]$ yielding a perturbed amplitude $a_h(z)(x,y,\xi) = (h+|\xi|)^z$ which has no singularities. Showing that the compact limit $h \searrow 0$ exists, then, justified our calculations. Using Vitali's theorem (cf. e.g. chapter 1 in [**42**]), we can largely generalize this idea.

THEOREM 6.1 (Vitali). *Let* $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$*,* $f \in C^\omega(\Omega)^{\mathbb{N}}$ *locally bounded*[1]*, and let*

$$\{z \in \Omega;\ (f_n(z))_{n\in\mathbb{N}} \text{ converges}\}$$

have an accumulation point in Ω*. Then,* f *is compactly convergent.*

We will consider two approaches to mollification. First, we will discuss a spectral approach in generalized convergence (cf. Chapter IV in [**44**], also known as gap topology; the most important results can also be found in appendix B). Then, we will generalize the shift $H \rightsquigarrow G$ to poly-log-homogeneous distributions.

Spectral mollification

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of gauged Fourier Integral Operators with C^∞-amplitudes and A a gauged Fourier Integral Operator whose amplitudes may contain singularities. Furthermore, let $A_n(z) \to A(z)$ for every z in gap topology (cf. appendix B). Let $d \in \mathbb{R}$ such that

$$\forall z \in \mathbb{C}:\ (\Re(z) < d \ \Rightarrow\ A(z) \text{ is of trace-class})$$

and $\Omega := \mathbb{C}_{\Re(\cdot)<d-1}$. Then, for every $z \in \Omega$, $(A_n(z))_{n\in\mathbb{N}}$ is eventually a sequence of bounded operators and $A_n|_\Omega \to A|_\Omega$ converges pointwise in norm (cf. Theorem

[1] $f \in C^\omega(\Omega)^{\mathbb{N}}$ is called locally bounded if and only if for every $z \in \Omega$ there exists a neighborhood of z such that f is uniformly bounded on that neighborhood.

B.13). Furthermore, let $(\lambda_k(z))_{k\in\mathbb{N}}$ be the sequence of eigenvalues of $A(z)$ counting multiplicities and $(\lambda_k(z)+h_k^n(z))_{k\in\mathbb{N}}$ be the sequence of eigenvalues of $A_n(z)$ counting multiplicities. Suppose that $h^n(z) := \sum_{k\in\mathbb{N}} |h_k^n(z)|$ exists and converges to zero for $z \in \Omega$.

Remark Note that $A_n(z) \to A(z)$ in the gap topology implies that the $h_k^n(z)$ exist and for every k and z we have $\lim_{n\to\infty} h_k^n(z) \to 0$. However, in general, we will not have any uniform bound on them, let alone find an $h^n(z)$; cf. the discussion following Theorem B.21.

■

Then,

$$|\zeta(A_n)(z) - \zeta(A)(z)| = \left|\sum_{k\in\mathbb{N}} (\lambda_k(z) + h_k^n(z)) - \sum_{k\in\mathbb{N}} \lambda_k(z)\right| = \left|\sum_{k\in\mathbb{N}} h_k^n(z)\right| \leq h^n(z) \to 0$$

for $z \in \Omega$ shows

$$\{z \in \Omega;\ (\zeta(A_n)(z))_{n\in\mathbb{N}} \text{ converges}\} = \Omega.$$

Let $\tilde{\Omega} \subseteq \mathbb{C}$ be open and connected with $\Omega \subseteq \tilde{\Omega}$ such that all $\zeta(A_n)|_{\tilde{\Omega}}$ are holomorphic and $\left\{\zeta(A_n)|_{\tilde{\Omega}};\ n \in \mathbb{N}\right\}$ is locally bounded. Then,

$$\lim_{n\to\infty} \zeta(A_n)|_{\tilde{\Omega}} = \zeta(A)|_{\tilde{\Omega}} \qquad \text{compactly.}$$

In particular, if h^n admits an analytic continuation to $\tilde{\Omega}$, then $\lim_{n\to\infty} \zeta(A_n)|_{\tilde{\Omega}} = \zeta(A)|_{\tilde{\Omega}}$ compactly.

Definition 6.2. *Let A be an operator with purely discrete spectrum. For every $\lambda \in \sigma(A)$ let μ_λ be the multiplicity of λ. Then, we define the spectral ζ-function $\zeta_\sigma(A)$ to be the meromorphic extension of*

$$\zeta_\sigma(A)(s) := \sum_{\lambda\in\sigma(A)\setminus\{0\}} \mu_\lambda \lambda^{-s}$$

and the spectral Θ*-function* $\Theta_\sigma(A)$

$$\forall t \in \mathbb{R}_{>0}: \ \Theta_\sigma(A)(t) := \sum_{\lambda\in\sigma(A)} \mu_\lambda \exp(-t\lambda)$$

if they exist.

DEFINITION 6.3. *Let* $T \in \mathbb{R}_{>0}$ *and* $\varphi \in C(\mathbb{R}_{>0})$*. We define the upper Mellin transform as*

$$\mathcal{M}^T(\varphi)(s) := \int_{(0,T)} \varphi(t)t^{s-1}dt$$

and the lower Mellin transform

$$\mathcal{M}_T(\varphi)(s) := \int_{\mathbb{R}_{\geq T}} \varphi(t)t^{s-1}dt$$

(if the integrals exist). If both integrals exist and have non-empty intersection Ω *of domains of holomorphy (that is, the maximal connected and open subset admitting an analytic continuation of the function), then we define the generalized Mellin transform of* φ *to be the meromorphic extension of*

$$\mathcal{M}(\varphi) := \mathcal{M}^T(\varphi)|_\Omega + \mathcal{M}_T(\varphi)|_\Omega.$$

EXAMPLE Let $\varphi(t) := t^\alpha$ for some $\alpha \in \mathbb{C}$. Then

$$\mathcal{M}^T(\varphi)(s) = \int_{(0,T)} t^{s+\alpha-1}dt = \frac{T^{s+\alpha}}{s+\alpha}$$

for $\Re(s) > \alpha$ extending to $\mathbb{C} \setminus \{-\alpha\}$ and

$$\mathcal{M}_T(\varphi)(s) = \int_{\mathbb{R}_{\geq T}} t^{s+\alpha-1}dt = -\frac{T^{s+\alpha}}{s+\alpha}$$

for $\Re(s) < \alpha$ extending to $\mathbb{C} \setminus \{-\alpha\}$. Hence, $\mathcal{M}(\varphi)$ exists with

$$\mathcal{M}(\varphi)(s) = \frac{T^{s+\alpha}}{s+\alpha} - \frac{T^{s+\alpha}}{s+\alpha} = 0$$

on $\mathbb{C} \setminus \{-\alpha\}$, i.e. $\mathcal{M}(\varphi) = 0$.

■

REMARK The example above is very important for pseudo-differential operators or, more generally, Fourier Integral Operators whose phase function ϑ satisfies $\forall x:\ \vartheta(x,x,\cdot)=0$. It means that homogeneous terms in the asymptotic expansion, which are not of critical degree, vanish under regularization in the Kontsevich-Vishik trace, i.e. it is the reason why we can split off finitely many terms in the Kontsevich-Vishik density.

■

EXAMPLE Let $\lambda\in\mathbb{R}_{>0}$ and $s\in\mathbb{C}$ with $\Re(s)>0$. Then

$$\int_{\mathbb{R}_{>0}} e^{-\lambda t}t^{s-1}dt=\int_{\mathbb{R}_{>0}} e^{-\tau}\tau^{s-1}\lambda^{-s}dt=\lambda^{-s}\Gamma(s)$$

shows that $\lambda\mapsto\int_{\mathbb{R}_{>0}} e^{-\lambda t}t^{s-1}dt$ extends analytically to $\mathbb{C}\setminus\mathbb{R}_{\le 0}$.

■

EXAMPLE Let A be an operator with purely discrete spectrum. For every $\lambda\in\sigma(A)$ let μ_λ be the multiplicity of λ and $\Re(\lambda)\ge 0$. $\mathcal{M}(1)=0$, then, implies

$$\begin{aligned}\mathcal{M}\left(\Theta_\sigma(A)\right)(s)&=\sum_{\lambda\in\sigma(A)}\mu_\lambda\mathcal{M}\left(t\mapsto\exp(-t\lambda)\right)(s)\\&=\sum_{\lambda\in\sigma(A)\setminus\{0\}}\mu_\lambda\mathcal{M}\left(t\mapsto\exp(-t\lambda)\right)(s)\\&=\sum_{\lambda\in\sigma(A)\setminus\{0\}}\mu_\lambda\lambda^{-s}\Gamma(s)\\&=\zeta_\sigma(A)(s)\Gamma(s).\end{aligned}$$

■

LEMMA 6.4. $\lim_{h\searrow 0}\mathcal{M}\left(t\mapsto\exp(-th)\right)=\mathcal{M}(1)=0$ *compactly.*

PROOF. For $\Re(s)>1$, we obtain

$$\frac{1}{\Gamma(s)}\mathcal{M}\left(t\mapsto\exp(-th)\right)(s)=\frac{1}{\Gamma(s)}\int_{\mathbb{R}_{>0}} e^{-th}t^{s-1}dt$$

$$
\begin{aligned}
&= h^{-s} \\
&= \sum_{k\in\mathbb{N}_0} (k+h)^{-s} - \sum_{k\in\mathbb{N}_0} (k+1+h)^{-s} \\
&= \zeta_H(s;h) - \zeta_H(s;1+h).
\end{aligned}
$$

Hence,

$$
\mathcal{M}\left(t \mapsto \exp(-th)\right)(s) = \Gamma(s)\zeta_H(s;h) - \Gamma(s)\zeta_H(s;1+h)
$$

holds on $\mathbb{C} \setminus \mathbb{Z}_{\leq 1}$. Furthermore, $\Gamma(s)\zeta_H(s;h) - \Gamma(s)\zeta_H(s;1+h)$ is locally bounded on $\mathbb{C} \setminus \mathbb{Z}_{\leq 1}$ for $h \searrow 0$ which implies

$$
\begin{aligned}
\lim_{h\searrow 0} \mathcal{M}\left(t \mapsto \exp(-th)\right)(s) &= \lim_{h\searrow 0} \left(\Gamma(s)\zeta_H(s;h) - \Gamma(s)\zeta_H(s;1+h)\right) \\
&= \Gamma(s)\zeta_H(s;0) - \Gamma(s)\zeta_H(s;1) \\
&= \Gamma(s)\zeta_R(s) - \Gamma(s)\zeta_R(s) \\
&= 0
\end{aligned}
$$

compactly, i.e. the compact limit $\lim_{h\searrow 0} \mathcal{M}\left(t \mapsto \exp(-th)\right)$ exists and vanishes on $\mathbb{C} \setminus \mathbb{Z}_{\leq 1}$.

□

Corollary 6.5. *Let A and A_h be operators with spectral ζ-functions. Let $\zeta_\sigma(A)$ be the meromorphic extension of $\sum_{k\in N} \lambda_k^{-s}$ for some $N \subseteq \mathbb{N}$ and $\zeta_\sigma(A_h)$ the meromorphic extension of $\sum_{j=1}^{n} \tilde{h}_j^{-s} + \sum_{k\in N}(\lambda_k + h_k)^{-s}$ where all $\tilde{h}_j \in \mathbb{R}_{>0}$ (the $\tilde{h}_j$ are the perturbations of the eigenvalue zero and n is the multiplicity of the zero in $\sigma(A)$). Suppose A_h converges to A in the gap topology and the meromorphic extension f_h of $\sum_{k\in N}(\lambda_k + h_k)^{-s}$ is locally bounded and converges to $\zeta_\sigma(A)$ pointwise.*

Then, $\zeta_\sigma(A_h)$ converges to $\zeta_\sigma(A)$ compactly.

PROOF. The assertion is a direct consequence of $\sum_{j=1}^{n} \tilde{h}_j^{-s} \to 0$ compactly (Lemma 6.4) and $f_h \to \zeta_\sigma(A)$ compactly (Vitali's theorem).

□

Mollification of poly-log-homogeneous distributions

The considerations regarding the spectral ζ-function have given us useful insights on the spectral level of the operator and contain some nice properties, e.g. that mollification will be essentially the generalized Mellin transform. However, it did not provide us with existence of a mollifying sequence of operators (and even if it did, it would only contain a rather restrictive sub-class of operators). In this section, we will consider gauged poly-log-homogeneous distributions which are poly-log-homogeneous everywhere on $\mathbb{R}_{>0} \times M$ and show that they can be mollified.

PROPOSITION 6.6. *Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ be a gauged poly-log-homogeneous distribution on $\mathbb{R}_{>0} \times M$ with I finite and α_0 regular. Then, $\zeta(\alpha)$ can be mollified.*

In particular,

$$\begin{aligned}\zeta(\alpha)(z) = &\int_{\mathbb{R}_{>0}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\ &+ \sum_{\iota\in I} \int_{(0,1)} r^{\dim M + d_\iota + z} (\ln r)^{l_\iota} dr \operatorname{res} \alpha_\iota(z)\end{aligned}$$

is the compact limit of

$$\begin{aligned}\zeta(\alpha_h)(z) = &\int_{\mathbb{R}_{>0}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\ &+ \sum_{\iota\in I} \int_{(0,1)} (h_\iota + r)^{\dim M + d_\iota + z} (\ln(h_\iota + r))^{l_\iota} dr \operatorname{res} \alpha_\iota(z)\end{aligned}$$

for $h_\iota \in \mathbb{R}_{>0}$, $h_\iota \searrow 0$.

Proof. The part

$$\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$

creates no problems in the formalism used to obtain the Laurent expansion. Hence, we only need to consider

$$\begin{aligned}
&\sum_{\iota\in I}\int_{(0,1)} r^{\dim M+d_\iota+z}(\ln r)^{l_\iota}\,dr\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\int_{(0,1)}\partial^{l_\iota}\left(s\mapsto r^{\dim M+d_\iota+s}\right)(z)dr\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(s\mapsto\int_{(0,1)} r^{\dim M+d_\iota+s}dr\right)(z)\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(s\mapsto\frac{1}{\dim M+d_\iota+s+1}\right)(z)\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\frac{(-1)^{l_\iota}l_\iota!}{(\dim M+d_\iota+z+1)^{l_\iota+1}}\,\mathrm{res}\,\alpha_\iota(z).
\end{aligned}$$

Introducing $h_\iota\in\mathbb{R}_{>0}$ we obtain

$$\begin{aligned}
&\sum_{\iota\in I}\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota}\,dr\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\int_{(0,1)}\partial^{l_\iota}\left(s\mapsto(h_\iota+r)^{\dim M+d_\iota+s}\right)(z)dr\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(s\mapsto\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+s}dr\right)(z)\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\partial^{l_\iota}\left(s\mapsto\frac{(1+h_\iota)^{\dim M+d_\iota+s+1}-h_\iota^{\dim M+d_\iota+s+1}}{\dim M+d_\iota+s+1}\right)(z)\,\mathrm{res}\,\alpha_\iota(z)\\
=&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\,\mathrm{res}\,\alpha_\iota(z)\\
&-\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\,\mathrm{res}\,\alpha_\iota(z).
\end{aligned}$$

Since each of the $(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}$ is locally bounded for $h_\iota\to 0$ (taking derivatives in Lemma 6.4) and

$$(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\to\begin{cases}0 & j\neq l_\iota\\ 1 & j=l_\iota\end{cases}$$

for $h_\iota \to 0$, we obtain

$$\begin{aligned}&\lim_{h_\iota\searrow 0}\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\operatorname{res}\alpha_\iota(z)\\ =&\sum_{\iota\in I}\frac{(-1)^{l_\iota}l_\iota!}{(\dim M+d_\iota+z+1)^{l_\iota+1}}\operatorname{res}\alpha_\iota(z)\end{aligned}$$

compactly. Furthermore,

$$h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}=h_\iota^{\dim M+d_\iota+z+1+j-l_\iota}(h_\iota\ln h_\iota)^{l_\iota-j}$$

being locally bounded for $h_\iota \to 0$ and converging to zero compactly (in z) (recall $\lim_{h\searrow 0}h^z=\lim_{h\searrow 0}\zeta_H(-z;h)-\zeta_R(-z)=0$ compactly) shows

$$\begin{aligned}&\zeta(\alpha_h)(z)\\ =&\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\ &+\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\operatorname{res}\alpha_\iota(z)\\ &-\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\operatorname{res}\alpha_\iota(z)\\ \to&\int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M}+\sum_{\iota\in I}\int_{\mathbb{R}_{\geq 1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}\\ &+\sum_{\iota\in I}\frac{(-1)^{l_\iota}l_\iota!}{(\dim M+d_\iota+z+1)^{l_\iota+1}}\operatorname{res}\alpha_\iota(z)\\ =&\zeta(\alpha)(z)\end{aligned}$$

where the convergence is compact by Vitali's theorem.

□

Example (re-visiting $\zeta\left(s\mapsto H^sH^\alpha\right)$) Let $\Gamma\subseteq\mathbb{R}^N$ be a discrete group generated by a basis of $\mathbb{R}^N$, $|\Delta|$ the Dirichlet Laplacian on $\mathbb{R}^N$, δ the Dirichlet Laplacian on $\mathbb{R}^N/\Gamma$, and $H:=\sqrt{\delta}$. Then,

$$\zeta\left(s\mapsto H^s\right)(z)=\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^N}e^{-i\langle\gamma,\xi\rangle_{\ell_2(N)}}(2\pi)^{-N}\|\xi\|_{\ell_2(N)}^z\ d\xi$$

where

$$\sum_{\gamma\in\Gamma\setminus\{0\}}\int_{\mathbb{R}^N} e^{-i\langle\gamma,\xi\rangle_{\ell_2(N)}}(2\pi)^{-N}\,\|\xi\|_{\ell_2(N)}^z\;d\xi$$

is regular, i.e.

$$\alpha_0(z)(\xi)=\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)\sum_{\gamma\in\Gamma\setminus\{0\}} e^{-i\langle\gamma,\xi\rangle_{\ell_2(N)}}(2\pi)^{-N}\,\|\xi\|_{\ell_2(N)}^z$$

and

$$\sum_{\iota\in I}\alpha_\iota(z)(\xi)=\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)(2\pi)^{-N}\,\|\xi\|_{\ell_2(N)}^z\,.$$

Hence, Proposition 6.6 is applicable.

■

In the following proposition, we will use Abel's summation.

Lemma 6.7 (Abel's summation). *Let G be a group, $a,b\in G^{\mathbb{N}}$, and*

$$\forall n\in\mathbb{N}:\;B_n:=\sum_{k=1}^n b_k.$$

Then,

$$\forall n\in\mathbb{N}:\;\sum_{k=1}^n a_kb_k=a_{n+1}B_n+\sum_{k=1}^n(a_k-a_{k+1})B_k.$$

Proof.

$$\begin{aligned}
a_{n+1}B_n+\sum_{k=1}^n(a_k-a_{k+1})B_k =&a_{n+1}B_n+\sum_{k=1}^n a_kB_k-\sum_{k=1}^n a_{k+1}B_k\\
=&\sum_{k=1}^n a_kB_k-\sum_{k=1}^{n-1}a_{k+1}B_k\\
=&\sum_{k=1}^n\sum_{j=1}^k a_kb_j-\sum_{k=1}^{n-1}\sum_{j=1}^k a_{k+1}b_j\\
=&a_1b_1+\sum_{k=2}^n\sum_{j=1}^k a_kb_j-\sum_{k=1}^{n-1}\sum_{j=1}^k a_{k+1}b_j\\
=&a_1b_1+\sum_{k=1}^{n-1}\sum_{j=1}^{k+1}a_{k+1}b_j-\sum_{k=1}^{n-1}\sum_{j=1}^k a_{k+1}b_j
\end{aligned}$$

$$=a_1 b_1 + \sum_{k=1}^{n-1} a_{k+1} b_{k+1}$$
$$= \sum_{k=1}^{n} a_k b_k$$

$\square$

PROPOSITION 6.8. *Let $\alpha = \alpha_0 + \sum_{\iota\in I} \alpha_\iota$ be a gauged poly-*log*-homogeneous distribution on $\mathbb{R}_{>0} \times M$ with $I \subseteq \mathbb{N}$, α_0 regular on $(0,1) \times M$,*

$$\alpha_\iota(z)(r,\xi) = r^{d_\iota+z}(\ln r)^{l_\iota} \tilde{\alpha}_\iota(z)(\xi),$$

where $(\Re(d_\iota))_{\iota\in I}$ is bounded from above, each $\left(\frac{1}{\dim M + d_\iota + z + 1}\right)_{\iota\in I} \in \ell_2(I)$, $(l_\iota)_{\iota\in I} \in \ell_\infty(I)$, $l := \|(l_\iota)_{\iota\in I}\|_{\ell_\infty(I)}$, and each $\sum_{\iota\in I} \tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$. Then, $\zeta(\alpha)$ can be mollified.

In particular,

$$\zeta(\alpha)(z) = \int_{\mathbb{R}_{>0}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$
$$+ \sum_{\iota\in I} \int_{(0,1)} r^{\dim M + d_\iota + z} (\ln r)^{l_\iota} dr \operatorname{res} \alpha_\iota(z)$$

is the compact limit of

$$\zeta(\alpha_h)(z) = \int_{\mathbb{R}_{>0}\times M} \alpha_0(z) d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I} \int_{\mathbb{R}_{\geq 1}\times M} \alpha_\iota(z) d\mathrm{vol}_{\mathbb{R}_{\geq 1}\times M}$$
$$+ \sum_{\iota\in I} \int_{(0,1)} (h_\iota + r)^{\dim M + d_\iota + z} (\ln(h_\iota + r))^{l_\iota} dr \operatorname{res} \alpha_\iota(z)$$

for $h := (h_\iota)_{\iota\in I} \in \ell_\infty(I; \mathbb{R}_{>0})$ and $h \searrow 0$ in $\ell_\infty(I)$ such that

$$Z_\iota(z) := |\zeta_H(l - d_\iota - z; h_\iota) - \zeta_H(l - d_\iota - z; 1 + h_\iota)|$$

defines $(Z_\iota(z))_{\iota\in I} \in \ell_\infty(I)$ which is uniformly bounded on an exhausting family of compacta as $h \searrow 0$,[2] i.e. there exists a family $(\Omega_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$:

[2]Note, this is a restraining property on the choice of $h \in \ell_\infty(I)$. It is possible to find such sequences because each Z_ι converges compactly to zero as $h_\iota \searrow 0$.

$\Omega_n \subseteq_{\text{compact}} \mathbb{C}$, $\forall n \in \mathbb{N}:\ \Omega_n \subseteq \Omega_{n+1}$, $\bigcup_{n\in\mathbb{N}} \Omega_n = \mathbb{C}$, *and*

$$\forall n \in \mathbb{N}:\ \limsup_{h\searrow 0} \left\| \left(\|Z_\iota\|_{L_\infty(\Omega_n)} \right)_{\iota\in I} \right\|_{\ell_\infty(I)} < \infty.$$

Proof. Proposition 6.6 yields the assertion for finite I. Hence, we may assume $I = \mathbb{N}$ without loss of generality. Furthermore, we only need to consider the part

$$\begin{aligned} A(h) :=& \sum_{\iota\in I} \int_{(0,1)} (h_\iota + r)^{\dim M + d_\iota + z} (\ln(h_\iota + r))^{l_\iota} dr \operatorname{res} \alpha_\iota(z) \\ =& \sum_{\iota\in I} \sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res} \alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}} (1 + h_\iota)^{\dim M + d_\iota + z + 1} (\ln(1 + h_\iota))^{l_\iota - j} \\ &- \sum_{\iota\in I} \sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res} \alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}} h_\iota^{\dim M + d_\iota + z + 1} (\ln h_\iota)^{l_\iota - j}, \end{aligned}$$

i.e. show that it converges compactly to zero. Recall that $\sum_{\iota\in I} \frac{\operatorname{res} \alpha_\iota(z)}{\dim M + d_\iota + z + 1}$ converges absolutely and $|\dim M + d_\iota + z + 1| \to \infty$ $(\iota \to \infty)$. Hence, we will assume, without loss of generality, $\forall \iota \in I:\ |\dim M + d_\iota + z + 1| \geq 1$ (as there can only be finitely many with $|\dim M + d_\iota + z + 1| < 1$ which is handled by Proposition 6.6). Then, we observe (for $h_0 := \|h\|_{\ell_\infty(I)} < e - 1$)

$$\begin{aligned} &\left| \sum_{\iota\in I} \sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res} \alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}} (1 + h_\iota)^{\dim M + d_\iota + z + 1} (\ln(1 + h_\iota))^{l_\iota - j} \right| \\ \leq& \sum_{\iota\in I} \sum_{j=0}^{l_\iota} \frac{j! |\operatorname{res} \alpha_\iota(z)|}{|\dim M + d_\iota + z + 1|^{j+1}} \left| (1 + h_\iota)^{\dim M + d_\iota + z + 1} (\ln(1 + h_\iota))^{l_\iota - j} \right| \\ \leq& l! \sum_{\iota\in I} \sum_{j=0}^{l_\iota} \frac{|\operatorname{res} \alpha_\iota(z)|}{|\dim M + d_\iota + z + 1|} \left| (1 + h_\iota)^{\dim M + d_\iota + z + 1} \right| \underbrace{(\ln(1 + h_0))^{l_\iota - j}}_{\leq 1\ (h_0 < e - 1)} \\ \leq& l! \cdot l \sum_{\iota\in I} \frac{|\operatorname{res} \alpha_\iota(z)|}{|\dim M + d_\iota + z + 1|} (1 + h_0)^{\dim M + \Re(d_\iota + z) + 1} \\ \leq& l! \cdot l \underbrace{(1 + h_0)^{\max\{\dim M + \Re(z) + 1 + \sup_{\iota\in I} \Re(d_\iota), 0\}}}_{\to 1\ (h \searrow 0)} \sum_{\iota\in I} \frac{|\operatorname{res} \alpha_\iota(z)|}{|\dim M + d_\iota + z + 1|} \end{aligned}$$

which is locally bounded by absolute convergence of $\sum_{\iota\in I}\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}$ and compact convergence of $(1+h_\iota)^{\dim M+d_\iota+z+1}$. Furthermore, we obtain (for $h_0\le e^{-1}$)

$$\begin{aligned}
&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\right|\\
\le&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{l!\left|\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}\left(h_\iota\ln h_\iota\right)^l\right|}{|\dim M+d_\iota+z+1|}h_\iota^{\dim M+1}\\
\le& l\cdot l!h_0^{\dim M+1}\sum_{\iota\in I}\frac{\left|\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}\left(h_\iota\ln h_\iota\right)^l\right|}{|\dim M+d_\iota+z+1|}.
\end{aligned}$$

Note that

$$|h_\iota\ln h_\iota|^l\to\begin{cases}1 & ,\ l=0\\ 0 & ,\ l\neq 0\end{cases}$$

for $h_\iota\to 0$, i.e. it suffices to show that

$$\sum_{\iota\in I}\frac{\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}}{\dim M+d_\iota+z+1}$$

converges absolutely. Since

$$\left|h_\iota^{d_\iota+z-l}\right|=\left|\zeta_H(l-d_\iota-z;h_\iota)-\zeta_H(l-d_\iota-z;1+h_\iota)\right|=Z_\iota(z)$$

holds (we can choose $(Z_\iota(z))_{\iota\in I}$ locally bounded because $z\mapsto\zeta_H(l-d_\iota-z;h_\iota)-\zeta_H(l-d_\iota-z;1+h_\iota)$ converges to zero compactly as $h_\iota\searrow 0$)[3], we observe

$$\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)h_\iota^{d_\iota+z-l}}{\dim M+d_\iota+z+1}\right|\le\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|Z_\iota(z)$$

which is bounded by absolute convergence of $\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|$ and the assumed boundedness of $(Z_\iota(z))_{\iota\in I}$. Furthermore, local boundedness (with respect to z)

[3]Since we have to construct a sequence $H\in\ell_\infty(I;\mathbb{R}_{>0})^{\mathbb{N}}$ where each element H_n is of the form h, it suffices to have uniform boundedness of $(Z_\iota)_{\iota\in I}$ on some compact set Ω_n for H_n and choose $(\Omega_n)_{n\in\mathbb{N}}$ to satisfy $\forall n\in\mathbb{N}\ \Omega_n\subseteq\Omega_{n+1}$ and $\bigcup_{n\in\mathbb{N}}\Omega_n=\mathbb{C}$.

follows from local boundedness of $\sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M+d_\iota+z+1}\right|$ and Z_ι. Observing

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}\underbrace{(\ln(1+h_\iota))^{l_\iota-j}}_{\to\delta_{j,l_\iota}}$$
$$\to\sum_{\iota\in I}\frac{(-1)^{l_\iota}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{l_\iota+1}}$$

and

$$\begin{aligned}
&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\right|\\
=&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+2+j-l_\iota}(h_\iota\ln h_\iota)^{l_\iota-j}h_\iota\right|\\
\leq h&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+2+j-l_\iota}(h_\iota\ln h_\iota)^{l_\iota-j}\right|\\
\to&0
\end{aligned}$$

for $\Re(\dim M+d_\iota+z+2-l)>0$ and $h\searrow 0$ shows

$$\begin{aligned}
A(h)=&\sum_{\iota\in I}\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota}dr\operatorname{res}\alpha_\iota(z)\\
=&\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}(1+h_\iota)^{\dim M+d_\iota+z+1}(\ln(1+h_\iota))^{l_\iota-j}\\
&-\sum_{\iota\in I}\sum_{j=0}^{l_\iota}\frac{(-1)^j j!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{j+1}}h_\iota^{\dim M+d_\iota+z+1}(\ln h_\iota)^{l_\iota-j}\\
\to&\sum_{\iota\in I}\frac{(-1)^{l_\iota}l_\iota!\operatorname{res}\alpha_\iota(z)}{(\dim M+d_\iota+z+1)^{l_\iota+1}}
\end{aligned}$$

compactly and, thus,

$$\zeta(\alpha_h)\to\zeta(\alpha)$$

compactly.

□

Remark Note that

$$\zeta_H(z;h)=\frac{1}{z-1}+\sum_{n\in\mathbb{N}_0}\frac{(-1)^n}{n!}\gamma_n(h)(z-1)^n$$

with

$$\gamma_n(h) := \lim_{N\to\infty}\left(-\frac{(\ln(N+h))^{n+1}}{n+1}+\sum_{k=1}^{N}\frac{(\ln(k+h))^n}{k+h}\right)$$

implies that

$$z \mapsto \zeta_H(z,h_0) - \zeta_H(z,h_1)$$

is an entire function for every $h_0, h_1 \in \mathbb{R}_{>0}$. Hence, each Z_ι is everywhere defined on $\mathbb{C}$.

■

Finally, we may also drop the assumption $(l_\iota)_{\iota\in I} \in \ell_\infty(I)$.

THEOREM 6.9. *Let $\alpha = \alpha_0 + \sum_{\iota\in I}\alpha_\iota$ be a gauged poly-*log*-homogeneous distribution on $\mathbb{R}_{>0}\times M$ with $I \subseteq \mathbb{N}$, α_0 regular on $(0,1)\times M$,*

$$\alpha_\iota(z)(r,\xi) = r^{d_\iota+z}(\ln r)^{l_\iota}\tilde{\alpha}_\iota(z)(\xi),$$

where $(\Re(d_\iota))_{\iota\in I}$ is bounded from above, each $\left(\frac{1}{\dim M+d_\iota+z+1}\right)_{\iota\in I} \in \ell_2(I)$, and each $\sum_{\iota\in I}\tilde{\alpha}_\iota(z)$ converges unconditionally in $L_1(M)$. Then, $\zeta(\alpha)$ can be mollified.

In particular,

$$\zeta(\alpha)(z) = \int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq1}\times M} + \sum_{\iota\in I}\int_{(0,1)} r^{\dim M+d_\iota+z}(\ln r)^{l_\iota}dr\,\mathrm{res}\,\alpha_\iota(z)$$

is the compact limit of

$$\zeta(\alpha_h)(z) = \int_{\mathbb{R}_{>0}\times M}\alpha_0(z)d\mathrm{vol}_{\mathbb{R}_{>0}\times M} + \sum_{\iota\in I}\int_{\mathbb{R}_{\geq1}\times M}\alpha_\iota(z)d\mathrm{vol}_{\mathbb{R}_{\geq1}\times M} + \sum_{\iota\in I}\int_{(0,1)}(h_\iota+r)^{\dim M+d_\iota+z}(\ln(h_\iota+r))^{l_\iota}dr\,\mathrm{res}\,\alpha_\iota(z)$$

for $h := (h_\iota)_{\iota\in I} \in \ell_\infty(I;\mathbb{R}_{>0})$ *and* $h \searrow 0$ *in* $\ell_\infty(I)$ *such that*

$$Z_\iota(z) := l_\iota \sum_{j=0}^{l_\iota} \left|\zeta_H(l_\iota - j - d_\iota - z; h_\iota) - \zeta_H(l_\iota - j - d_\iota - z; 1 + h_\iota)\right|$$

is uniformly bounded on an exhausting family of compacta as $h \searrow 0$.

Proof. The proof works precisely as the proof of Proposition 6.8. The only difference is that we have to show local boundedness of

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}} (1+h_\iota)^{\dim M + d_\iota + z + 1} (\ln(1+h_\iota))^{l_\iota - j}$$

and

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}} h_\iota^{\dim M + d_\iota + z + 1} (\ln h_\iota)^{l_\iota - j}$$

since the estimates do not hold anymore. Since

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}}$$

is a well-defined meromorphic function, it is locally bounded. Furthermore, $(1+h_\iota)^{\dim M + d_\iota + z + 1}(\ln(1+h_\iota))^{l_\iota - j}$ can be chosen uniformly bounded on any half plane $\{z\in\mathbb{C};\ \Re(z) < r\}$ for any $r\in\mathbb{R}$, i.e. we can construct a sequence that is eventually uniformly convergent on any given compactum. Hence,

$$\sum_{\iota\in I}\sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}} (1+h_\iota)^{\dim M + d_\iota + z + 1} (\ln(1+h_\iota))^{l_\iota - j}$$

is fine. Thus, choosing $|h_\iota \ln h_\iota| < 1$ and $|\dim M + d_\iota + z + 1| \geq 1$ without loss of generality, we observe

$$\begin{aligned}
&\left|\sum_{\iota\in I}\sum_{j=0}^{l_\iota} \frac{(-1)^j j! \operatorname{res}\alpha_\iota(z)}{(\dim M + d_\iota + z + 1)^{j+1}} h_\iota^{\dim M + d_\iota + z + 1} (\ln h_\iota)^{l_\iota - j}\right| \\
&\leq \sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M + d_\iota + z + 1}\right| l_\iota! \sum_{j=0}^{l_\iota} \left|h_\iota^{\dim M + d_\iota + z + 1 + j - l_\iota}\right| \left|(h_\iota \ln h_\iota)^{l_\iota - j}\right| \\
&\leq \sum_{\iota\in I}\left|\frac{\operatorname{res}\alpha_\iota(z)}{\dim M + d_\iota + z + 1}\right| l_\iota! \, \|h\|_{\ell_\infty(I)}^{\dim M + 1} \sum_{j=0}^{l_\iota} \left|h_\iota^{d_\iota + z + j - l_\iota}\right|
\end{aligned}$$

$$\leq \|h\|_{\ell_\infty(I)}^{\dim M+1} \sum_{\iota\in I} \left| \frac{\operatorname{res}\alpha_\iota(z)}{\dim M + d_\iota + z + 1} \right| Z_\iota(z)$$

which completes the proof.

□

Remark Note that the application to Fourier Integral Operators is not as trivial as for pseudo-differential operators because, even though we have an amplitude that is poly-log-homogeneous everywhere on $\mathbb{R}^N \setminus \{0\}$, going to the gauged poly-log-homogeneous distribution form means we do not know how the poly-log-homogeneous distributions look like on $B_{\mathbb{R}^N}$. In fact, we already know that homogeneous distributions regularize to zero by virtue of the generalized Mellin transform while we will see later (end of chapter 8), that there are Fourier Integral Operators with homogeneous amplitudes whose Kontsevich-Vishik traces don't vanish. In other words, we still owe an argument there.

The ζ-function of a gauged Fourier Integral Operator with an amplitude that is poly-log-homogeneous everywhere on $\mathbb{R}^N \setminus \{0\}$ can be written in the form

$$z \mapsto \left\langle x \mapsto \int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle_{\ell_2(N)}} v(z)(x,\xi)d\xi, \delta_0 \right\rangle$$

where $v = v_0 + \sum_{\iota\in I} v_\iota$ and each v_ι is log-homogeneous on $\mathbb{R}^N \setminus B_{\mathbb{R}^N}$. Re-parametrizing $\xi \rightsquigarrow -\xi$ yields

$$z \mapsto \left\langle x \mapsto \int_{\mathbb{R}^N} e^{-i\langle x,\xi\rangle_{\ell_2(N)}} w(z)(x,\xi)d\xi, \delta_0 \right\rangle$$

where $w = w_0 + \sum_{\iota\in I} w_\iota$ and each w_ι is log-homogeneous on $\mathbb{R}^N \setminus B_{\mathbb{R}^N}$.

Let $\hat{w} := w_0 + \sum_{\iota\in I} \hat{w}_\iota$ where each $\hat{w}_\iota$ is log-homogeneous on $\mathbb{R}^N \setminus \{0\}$ and coincides with w_ι on $\mathbb{R}^N \setminus B_{\mathbb{R}^N}$. Then,

$$z \mapsto \left\langle x \mapsto \int_{\mathbb{R}^N} e^{-i\langle x,\xi\rangle_{\ell_2(N)}} \hat{w}(z)(x,\xi)d\xi, \delta_0 \right\rangle$$

is a ζ-function of a gauged poly-log-homogeneous distribution and can, thus, be mollified. Furthermore,

$$\forall z \in \mathbb{C}\ \forall x \in X\ \forall \xi \in \mathbb{R}^N \setminus B_{\mathbb{R}^N}:\ w(z)(x,\xi) - \hat{w}(z)(x,\xi) = 0$$

shows that

$$\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle_{\ell_2(N)}} \left(w(z)(x,\xi) - \hat{w}(z)(x,\xi)\right) d\xi$$

is the Fourier transform of a compactly supported distribution (for every x), i.e.

$$z \mapsto \left\langle x \mapsto \int_{\mathbb{R}^N} e^{-i\langle x,\xi\rangle_{\ell_2(N)}} \left(w(z)(x,\xi) - \hat{w}(z)(x,\xi)\right) d\xi, \delta_0 \right\rangle$$

is a holomorphic function. In other words,

$$z \mapsto \left\langle x \mapsto \int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle_{\ell_2(N)}} v(z)(x,\xi) d\xi, \delta_0 \right\rangle$$

is of the form "holomorphic function + mollifiable" and, hence, mollifiable itself since, by construction, the holomorphic function precisely accounts for the difference in the limit (of the mollification).

■

CHAPTER 7

On structural singularities and the generalized Kontsevich-Vishik trace

In this chapter, we will discuss the integrals appearing in the Laurent coefficients. Most importantly, this will yield the generalized Kontsevich-Vishik density

$$
\begin{aligned}
&\int_{B_{\mathbb{R}^N}(0,1)} e^{i\vartheta(x,x,\xi)} a(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x) \\
&+ \int_{\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} a_0(0)(x,x,\xi)\ d\mathrm{vol}_{\mathbb{R}_{\ge 1}\times\partial B_{\mathbb{R}^N}}(\xi)\ d\mathrm{vol}_X(x) \\
&+ \sum_{\iota\in I\setminus I_0} \frac{-\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)} a_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)}{N+d_\iota}\ d\mathrm{vol}_X(x),
\end{aligned}
$$

as well as the fact that this density is globally defined in the $I_0 = \varnothing$ case, that is, in the absence of terms with critical degree of homogeneity, provided that the kernel is globally defined in the first place (rather than considering any locally finite sum of local representations in the form of oscillatory integrals); whenever we will talk about densities being globally defined, we will tacitly assume that the kernel is globally defined since the entire discourse would make no sense otherwise. We will be able to calculate interesting examples by the end of chapter 8 leading up to (and including) Theorem 8.5.

Considering classical pseudo-differential operators, it is common to construct the Kontsevich-Vishik trace by removing those terms from the asymptotic expansion which have degree of homogeneity with real part greater than or equal to $-\dim X$ where X denotes the underlying manifold, i.e. if k is the kernel of the pseudo-differential operator, then the regularized kernel is given by

$$k^{\mathrm{reg}} := k - \sum_{j=0}^{N} k_{d-j}$$

where $d-j$ is the degree of homogeneity of the corresponding term in the expansion of the amplitude $a \sim \sum_{j\in\mathbb{N}_0} a_{d-j}$ and N sufficiently large. Then, $k^{\mathrm{reg}} \in C(X\times X)$, i.e. $\int_X k^{\mathrm{reg}}(x,x)d\mathrm{vol}_X(x)$ is well-defined. In other words, k^{reg} and α_0 play the same role and we would like to interpret $\zeta(\alpha_0)(0)$ as a generalized version of the Kontsevich-Vishik trace. The term $\sum_{j=0}^{N}\int_X k_{d-j}(x,x)d\mathrm{vol}_X(x)$ would, hence, be analogous to spinning off $\sum_{\iota\in I}\zeta(\alpha_\iota)(0)$. Unfortunately, we have to issue a couple of caveats.

(i) The observation above is fine if we are in local coordinates. However, when patching things together some of the terms in our Laurent expansion will not patch to global densities on X. This is no problem for Fourier Integral Operators, per se, as they are simply defined as a sum of local representations and in each of these representations the Laurent expansion holds. It will become a problem if we want to write down formulae in terms of kernels, though (especially if we require local terms to patch together defining densities globally).

(ii) Since $\mathcal{F}(a_{d-j}(x,y,\cdot))(z)$ is homogeneous of degree $-\dim X - d + j$ (where $\mathcal{F}$ denotes the Fourier transform), we obtain $\mathcal{F}(a_{d-j}(x,y,\cdot))(0) = 0$ for $d-j < -\dim X$, i.e. $k_{d-j}(x,x) = \lim_{y\to x} k_{d-j}(x,y) = \lim_{y\to x}\mathcal{F}(a_{d-j}(x,y,\cdot))(y-x) = \mathcal{F}(a_{d-j}(x,x,\cdot))(0) = 0$. Thus, $k^{\mathrm{reg}}(x,x)$ is independent of N.

However, this property does not extend to Fourier Integral Operators as we can easily construct a counter-example. Let $a(x,y,\xi)$ be homogeneous of degree $d < -n$ in the third argument and the phase function

$\vartheta(x,y,\xi) = -\langle\Theta(x,y),\xi\rangle_{\ell_2(n)}$ such that $\Theta(x,x)$ has no zeros. Then,

$$k(x,y) = \int_{\mathbb{R}^n} e^{-i\langle\Theta(x,y),\xi\rangle_{\ell_2(n)}} a(x,y,\xi)d\xi = \mathcal{F}(a(x,y,\cdot))(\Theta(x,y))$$

shows that $k(x,x)$ is well-defined and continuous. Furthermore, since $\mathcal{F}(a(x,y,\cdot))$ is homogeneous, vanishing $k(x,x)$ implies $\mathcal{F}(a(x,x,\cdot)) = 0$ on $\{r\Theta(x,x);\ r \in \mathbb{R}_{>0}\}$.

On the other hand, for pseudo-differential operators the terms a_{d-j} with $d-j = -\dim X$ define a global density on the manifold giving rise to the residue trace. If this extends to poly-log-homogeneous distributions, then we obtain the residue trace globally from $\sum_{\iota\in I_0}\alpha_\iota$. Furthermore, this would imply that

$$\mathfrak{fp}_0\alpha = \alpha - \sum_{\iota\in I_0}\alpha_\iota$$

induces a global density, if α does and the contributions of the α_ι for $\iota \in I_0$ to the constant term Laurent coefficient vanish (in particular if $I_0 = \varnothing$), which allows us to interpret $\zeta(\mathfrak{fp}_0\alpha)(0)$ as the generalization of the Kontsevich-Vishik trace.

This, of course, needs to be interpreted in a gauged sense. $\zeta(\mathfrak{fp}_0\alpha)(0)$ corresponds to the kernel $k(x,y) - k_{d-j}(x,y)$ where $d-j = -\dim X$. Hence, all terms k_{d-j} with $j \in \mathbb{N}_{0,<d+\dim X}$ still appear in $\mathfrak{fp}_0\alpha$ but not in k^{reg}. Since $\zeta(\mathfrak{fp}_0\alpha)(0)$ is but constructed by gauging, we should do the same for k_{d-j}, i.e. consider k_{d-j+z} which is continuous for $\Re(z)$ sufficiently small and vanishes along the diagonal. Therefore,

$$\zeta(\mathfrak{fp}_0\alpha)(0) = \int_X k^{\mathrm{reg}}(x,x)d\mathrm{vol}_X(x).$$

holds in the regularized sense for pseudo-differential operators; particularly so since Corollary 2.8 guarantees that $\zeta(\mathfrak{fp}_0\alpha)(0)$ is independent of the gauge. In other words, the objective is to show that

$$\begin{aligned}\sum_\chi \operatorname{res}\alpha^\chi(0) &= \sum_\chi \left\langle \int_{\partial B_{\mathbb{R}^N}} \tilde{\alpha}^\chi(0) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}, f \right\rangle \\ &= \sum_\chi \left\langle P \int_{\partial B_{\mathbb{R}^N}} \tilde{\alpha}^\chi(0) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}, \delta_0 \right\rangle \\ &= \sum_\chi \left\langle \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} \tilde{a}^\chi(0)(x,y,\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi), \delta_0 \right\rangle\end{aligned}$$

is globally well-defined ($\sum_\chi$ denotes a partition of unity and P is a suitable pseudo-differential operator) if the a^χ are log-homogeneous with degree of homogeneity $-N$.

At this point, we return to the fact that we can find a representation

$$\int_{\mathbb{R}^{2\dim X} \setminus B_{\mathbb{R}^{2\dim X}}} e^{i\langle (x,y),\xi\rangle_{\ell_2(2\dim X)}} \hat{a}((x,y),\xi)\ d\mathrm{vol}_{\mathbb{R}^{2\dim X} \setminus B_{\mathbb{R}^{2\dim X}}}(\xi)$$

of

$$\int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi)\ d\mathrm{vol}_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}}(\xi)$$

where $\hat{a}$ is poly-log-homogeneous with degree of homogeneity $-2\dim X$ and logarithmic order l if a has degree of homogeneity $-N$ and logarithmic order l. Thus, we want to show that the locally defined

$$\int_{\partial B_{\mathbb{R}^{2\dim X}}} e^{i\langle (x,y),\xi\rangle_{\ell_2(2\dim X)}} \tilde{a}^\chi((x,y),\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi)$$

patch together if a^χ is log-homogeneous with degree of homogeneity $-2\dim X$.

Let φ be a suitable test function, and

$$\int_{X^2} \int_{\mathbb{R}^{2\dim X}} e^{i\langle (x,y),\xi\rangle_{\ell_2(2\dim X)}} a(x,y,\xi)\varphi(x,y)\ d\xi\ d\mathrm{vol}_{X^2}(x,y)$$

and

$$\int_{X^2} \int_{\mathbb{R}^{2\dim X}} e^{i\vartheta(x,y,\xi)} a^\chi(x,y,\xi)\varphi(x,y)\ d\xi\ d\mathrm{vol}_{X^2}(x,y)$$

be two representations of $\langle u, f\rangle$ where ϑ is another linear phase function.[1] Proposition 2.4.1 in [**39**] warrants the existence of a C^∞-map Θ taking values $\Theta(x,y) \in GL(\mathbb{R}^{2\dim X})$ such that

$$\vartheta(x,y,\xi) = \langle (x,y), \Theta(x,y)\xi\rangle_{\ell_2(2\dim X)}$$

holds. Hence,

$$\begin{aligned}
&\int_{X^2}\int_{\mathbb{R}^{2\dim X}} e^{i\vartheta(x,\xi)} a^\chi(x,\xi)\varphi(x)\ d\xi\ d\mathrm{vol}_{X^2}(x) \\
&= \int_{X^2}\int_{\mathbb{R}^{2\dim X}} e^{i\langle x,\Theta(x)\xi\rangle_{\ell_2(2\dim X)}} a^\chi(x,\xi)\varphi(x)\ d\xi\ d\mathrm{vol}_{X^2}(x) \\
&= \int_{X^2}\int_{\mathbb{R}^{2\dim X}} e^{i\langle x,\xi\rangle_{\ell_2(2\dim X)}} a^\chi(x,\Theta(x)^{-1}\xi)\varphi(x)\left|\det\Theta(x)^{-1}\right|\ d\xi\ d\mathrm{vol}_{X^2}(x).
\end{aligned}$$

In other words, the amplitude a transforms into $a^\chi(x,\Theta(x)^{-1}\xi)\left|\det\Theta(x)^{-1}\right|$ for some C^∞-function Θ taking values in $GL(\mathbb{R}^{2\dim X})$, more precisely

$$a(x,y,\xi) = a^\chi(\chi(x,y),\Theta(x,y)^{-1}\xi)\left|\det\Theta(x,y)^{-1}\right|\left|\det\chi'(x,y)\right|$$

for some diffeomorphism χ, and we need to show

$$\begin{aligned}
\operatorname{res}\alpha(0) &= \int_{\partial B_{\mathbb{R}^{2\dim X}}} \hat{a}(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi) \\
&= \int_{\partial B_{\mathbb{R}^{2\dim X}}} \hat{a}^\chi(\Theta^{-1}\xi)\left|\det\Theta^{-1}\right| d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi) \\
&\overset{?}{=} \int_{\partial B_{\mathbb{R}^{2\dim X}}} \hat{a}^\chi(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^{2\dim X}}}(\xi) \\
&= \operatorname{res}\alpha^\chi(0)
\end{aligned}$$

where α and α^χ are the corresponding log-homogeneous distributions, and $\hat{a}$ and $\hat{a}^\chi$ are the restrictions to the homogeneous part of α and α^χ, i.e. $\hat{a}(r\xi) = r^{d_\iota}\tilde{\alpha}(\xi)$.

[1] Since we are considering representations with phase function $\langle (x,y),\xi\rangle_{\ell_2(2\dim X)}$, changing charts yields a phase function $\langle \chi(x,y),\xi\rangle_{\ell_2(2\dim X)}$. Hence, it suffices to consider replacements by linear phase functions only.

LEMMA 7.1. *Let $a \in C(\mathbb{R}^n \smallsetminus \{0\})$ be homogeneous of degree d, $k \in \mathbb{N}_0$, $z \in \mathbb{C}$, and $T \in GL(\mathbb{R}^n)$. Then*

$$\begin{aligned}&\int_{\partial B_{\mathbb{R}^n}} a(T\xi)\,\|T\xi\|^z\,(\ln\|T\xi\|)^k\,d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)\\ =&\frac{(-1)^k}{|\det T|}\int_{\partial B_{\mathbb{R}^n}} a(\xi)\,\left\|T^{-1}\xi\right\|^{-n-d-z}\left(\ln\left\|T^{-1}\xi\right\|\right)^k d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi).\end{aligned}$$

PROOF. Let $D := (2B_{\mathbb{R}^n}) \smallsetminus B_{\mathbb{R}^n}(0,1) = \{\xi \in \mathbb{R}^n;\ \|\xi\|_{\ell_2(n)} \in [1,2]\}$. Then, we observe for $z \neq -n-d$

$$\begin{aligned}\int_D a(T\xi)\,\|T\xi\|^z\,d\xi =& \int_{[1,2]} r^{n-1}\int_{\partial B_{\mathbb{R}^n}} a(rT\xi)\,\|rT\xi\|^z\,d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)dr\\ =&\int_{[1,2]} r^{n+d+z-1}dr\int_{\partial B_{\mathbb{R}^n}} a(T\xi)\,\|T\xi\|^z\,d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)\\ =&\frac{2^{n+d+z}-1}{n+d+z}\int_{\partial B_{\mathbb{R}^n}} a(T\xi)\,\|T\xi\|^z\,d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi),\end{aligned}$$

as well as,

$$\begin{aligned}\int_D a(T\xi)\,\|T\xi\|^z\,d\xi =& \int_{T[D]} a(\xi)\,\|\xi\|^z\left|\det T^{-1}\right|d\xi\\ =&\frac{1}{|\det T|}\int_{\{\xi\in\mathbb{R}^n;\ \|T^{-1}\xi\|\in[1,2]\}} a(\xi)\,\|\xi\|^z\,d\xi\\ =&\frac{1}{|\det T|}\int_{\partial B_{\mathbb{R}^n}}\int_{\left[\frac{1}{\|T^{-1}\xi\|},\frac{2}{\|T^{-1}\xi\|}\right]} a(r\xi)\,\|r\xi\|^z\,r^{n-1}\ dr\ d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)\\ =&\frac{1}{|\det T|}\int_{\partial B_{\mathbb{R}^n}} a(\xi)\int_{\left[\frac{1}{\|T^{-1}\xi\|},\frac{2}{\|T^{-1}\xi\|}\right]} r^{n+d+z-1}\ dr\ d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)\\ =&\frac{1}{|\det T|}\int_{\partial B_{\mathbb{R}^n}} a(\xi)\frac{1}{n+d+z}\frac{2^{n+d+z}-1}{\left\|T^{-1}\xi\right\|^{n+d+z}}\ d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi).\end{aligned}$$

In other words,

$$\int_{\partial B_{\mathbb{R}^n}} a(T\xi)\,\|T\xi\|^z\,d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi) = \frac{1}{|\det T|}\int_{\partial B_{\mathbb{R}^n}} a(\xi)\,\left\|T^{-1}\xi\right\|^{-n-d-z}\ d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)$$

holds for $z \in \mathbb{C} \smallsetminus \{-n-d\}$ and by holomorphic extension for every $z \in \mathbb{C}$.

For $k \in \mathbb{N}$ we, thus, obtain

$$\begin{aligned}
&\int_{\partial B_{\mathbb{R}^n}} a(T\xi) \|T\xi\|^z (\ln \|T\xi\|)^k \, d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi) \\
=&\partial^k \left(s \mapsto \int_{\partial B_{\mathbb{R}^n}} a(T\xi) \|T\xi\|^s \, d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi) \right)(z) \\
=&\partial^k \left(s \mapsto \frac{1}{|\det T^{-1}|} \int_{\partial B_{\mathbb{R}^n}} a(\xi) \left\|T^{-1}\xi\right\|^{-n-d-s} \, d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi) \right)(z) \\
=&\frac{(-1)^k}{|\det T|} \int_{\partial B_{\mathbb{R}^n}} a(\xi) \left\|T^{-1}\xi\right\|^{-n-d-z} \left(\ln \left\|T^{-1}\xi\right\|\right)^k d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)
\end{aligned}$$

which completes the proof.

□

Lemma 7.1 (first observed by Lesch; equation (2.13) in [**51**]), and the fact that $\tilde{a}$ is a homogeneous function with degree of homogeneity $-N$ if a is log-homogeneous with degree of homogeneity $-N$, yield (using $N = 2\dim X$, a suitable $U \subseteq_{\text{open}} \mathbb{R}^N$, a diffeomorphism $\chi\colon\ U \to \chi[U]$, and a $\varphi \in C_c^\infty(\chi[U])$)

$$\begin{aligned}
&\int_U \int_{\partial B_{\mathbb{R}^N}} \hat{a}(x,\xi)\varphi(\chi(x)) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) dx \\
=&\int_U \int_{\partial B_{\mathbb{R}^N}} \hat{a}^\chi(\chi(x), \Theta(x)^{-1}\xi) \left|\det \Theta(x)^{-1}\right| |\det \chi'(x)| \varphi(\chi(x)) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) dx \\
=&\int_U \int_{\partial B_{\mathbb{R}^N}} \hat{a}^\chi(\chi(x), \Theta(x)^{-1}\xi) \left|\det \Theta(x)^{-1}\right| |\det \chi'(x)| \varphi(\chi(x)) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) dx \\
=&\int_U \left|\det \Theta(x)^{-1}\right| \int_{\partial B_{\mathbb{R}^N}} \hat{a}^\chi(\chi(x), \Theta(x)^{-1}\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) |\det \chi'(x)| \varphi(\chi(x)) dx \\
=&\int_U \int_{\partial B_{\mathbb{R}^N}} \hat{a}^\chi(\chi(x), \xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) |\det \chi'(x)| \varphi(\chi(x)) dx \\
=&\int_{\chi[U]} \int_{\partial B_{\mathbb{R}^N}} \hat{a}^\chi(x,\xi)\varphi(x) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) dx,
\end{aligned}$$

i.e. the following theorem.

THEOREM 7.2. $\mathrm{res}\langle u, f\rangle = \mathrm{res}\,\alpha(0) = \int_{\partial B_{\mathbb{R}^N}} \tilde{\alpha}(0) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ *is form-invariant under change of coordinates if* $\alpha(0)$ *has degree of homogeneity* $-N$.

In particular, $\sum_\chi \sum_{\iota\in I_0^\chi} \operatorname{res}\alpha_\iota^\chi(0)$ *and* $\sum_\chi \zeta\left(\mathfrak{fp}_0\alpha^\chi\right)(0) + \sum_\chi \sum_{\iota\in I_0^\chi} \partial \operatorname{res}\alpha_\iota^\chi(0)$ *induce globally defined densities provided that* $\forall \iota \in I_0 : \ l_\iota = 0$.

Proof. Note that $\zeta(\alpha)$ induces a globally defined density through the implicit assumption of the kernel being globally defined and, given $\forall \iota \in I_0 : \ l_\iota = 0$, $\sum_\chi \sum_{\iota\in I_0^\chi} \operatorname{res}\alpha_\iota^\chi(0)$ being form-invariant implies that the principal part of $\zeta(\alpha)$ induces a globally defined density. Hence, their difference (here evaluated at zero) $\sum_\chi \zeta\left(\mathfrak{fp}_0\alpha^\chi\right)(0) + \sum_\chi \sum_{\iota\in I_0^\chi} \partial \operatorname{res}\alpha_\iota^\chi(0)$ must induce a globally defined density, as well.

□

Remark Note that this means that if a is polyhomogeneous and ι_0 is the index such that a_{ι_0} is homogeneous of degree $-N$, then

$$\begin{aligned}&\sum_{\iota\in I_0}\int_X\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}a_\iota(x,x,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)d\mathrm{vol}_X(x)\\ =&\int_X\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}a_{\iota_0}(x,x,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)d\mathrm{vol}_X(x).\end{aligned}$$

This, of course, extends to higher order residues

$$\int_X\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}a_\iota(x,x,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)d\mathrm{vol}_X(x)$$

with $\iota \in I_0$ and $l_\iota > 0$; this generalizes Corollary 4.8 in [**51**] on the residue traces for log-polyhomogeneous pseudo-differential operators; that is, the k^{th} residues (Laurent coefficients of order $-k-1$) are well-defined and induce globally defined densities.

■

Uniqueness of the residue trace, then, directly implies the following proposition.

Proposition 7.3. *Let* $a \sim \sum_{j\in\mathbb{N}_0} a_{m-j}$ *be the amplitude of a Fourier Integral Operator where* $m \in \mathbb{Z}$ *and* a_{m-j} *is homogeneous of degree* $m-j$*. If the residue* *trace is the (projectively) unique non-trivial continuous trace, then none of the* $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a_{m-j}(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$ *with* $m-j \neq -N$ *can define a global density, in general, unless they are trivial (i.e. vanish constantly).*

In particular, removing non-trivial terms from $\zeta(\mathfrak{fp}_0\alpha)$ *will, in general, destroy global well-definedness of the induced density.*

PROOF. Let $j \in \mathbb{Z} \setminus \{-N\}$ and suppose $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a_j(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$ defines a global density. Then, it defines a continuous trace functional τ. On the other hand, we know that $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a_{-N}(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$ defines a continuous trace functional $\operatorname{res}\operatorname{tr}$. Since $\operatorname{res}\operatorname{tr}$ is the unique trace, τ must be a constant multiple of $\operatorname{res}\operatorname{tr}$, i.e. $\exists t \in \mathbb{C}: \ \tau = t \operatorname{res}\operatorname{tr}$. Hence, there are two cases; $t = 0$ or $t \neq 0$.

If $t = 0$, then $\tau = 0$, i.e. τ is trivial. If $t \neq 0$, we might replace a_j by zero and leave a_{-N} unchanged. Let A be the unchanged Fourier Integral Operator and B the changed. Then,

$$0 = \frac{1}{t}\tau(B) = \operatorname{res}\operatorname{tr}(B) = \operatorname{res}\operatorname{tr}(A) = \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a_{-N}(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$$

holds independently of the choice of a_{-N}, i.e. $\operatorname{res}\operatorname{tr} = 0$, contradicting the assumption that $\operatorname{res}\operatorname{tr}$ is non-trivial.

□

The proposition above can be extended to the formulation

PROPOSITION 7.3'. *Let* $\mathcal{A}$ *be an algebra of polyhomogeneous Fourier Integral Operators such that the residue trace is the unique non-trivial continuous trace. Let* $a = a_0 + \sum_{\iota \in I} a_\iota$ *be the amplitude of a Fourier Integral Operator* $A \in \mathcal{A}$*. Then, none of the* $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a_\iota(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)$ *with* $d_\iota \neq -N$ *can define a global density, in general, unless they are trivial.*

In particular, removing non-trivial terms from $\zeta(\mathfrak{fp}_0\alpha)$ *will, in general, destroy global well-definedness of the induced density.*

using the same proof.

Now, we may ask when the residue vanishes. As a first result we obtain the well-known fact that the residue trace vanishes for odd-class operators on odd-dimensional manifolds.

OBSERVATION 7.4. *Let* $\alpha(-\xi) = -\alpha(\xi)$. *Then,*

$$\operatorname{res}\alpha = \int_{\partial B_{\mathbb{R}^N}} \alpha(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) = 0.$$

PROOF. Using Lemma 7.1, we obtain

$$\begin{aligned}
\operatorname{res}\alpha &= \int_{\partial B_{\mathbb{R}^N}} \alpha(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= -\int_{\partial B_{\mathbb{R}^N}} \alpha(-\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= -\int_{\partial B_{\mathbb{R}^N}} \alpha\left(\frac{-\xi}{\|-\xi\|_{\ell_2(N)}}\right) \|-\xi\|_{\ell_2(N)}\, d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= -\int_{\partial B_{\mathbb{R}^N}} \alpha\left(\frac{\xi}{\|\xi\|_{\ell_2(N)}}\right) \|\xi\|_{\ell_2(N)}^{-N-1}\, d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= -\int_{\partial B_{\mathbb{R}^N}} \alpha(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= -\operatorname{res}\alpha,
\end{aligned}$$

i.e. the assertion.

□

Note that the property $\alpha(-\xi) = -\alpha(\xi)$ is invariant under change of linear phase functions with the same "N". Choosing non-linear phase functions or changing N might destroy this property. In fact, having phase functions with $\vartheta(-\xi) = -\vartheta(\xi)$

will only yield

$$\begin{aligned}
\operatorname{res}(a,\vartheta) &:= \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(\xi)} a(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= -\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(\xi)} a(-\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= -\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(-\xi)} a(-\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right)^* \\
&= -\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta\left(\frac{-\xi}{\|-\xi\|_{\ell_2(N)}}\right)} a\left(\frac{-\xi}{\|-\xi\|_{\ell_2(N)}}\right) \|-\xi\|_{\ell_2(N)} d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right)^* \\
&= -\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta\left(\frac{\xi}{\|\xi\|_{\ell_2(N)}}\right)} a\left(\frac{\xi}{\|\xi\|_{\ell_2(N)}}\right) \|\xi\|_{\ell_2(N)}^{-N-1} d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right)^* \\
&= -\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(\xi)} a(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right)^* \\
&= -\left(\operatorname{res}(a,\vartheta)\right)^*,
\end{aligned}$$

i.e. $\mathfrak{R}\left(\operatorname{res}(a,\vartheta)\right) = 0$ but not necessarily $\mathfrak{I}\left(\operatorname{res}(a,\vartheta)\right) = 0$.

On the other hand, if $N = 1$, then

$$\int_{\partial B_{\mathbb{R}}} \alpha(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi) = \alpha(1) + \alpha(-1)$$

shows that $\operatorname{res}\alpha$ vanishes if and only if α is odd. Equivalently, we obtain

$$\int_{\partial B_{\mathbb{R}}} e^{i\vartheta(x,\xi)} a(x,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}}}(\xi) = e^{i\vartheta(x,1)} a(x,1) + e^{i\vartheta(x,-1)} a(x,-1).$$

Note, this implies there are two residue traces for $N = 1$; namely, $\alpha_{-1}(1)$ and $\alpha_{-1}(-1)$.

Remark Boutet de Monvel [**7**] considers Fourier Integral Operators on the half-line bundle only, since for $N = 1$ the residue trace is not unique. Hence, in his case, the amplitude and phase function are defined on $X \times X \times \mathbb{R}_{>0}$ which can easily be modeled using $\forall (x,y,\xi) \in X \times X \times \mathbb{R}_{<0}: \ a(x,y,\xi) = 0$. However, using the gauged poly-log-homogeneous distributions, no such trick is necessary since we can simply choose M to be a single point, i.e. $\mathbb{R}_{>0} \times M \cong \mathbb{R}_{>0}$.

In chapter 8, following Proposition 8.4, we will have a closer look at the case Boutet de Monvel studied in [**7**]. In particular, we will re-obtain the kernel singularity structure and the residue trace as the logarithmic coefficient, as well as, calculate the generalized Kontsevich-Vishik trace which will turn out to be form-equivalent to the pseudo-differential case.

■

For $N > 1$, the de Rham co-homology of $\partial B_{\mathbb{R}^N}$ is given by

$$\forall k \in \mathbb{N}_0 : \ H^k_{\mathrm{dR}}(\partial B_{\mathbb{R}^N}) \cong \begin{cases} \mathbb{R} & , \ k \in \{0, N-1\} \\ 0 & , \ k \in \mathbb{N} \setminus \{N-1\} \end{cases}$$

(cf. Example 9.29 in [**52**]). Let d be the exterior derivative and $d_k := d|_{\Omega^k(\partial B_{\mathbb{R}^N})}^{\Omega^{k+1}(\partial B_{\mathbb{R}^N})}$. Then,

$$\forall \omega \in \Omega^{N-1}(\partial B_{\mathbb{R}^N}) : \ d\omega = d_{N-1}\omega = 0$$

implies

$$\mathbb{R} \cong H^{N-1}_{\mathrm{dR}}(\partial B_{\mathbb{R}^N}) = [\{0\}]d_{N-1} / _{d_{N-2}[\Omega^{N-2}(\partial B_{\mathbb{R}^N})]} = \Omega^{N-1}(\partial B_{\mathbb{R}^N}) / _{d_{N-2}[\Omega^{N-2}(\partial B_{\mathbb{R}^N})]}.$$

Hence, for every $(N-1)$-form ω there exists an $r \in \mathbb{R}$ and an $(N-2)$-form $\tilde{\omega}$ such that $\omega = r\omega_0 + d\tilde{\omega}$ where[2] $\omega_0 := \mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})^{-1} \, d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$, i.e.

$$\int_{\partial B_{\mathbb{R}^N}} \omega = r \underbrace{\int_{\partial B_{\mathbb{R}^N}} \omega_0}_{=1} + \int_{\partial B_{\mathbb{R}^N}} d\tilde{\omega} = r + \int_{\underbrace{\partial\partial B_{\mathbb{R}^N}}_{=\emptyset}} \tilde{\omega} = r.$$

If ω is complex valued, then there are $r, s \in \mathbb{R}$ and ω_r, ω_s such that $\mathfrak{R} \circ \omega = r\omega_0 + d\omega_r$ and $\mathfrak{I} \circ \omega = s\omega_0 + d\omega_s$ hold and, therefore,

$$\int_{\partial B_{\mathbb{R}^N}} \omega = \int_{\partial B_{\mathbb{R}^N}} \mathfrak{R} \circ \omega + i \int_{\partial B_{\mathbb{R}^N}} \mathfrak{I} \circ \omega = r + is.$$

[2] Note that $\omega_0 \notin d_{N-2}[\Omega^{N-2}(\partial B_{\mathbb{R}^N})]$ since $\int_{\partial B_{\mathbb{R}^N}} \omega_0 = 1$ and $\forall \omega \in \Omega^{N-2}(\partial B_{\mathbb{R}^N})$: $\int_{\partial B_{\mathbb{R}^N}} d\omega = \int_{\underbrace{\partial\partial B_{\mathbb{R}^N}}_{=\emptyset}} \omega = 0$.

In other words, $H^{N-1}_{\mathrm{dR}}(\partial B_{\mathbb{R}^N}, \mathbb{C}) \cong \mathbb{C}$ or

$$\forall \omega \in \Omega^{N-1}(\partial B_{\mathbb{R}^N}, \mathbb{C}) \ \exists c \in \mathbb{C} \ \exists \tilde{\omega} \in \Omega^{N-2}(\partial B_{\mathbb{R}^N}, \mathbb{C}) : \ \omega = c\omega_0 + d\tilde{\omega}.$$

Thus, we obtain the following statements.

(i) $\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) = 0$ if and only if the differential form $e^{i\vartheta(x,y,\cdot)} a(x,y,\cdot) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ is exact.

(ii) $\Re\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right) = 0$ if and only if the differential form $\cos(\vartheta(x,y,\cdot))\, a(x,y,\cdot) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ is exact.

(iii) $\Im\left(\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\right) = 0$ if and only if the differential form $\sin(\vartheta(x,y,\cdot))\, a(x,y,\cdot) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$ is exact.

REMARK Since we are integrating $\dim M$-forms over a manifold M, we assume that all manifolds are orientable as we can only integrate pseudo-$\dim M$-forms if M is non-orientable. So far everything can be re-formulated for pseudo-forms and, thus, on non-orientable manifolds. From this point onwards (until the end of the chapter), though, statements will need orientability; in particular with respect to uniqueness of residue traces and the commutator structure since

$$H^{\dim M}_{\mathrm{dR}}(M) \cong \begin{cases} \mathbb{R} & , \ M \text{ orientable, connected} \\ 0 & , \ M \text{ non-orientable, connected} \end{cases}$$

(cf. Theorem 10.13 and Corollary 10.14 in [**52**] for the orientable case, that is the case we are going to use).

■

The case above allows us to treat Laurent coefficients of the form $\int_{\partial B_{\mathbb{R}^N}} \tilde{\alpha} d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$. However, considering more general poly-log-homogeneous distributions means we will want to replace $\partial B_{\mathbb{R}^N}$ by some other manifold M. Similarly, if we want to choose more suitable coordinates, then our Laurent coefficients are integrals over

$X \times M$ where X is the underlying manifold and $M = \partial B_{\mathbb{R}^N}$ in the canonical Fourier Integral Operator case.

Using the fact that the de Rham co-homology is additive on disjoint unions, i.e. $\forall k \in \mathbb{N}_0 : H^k_{\mathrm{dR}}(M \uplus M') = H^k_{\mathrm{dR}}(M) \oplus H^k_{\mathrm{dR}}(M')$, and splitting in real and imaginary parts again, we obtain for a smooth, compact, orientable manifold M

$$H^{\dim M}_{\mathrm{dR}}(M, \mathbb{C}) \cong \mathbb{C}^k$$

where k is number of connected components of M.

Definition 7.5. *Let A be a polyhomogeneous Fourier Integral Operator on a compact manifold X and* $\operatorname{res}_0 \zeta(A)$ *be locally given by*

$$\int_X \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,\xi)} a(x,\xi) \ \mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\xi) \ \mathrm{dvol}_X(x).$$

Then, we call the $(N-1+\dim X)$-form $\varrho(A)$ on $X \times \partial B_{\mathbb{R}^N}$ locally defined as

$$\varrho(A) := \exp \circ (i\vartheta) \cdot a \ \mathrm{dvol}_{X \times \partial B_{\mathbb{R}^N}}$$

the residue form of A (in other words, $*\varrho(A) = e^{i\vartheta} a$ where $*$ denotes the Hodge-$*$-operator).

Proposition 7.6. *Let $Y \subseteq X$ be a connected component and $\varrho(A)$ a residue form. Then, $\int_{Y \times \partial B_{\mathbb{R}^N}} \varrho(A) = 0$ if and only if $\varrho(A)$ is exact on $Y \times \partial B_{\mathbb{R}^N}$.*

More precisely, let $X = Y_1 \uplus \ldots \uplus Y_k$ be composed of finitely many connected components ($\uplus$ denotes the disjoint union) and let $\varrho(A)|_{Y_j \times \partial B_{\mathbb{R}^N}} = c_j \omega_j + d\tilde{\omega}_j$ be the corresponding decomposition of $\varrho(A)$ with

$$\omega_j = \mathrm{vol}_{Y_j \times \partial B_{\mathbb{R}^N}}(Y_j \times \partial B_{\mathbb{R}^N})^{-1} \mathrm{dvol}_{Y_j \times \partial B_{\mathbb{R}^N}}.$$

Then,

$$\int_{X\times\partial B_{\mathbb{R}^N}} \varrho(A) = \sum_{j=1}^{k} c_j.$$

Using the Hodge-$*$-operator $*$, the co-derivative $d^* := (-1)^{N_X(N_X-1)+1} * d *$ with $N_X := N + \dim X - 1$, as well as

$$\begin{aligned}\varrho(A) = d\omega \;\Leftrightarrow\; e^{i\vartheta}a &= * \, d\omega \\ &= * \, d * (-1)^{N_X-1} * \omega \\ &= d^* (-1)^{N_X(N_X-1)+1} (-1)^{N_X-1} * \omega \\ &= d^* \left((-1)^{N_X^2} * \omega \right),\end{aligned}$$

and the divergence $\operatorname{div} F = *d * F^\flat = (-1)^{N_X(N_X-1)+1} d^* F^\flat$ with the musical isomorphism

$$\cdot^\flat :\; T\left(X \times \partial B_{\mathbb{R}^N}\right) \to T^*\left(X \times \partial B_{\mathbb{R}^N}\right);\; \sum_i F_i \partial_i \mapsto \sum_i F_i dx_i,$$

we can re-formulate Proposition 7.6.

THEOREM 7.7. *Let $X \times M$ be connected and orientable. Then, the following are equivalent.*

(i) $\int_{x\times M} e^{i\vartheta(x,\xi)} a(x,\xi)\ \mathrm{dvol}_{x\times M}(x,\xi) = 0.$

(ii) *There exists an* $(\dim M + \dim X - 1)$*-form ω on $X \times M$ such that* $d\omega = e^{i\vartheta} a\ \mathrm{dvol}_{X\times M}$ *locally.*

(iii) *There exists a* 1*-form ω on $X \times M$ such that* $d^*\omega = e^{i\vartheta} a$ *locally.*

(iv) *There exists a vector field F on $X \times M$ such that* $\operatorname{div} F = e^{i\vartheta} a$ *locally.*

COROLLARY 7.8. *Let α be a poly-*log*-homogeneous distribution and* $\operatorname{res}\alpha = \int_M \hat{\alpha}\mathrm{dvol}_M$. *Then,* $\operatorname{res}\alpha = 0$ *if and only if there exists a vector field F on M such that* $\hat{\alpha} = \operatorname{div} F$.

REMARK *Condition (iv) can be extended to* $X \times \left(\mathbb{R}^N \smallsetminus \{0\}\right)$. Let $\hat{M} := X \times M$, $(g_i)_i$ the local frame in which $e^{i\vartheta}a$ is given by α, and $(g^i)_i$ the dual frame. Let $\tilde{M} := \mathbb{R}_{>0} \times \hat{M}$ such that the metric tensor is of the form

$$\tilde{g}(r,\xi) = \begin{pmatrix} 1 & 0 \\ 0 & r^{2\dim \hat{M}} g(\xi) \end{pmatrix},$$

i.e. $d\mathrm{vol}_{\tilde{M}}(r,\xi) = \sqrt{\det \tilde{g}(r,\xi)}dr \wedge d\xi = r^{\dim \hat{M}} \sqrt{\det g(\xi)} dr \wedge d\xi = r^{\dim \hat{M}} dr \wedge d\mathrm{vol}_{\hat{M}}(\xi)$. Let F be a vector field on $\hat{M}$ and $\tilde{F}$ be a vector field on $\tilde{M}$. Then,

$$\begin{aligned} \operatorname{div} F(\xi) = \operatorname{tr} \operatorname{grad} F(\xi) &= \operatorname{tr} \sum_{j=1}^{\dim \hat{M}} \sum_{i=1}^{\dim \hat{M}} \partial_j F_i(\xi) g^j(\xi) \otimes g^i(\xi) \\ &= \sum_{j=1}^{\dim \hat{M}} \sum_{i=1}^{\dim \hat{M}} \partial_j F_i(\xi) g^{ji}(\xi) \end{aligned}$$

and

$$\begin{aligned} \operatorname{div} \tilde{F}(r,\xi) &= \operatorname{tr} \sum_{j=0}^{\dim \hat{M}} \sum_{i=0}^{\dim \hat{M}} \partial_j \tilde{F}_i(r,\xi) \tilde{g}^j \otimes \tilde{g}^i \\ &= \partial_0 \tilde{F}_0(r,\xi) + r^{2\dim \hat{M}} \sum_{j=1}^{\dim \hat{M}} \sum_{i=1}^{\dim \hat{M}} \partial_j \tilde{F}_i(r,\xi) g^{ji}(\xi). \end{aligned}$$

In other words, we obtain $\operatorname{div} \tilde{F}(1,\xi) = \operatorname{div} F(\xi)$ if $\partial_0 \tilde{F}_0(1,\xi) = 0$ and $\partial_j \tilde{F}_i(1,\xi) = \partial_j F_i(\xi)$. On the other hand, we want $\operatorname{div} F(\xi) = \tilde{\alpha}(\xi)$ and $\operatorname{div} \tilde{F}(r,\xi) = f(r)\tilde{\alpha}(\xi)$ with $f(1) = 1$. Choosing $\tilde{F}_0 = 0$ and $\tilde{F}_i(r,\xi) = f(r)F_i(\xi)$ implies $\operatorname{div} \tilde{F}(r,\xi) = f(r)\tilde{\alpha}(\xi)$ and $\operatorname{div} \tilde{F}(1,\xi) = \operatorname{div} F(\xi)$.

Thus, knowing (iv) we can construct a vector field $\tilde{F}$ such that $e^{i\vartheta} = \operatorname{div} \tilde{F}$ on $X \times (\mathbb{R}_{>0} \times M)$ and $\tilde{F}$ satisfies the conditions above. Conversely, if $\tilde{F}$ has the described properties, then $\tilde{F}|_{X \times M}$ satisfies (iv).

■

At this point, using the framework of gauged poly-log-homogeneous distributions, we can follow the lines of Theorem 1.1 in **[34]** to obtain the following theorem (Theorem 1.2 in **[34]**).

THEOREM 7.9. *Let $\mathcal{A}_\Gamma$ be an algebra of classical Fourier Integral Operators associated with the canonical relation Γ such that the twisted relation Γ' ($A \in \mathcal{A}_\Gamma \Leftrightarrow k_A \in I(X^2, \Gamma')$) has clean and connected intersection with the co-normal bundle of diagonal in X^2. Then, the residue-trace of $A \in \mathcal{A}_\Gamma$ vanishes if and only if A is a smoothing operator plus a sum of commutators $[P_i, A_i]$ where the P_i are pseudo-differential operators and the $A_i \in \mathcal{A}_\Gamma$.*

PROOF. If $A = S + \sum_{i=1}^k [P_i, A_i] \in \mathcal{A}_\Gamma$ where S is a smoothing operator, $A_i \in \mathcal{A}_\Gamma$, and the P_i are pseudo-differential operators, then $\zeta(A) = \zeta(S)$ which is an entire function choosing any appropriate gauge, i.e. $\operatorname{res}_0 \zeta(A) = 0$. The interesting direction is, therefore, the other implication. Let $I_{\text{compact}}(X, \Lambda)$ be the set of compactly supported Lagrangian distributions on X with micro-support in a closed conic Lagrangian sub-manifold Λ of $T^*X \setminus \{X \times \{0\}\}$. Let $f \in I_{\text{compact}}(X, \tilde{\Lambda})$ such that the intersection of Λ and $\tilde{\Lambda}$ is clean and connected. Furthermore, let f be non-vanishing on $\Lambda \cap \tilde{\Lambda}$. Let $\psi DO(X)$ be the ring of properly supported pseudo-differential operators on X, that is, pseudo-differential operators mapping $C_c^\infty(X)$ into itself. We will define the transposed annihilator of f to be

$$\operatorname{ann}(f)^t := \{P \in \psi DO(X);\ P^t f \in C^\infty(X)\}$$

and we say $u_1, u_2 \in I_{\text{compact}}(X, \Lambda)$ are equivalent ($u_1 \sim u_2$) if and only if there are $k \in \mathbb{N}$, $v_i \in I_{\text{compact}}(X, \Lambda)$, and $P_i \in \operatorname{ann}(f)^t$ such that

$$u_1 - u_2 \equiv \sum_{i=1}^k P_i v_i$$

modulo smoothing terms.

- As we are interested in traces, we will need to consider $f = \delta_{\text{diag}}$ and, since $\operatorname{ann}(\delta_{\text{diag}})^t$ is generated by operators of the form $P(x, D_x) - P(y, D_y)^t$, we obtain that two kernels K and K' are equivalent if and only if

$$K(x,y) - K'(x,y) \equiv \sum_{i=1}^{k} \left(P_i(x,D_x) - P_i(y,D_y)^t\right) K_i(x,y)$$

(modulo smoothing terms) which implies that the corresponding Fourier Integral Operators are a smoothing operator plus a sum of commutators $[P_i, A_i]$ (K_i is the kernel of A_i) as

$$\begin{aligned}
&\int_X \left(P_i(x,D_x) - P_i(y,D_y)^t\right) K_i(x,y) f(y) d\mathrm{vol}_X(y) \\
=&P_i A_i f(x) - \langle P_i^t K_i(x,\cdot), f\rangle \\
=&P_i A_i f(x) - \langle K_i(x,\cdot), P_i f\rangle \\
=&(P_i A_i - A_i P_i) f.
\end{aligned}$$

Since $A \in \mathcal{A}_\Gamma$ and the Lagrangian sub-manifold associated with pseudo-differential operators is the co-normal bundle of the diagonal in X^2, we need to assume that Γ' has clean and connected intersection with the co-normal bundle of the diagonal in X^2 for this calculation to be senseful.

- Let $p \in \Lambda \cap \tilde{\Lambda}$. By assumption f does not vanish at p, hence, there is a gauged distribution $u \in C^\infty(\mathbb{C}, I_{\mathrm{compact}}(X,\Lambda))$ such that $\mathrm{res}_0\langle u, f\rangle = 1$ (we can freely choose the amplitude of critical degree of homogeneity) and we will have to show

$$\forall u' \in C^\infty(\mathbb{C}, I_{\mathrm{compact}}(X,\Lambda))_{\mathrm{gauged}} : \; u' \sim \left(\mathrm{res}_0\langle u', f\rangle\right) u.$$

We may assume that u has micro-support in a very small conic neighborhood U of p. Now, we may localize. Suppose the assertion holds and let u' have micro-support in a small conic neighborhood U' of a point $p' \in \Lambda \cap \tilde{\Lambda}$.

(i) If $U \cap U' \neq \varnothing$, then we may assume that u has micro-support in $U \cap U'$ and we have the assertion on U'.

(ii) If $U \cap U' = \varnothing$, then we can find a sequence of points $p_1, \dots, p_k \in \Lambda \cap \tilde{\Lambda}$ and sufficiently small neighborhoods $U_1, \dots, U_k$ such that $p_1 = p$, $p_k = p'$, each $p_i \in U_i \cap U_{i+1}$, and there are $u_i \in C^\infty(\mathbb{C}, I_{\mathrm{compact}}(X, \Lambda))_{\mathrm{gauged}}$ with $\mathrm{res}_0\langle u_i, f\rangle = 1$ and micro-support in U_i.

If the local version of the assertion holds, then we directly obtain $u' \sim (\mathrm{res}_0\langle u', f\rangle)\, u$ in case (i) and $u \sim u_1 \sim \dots \sim u_k$ and $u' \sim (\mathrm{res}_0\langle u', f\rangle)\, u_k$ in case (ii). Using this localization, we may introduce charts to obtain $X = \mathbb{R}^n$ and $f = P\delta_0$. Hence, it suffices to show that $\mathrm{res}_0\langle u, \delta_0\rangle = 0$ implies

$$\exists P_i \in \mathrm{ann}(\delta_0)^t \ \exists v_i \in C^\infty(\mathbb{C}, I_{\mathrm{compact}}(X, \Lambda))_{\mathrm{gauged}}: \ u = w + \sum_{i=1}^{k} P_i v_i$$

where $w \in C^\infty(\mathbb{C}, C_c^\infty(X))$ and $\mathrm{ann}(\delta_0)^t$ is generated by smoothing operators and multiplications with the argument x_i. Furthermore, u is given by an expression of the form

$$u(z)(x) \equiv \int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\langle x, \xi\rangle} \alpha(z)(\xi) d\xi$$

modulo smooth functions. Thus, we will have to find distributions

$$u_s(z)(x) = \int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\langle x, \xi\rangle} \alpha_s(z)(\xi) d\xi$$

such that $\sum_{s=1}^N x_s u_s(x) \equiv u(x)$ modulo smooth functions.

- Let $\chi \in C_c^\infty\left(\mathbb{R}^N\right)$ with $\chi = 1$ in a open neighborhood of zero and $\alpha_s^\varepsilon(\xi) := \alpha_s(\xi)\chi(\varepsilon\xi)$ for $\varepsilon \in (0,1)$. Then, Proposition 1.1.11 in [**39**] yields that $\alpha_s^\varepsilon \to \alpha_s$ in every Hörmander class $S^{m'}$ with $m' > m$ if $\alpha_s \in S^m$, i.e.

$$\int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\langle x, \xi\rangle} \alpha_s(\xi) d\xi = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\langle x, \xi\rangle} \alpha_s^\varepsilon(\xi) d\xi.$$

Let $R_\varepsilon \in \mathbb{R}_{>0}$ be such that $\alpha_s^\varepsilon|_{\mathbb{R}^N \setminus R_\varepsilon B_{\mathbb{R}^N}} = 0$. Then,

$$\begin{aligned} &x_s \int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} e^{i\langle x, \xi\rangle} \alpha_s(\xi) d\xi \\ =&\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_{\mathbb{R}^N}} x_s e^{i\langle x, \xi\rangle} \alpha_s^\varepsilon(\xi) d\xi \end{aligned}$$

$$=\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\smallsetminus B_{\mathbb{R}^N}} -i\partial_{2,s}e^{i\langle x,\xi\rangle}\alpha_s^\varepsilon(\xi)d\xi$$

$$=\lim_{\varepsilon\searrow 0}\int_{R_\varepsilon B_{\mathbb{R}^N}\smallsetminus B_{\mathbb{R}^N}} -i\partial_{2,s}e^{i\langle x,\xi\rangle}\alpha_s^\varepsilon(\xi)d\xi$$

$$=i\lim_{\varepsilon\searrow 0}\int_{R_\varepsilon B_{\mathbb{R}^N}\smallsetminus B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\partial_s\alpha_s^\varepsilon(\xi)d\xi + i\lim_{\varepsilon\searrow 0}\int_{\partial B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\alpha_s^\varepsilon(\xi)\xi_s d\xi$$

$$=i\int_{\mathbb{R}^N\smallsetminus B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\partial_s\alpha_s(\xi)d\xi + i\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\smallsetminus B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\alpha_s(\xi)\varepsilon\left(\partial_s\chi\right)(\varepsilon\xi)d\xi$$

$$+ i\int_{\partial B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\alpha_s(\xi)\xi_s d\xi$$

$$=i\int_{\mathbb{R}^N\smallsetminus B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\partial_s\alpha_s(\xi)d\xi + i\int_{\partial B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\alpha_s(\xi)\xi_s d\xi.$$

As $i\int_{\partial B_{\mathbb{R}^N}} e^{i\langle x,\xi\rangle}\alpha_s(\xi)\xi_s d\xi$ is smooth again, we are looking for α_s such that (modulo smooth functions) $\sum_{i=1}^N i\partial_s\alpha_s \equiv \alpha$. Since α has an asymptotic expansion (and the smoothing terms are irrelevant), we may also assume that α is homogeneous of degree d (in a neighborhood of ξ). For $d \neq -N$, we observe

$$\begin{aligned}\sum_{s=1}^N i\partial_s\left(\frac{-i\xi_s\alpha(\xi)}{N+d}\right) =& \frac{1}{N+d}\sum_{s=1}^N \partial_s\left(\xi_s\alpha(\xi)\right)\\ =& \frac{1}{N+d}\sum_{s=1}^N \xi_s\partial_s\alpha(\xi) + \frac{1}{N+d}\sum_{s=1}^N \alpha(\xi)\\ =& \frac{1}{N+d}\langle\xi,\operatorname{grad}\alpha(\xi)\rangle + \frac{N\alpha(\xi)}{N+d}\\ =& \frac{d\alpha(\xi)}{N+d} + \frac{N\alpha(\xi)}{N+d}\\ =& \alpha(\xi).\end{aligned}$$

For $d = -N$ we actually have a residue to consider. However, the remark above warrants the existence of a vector field F on $\mathbb{R}^N \smallsetminus B_{\mathbb{R}^N}$ such that $\operatorname{div} F = \alpha$ if the residue vanishes and, thus, the assertion follows from

$$\alpha = \operatorname{div} F = \sum_{s=1}^N \partial_s F_s = \sum_{s=1}^N i\partial_s\left(-iF_s\right).$$

$\square$

Example Considering $G := \left(h + \sqrt{|\Delta|}\right)^\alpha$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$ we are interested in integrals

$$\int_0^{2\pi} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-2\pi in\xi} \frac{(h+|\xi|)^\alpha}{2\pi}\ d\xi\ dx = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-2\pi in\xi} (h+|\xi|)^\alpha\ d\xi$$
$$= \sum_{n\in\mathbb{Z}} \left\langle x \mapsto \int_{\mathbb{R}} e^{i(x-2\pi n)\xi} (h+|\xi|)^\alpha\ d\xi, \delta_0 \right\rangle.$$

Hence, we are looking for $v(x) = \int_{\mathbb{R}} e^{i(x-2\pi n)\xi} a(\xi)\ d\xi$ such that

$$u_{n,h}(x) := \int_{\mathbb{R}} e^{i(x-2\pi n)\xi} (h+|\xi|)^\alpha\ d\xi$$

is equivalent to $xv(x)$ modulo smoothing terms. Now,

$$\begin{aligned} xv(x) &= \int_{\mathbb{R}} xe^{ix\xi} e^{-2\pi in\xi} a(\xi)\ d\xi \\ &= \int_{\mathbb{R}} (-i)\partial_\xi \left(e^{ix\xi}\right) e^{-2\pi in\xi} a(\xi)\ d\xi \\ &= \int_{\mathbb{R}} e^{ix\xi} i\partial_\xi \left(e^{-2\pi in\xi} a(\xi)\right)\ d\xi \end{aligned}$$

shows that we are looking for a such that $i\partial\left(e^{-2\pi in\cdot} a\right) \equiv e^{-2\pi in\cdot}(h+|\cdot|)^\alpha$.

Let Γ_{ui} be the upper incomplete Γ-function given by the meromorphic extension of

$$\Gamma_{ui}(s,x) := \int_x^\infty t^{s-1}e^{-t}dt \qquad (\Re(s) > 0,\ x \in \mathbb{R}_{\geq 0}).$$

Recall that Γ_{ui} satisfies $\Gamma_{ui}(s,0) = \Gamma(s)$ where Γ denotes the (usual) Γ-function, $\Gamma(s,\infty) = 0$, and $\partial_2\Gamma_{ui}(s,x) = -x^{s-1}e^{-x}$. For $\xi > 0$ and $n \neq 0$, we obtain

$$i\partial\left(\eta \mapsto \frac{ie^{2\pi inh}\Gamma_{ui}(1+\alpha, 2\pi in(h+\eta))}{(2\pi in)^{1+\alpha}}\right)(\xi) = \frac{e^{2\pi inh}\left(2\pi in(h+\xi)\right)^\alpha e^{-2\pi in(h+\xi)}}{(2\pi in)^\alpha}$$
$$= (h+\xi)^\alpha e^{-2\pi in\xi},$$

i.e.

$$i\partial\left(\eta \mapsto e^{-2\pi in\eta}\frac{ie^{2\pi in(h+\eta)}\Gamma_{ui}(1+\alpha, 2\pi in(h+|\eta|))}{(2\pi in)^{1+\alpha}}\right)(\xi) = (h+|\xi|)^\alpha e^{-2\pi in\xi},$$

that is,

$$a|_{\mathbb{R}_{>0}}(\xi) = \frac{ie^{2\pi in(h+\xi)}\Gamma_{ui}(1+\alpha, 2\pi in(h+|\xi|))}{(2\pi in)^{1+\alpha}}.$$

For $\xi < 0$ we obtain

$$\begin{aligned} i\partial\left(\eta \mapsto \frac{ie^{-2\pi inh}\Gamma_{ui}(1+\alpha,-2\pi in(h-\eta))}{(2\pi in)(-2\pi in)^{\alpha}}\right)(\xi) =& \frac{e^{-2\pi inh}\left(-2\pi in(h-\xi)\right)^{\alpha} e^{2\pi in(h-\xi)}}{(-2\pi in)^{\alpha}} \\ =& (h-\xi)^{\alpha} e^{-2\pi in\xi}, \end{aligned}$$

i.e.

$$i\partial\left(\eta \mapsto e^{-2\pi in\eta}\frac{ie^{-2\pi in(h-\eta)}\Gamma_{ui}(1+\alpha,-2\pi in(h-\eta))}{(2\pi in)(-2\pi in)^{\alpha}}\right)(\xi) = (h+|\xi|)^{\alpha} e^{-2\pi in\xi},$$

that is,

$$a|_{\mathbb{R}_{<0}}(\xi) = \frac{ie^{-2\pi in(h-\xi)}\Gamma_{ui}(1+\alpha,-2\pi in(h+|\xi|))}{(2\pi in)(-2\pi in)^{\alpha}}.$$

In other words,

$$a(\xi) = \frac{ie^{2\pi in\,\mathrm{sgn}(\xi)(h+|\xi|)}\Gamma_{ui}(1+\alpha, 2\pi in\,\mathrm{sgn}(\xi)(h+|\xi|))}{(2\pi in)(2\pi in\,\mathrm{sgn}(\xi))^{\alpha}}$$

for large values of $|\xi|$ where sgn denotes the sign-function.

Let $\chi_n \in C_c^{\infty}(\mathbb{R})$ with $\chi_n = 1$ in a neighborhood of zero. Let $v_{\mathfrak{R}}^{n,h}(x)$ be given by

$$\int_{\mathbb{R}} e^{i(x-2\pi n)\xi}(1-\chi_n(\xi))\mathfrak{R}\left(\frac{ie^{2\pi in\,\mathrm{sgn}(\xi)(h+|\xi|)}\Gamma_{ui}(1+\alpha, 2\pi in\,\mathrm{sgn}(\xi)(h+|\xi|))}{(2\pi in)(2\pi in\,\mathrm{sgn}(\xi))^{\alpha}}\right)\,d\xi$$

and $v_{\mathfrak{I}}^{n,h}(x)$ be given by

$$\int_{\mathbb{R}} e^{i(x-2\pi n)\xi}(1-\chi_n(\xi))\mathfrak{I}\left(\frac{ie^{2\pi in\,\mathrm{sgn}(\xi)(h+|\xi|)}\Gamma_{ui}(1+\alpha, 2\pi in\,\mathrm{sgn}(\xi)(h+|\xi|))}{(2\pi in)(2\pi in\,\mathrm{sgn}(\xi))^{\alpha}}\right)\,d\xi.$$

For $n = 0$, we obtain

$$u_{0,h}(x) = \int_{\mathbb{R}} e^{ix\xi}(h+|\xi|)^{\alpha}d\xi$$

and

$$xv(x)=\int_{\mathbb{R}}(-i)\partial_\xi\left(e^{ix\xi}\right)a(\xi)d\xi=\int_{\mathbb{R}}e^{ix\xi}ia'(\xi)d\xi,$$

i.e. we are looking for a such that $ia'(\xi)=(h+|\xi|)^\alpha$ which is

$$a(\xi)=(h+|\xi|)^{1+\alpha}\frac{-i\,\mathrm{sgn}(\xi)}{1+\alpha}$$

and, hence,

$$v_{\mathfrak{R}}^{0,h}(x)=\int_{\mathbb{R}}e^{ix\xi}(1-\chi_0(\xi))\mathfrak{R}\left((h+|\xi|)^{1+\alpha}\frac{-i\,\mathrm{sgn}(\xi)}{1+\alpha}\right)d\xi,$$

as well as,

$$v_{\mathfrak{J}}^{0,h}(x)=\int_{\mathbb{R}}e^{ix\xi}(1-\chi_0(\xi))\mathfrak{J}\left((h+|\xi|)^{1+\alpha}\frac{-i\,\mathrm{sgn}(\xi)}{1+\alpha}\right)d\xi.$$

Then,

$$u_{n,h}(x)\equiv xv_{\mathfrak{R}}^{n,h}(x)+ixv_{\mathfrak{J}}^{n,h}(x)$$

modulo smoothing terms; in fact,

$$u_{n,h}(x)-x\left(v_{\mathfrak{R}}^{n,h}(x)+iv_{\mathfrak{J}}^{n,h}(x)\right)=\int_{\mathbb{R}}e^{i(x-2\pi n)\xi}a_{n,h}(\xi)d\xi$$

with $a_{n,h}\in C_c^\infty(\mathbb{R})$ and

$$\begin{aligned}\zeta_R(-\alpha)=&\frac{1}{2}\lim_{h\searrow 0}\sum_{n\in\mathbb{Z}}\langle u_{n,h},\delta_0\rangle\\=&\frac{1}{2}\lim_{h\searrow 0}\sum_{n\in\mathbb{Z}}\left\langle x\mapsto\left(u_{n,h}(x)-xv_{\mathfrak{R}}^{n,h}(x)-ixv_{\mathfrak{J}}^{n,h}(x)\right),\delta_0\right\rangle\\=&\frac{1}{2}\lim_{h\searrow 0}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2\pi in\xi}a_{n,h}(\xi)d\xi\\=&\frac{1}{2}\lim_{h\searrow 0}\sum_{n\in\mathbb{Z}}\mathcal{F}\left(a_{n,h}\right)(n).\end{aligned}$$

■

Guillemin also proved the following (more general) version of Theorem 7.9 (cf. Proposition 4.11 in [**35**]).

PROPOSITION 7.10. *Let Γ be connected. Then, the commutator of $\mathcal{A}_\Gamma$ is of co-dimension one in $\mathcal{A}_\Gamma$ modulo smoothing operators.*

Hence, $\operatorname{res}_0 \circ \zeta$ is either zero or the unique continuous trace on $\mathcal{A}_\Gamma$ up to a constant factor provided that Γ is connected. Regarding the trace of smoothing operators, Theorems A.1 and A.2 in [**35**] yield the commutator structure of smoothing operators (the following two definitions, the theorem, and the remark can all be found in the appendix of [**35**]).

DEFINITION 7.11. *Let H be a separable Hilbert space and $e := (e_i)_{i\in\mathbb{N}}$ an orthonormal basis of H. An operator $A \in L(H)$ is called smoothing with respect to e if and only if*

$$\forall n \in \mathbb{N}\ \exists c \in \mathbb{R} :\ |\langle Ae_i, e_j\rangle_H| \leq c(i+j)^{-n}.$$

DEFINITION 7.12. *Let H be a separable Hilbert space, e an orthonormal basis, $\Omega \subseteq_{\text{open}} \mathbb{K}^n$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $A \in L(H)^\Omega$ such that each $A(s)$ is smoothing with respect to e. Then, A is said to be (scalarly) smooth/holomorphic if and only if all $s \mapsto \langle A(s)e_i, e_j\rangle_H$ are $C^\infty(\Omega)$.*

THEOREM 7.13.

(i) *If A is smoothing with respect to e and $\operatorname{tr} A = 0$, then A can be written as a finite sum of commutators $[B_i, C_i]$ where the B_i and C_i are smoothing with respect to e.*

(ii) *If a family $A \in L(H)^\Omega$ of smoothing operators is smooth/holomorphic, then A can be written as a finite sum of commutators $s \mapsto [B_i(s), C_i]$ on every compact $K \subseteq \Omega$ where the $B_i(s)$ and C_i are smoothing, and the B_i are smooth/holomorphic.*

REMARK

(i) Let X be a compact Riemannian manifold, $H = L_2(X)$, and e the family of eigenfunctions of the Laplacian on X. An operator $A \in$

$L(L_2(X))$ is smoothing with respect to e if it is smoothing with respect to the Sobolev norms.

(ii) Let $H = L_2(\mathbb{R}^n)$ and e the family of Hermite functions. An operator $A \in L(H)$ is e-smoothing if it is smoothing with respect to the Schwartz semi-norms.

■

These theorems yield the following table assuming that the residue trace $\mathrm{res}_0 \circ \zeta$ is non-trivial and unique, and $\mathcal{A}_\Gamma = \langle \mathfrak{A} \rangle + \langle [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma] \rangle + \{\text{smoothing operators}\}$ for some $\mathfrak{A} \in \mathcal{A}_\Gamma$ with $\mathrm{res}_0 \, \zeta(\mathfrak{A}) \neq 0$.

<table>
<tr><th colspan="2">$I_0 \neq \varnothing$</th><th colspan="2">$I_0 = \varnothing$</th></tr>
<tr><th>$\mathrm{res}_0 \, \zeta(A) \neq 0$</th><th>$\mathrm{res}_0 \, \zeta(A) = 0$</th><th>$\zeta(A)(0) \neq 0$</th><th>$\zeta(A)(0) = 0$</th></tr>
<tr><td>$A = \alpha\mathfrak{A} + S + \sum_{i=1}^k C_i$
$C_i \in [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma]$
$\alpha = (\mathrm{res}_0 \, \zeta(\mathfrak{A}))^{-1} \, \mathrm{res}_0 \, \zeta(A)$
S smoothing</td><td colspan="2">$A = S + \sum_{i=1}^k C_i$
$C_i \in [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma]$
S smoothing</td><td>$A = \sum_{i=1}^k C_i$
C_i commutators</td></tr>
</table>

Remark Note that the obstruction to the generalized Kontsevich-Vishik trace is given by the derivatives of the a_ι for $\iota \in I_0$. Using the example above Theorem 2.15, we obtain that these are residue traces themselves if the operator is polyhomogeneous. These residues are explicitly calculated for gauged families $A(z) = BQ^z$ in Proposition 4.5.

■

Remark Recall that $\zeta_R(\alpha) = \frac{1}{2}\zeta\left(s \mapsto H^s H^{-\alpha}\right)(0)$ holds. Since $I_0 = \varnothing$ for $\Re(\alpha) \in (0,1)$, we obtain $H^{-\alpha} = S_\alpha + \sum_{i=1}^k [B_i, C_i]$ where S_α is a smoothing operator. Hence, the following are equivalent.

(i) Riemann's Hypothesis

(ii) $\Re(\alpha) \in (0,1) \ \wedge \ H^{-\alpha} \in \langle [\mathcal{A}_\Gamma, \mathcal{A}_\Gamma] \rangle \ \Rightarrow \ \Re(\alpha) = \frac{1}{2}$

(iii) $\mathfrak{R}(\alpha) \in (0,1) \ \wedge \ S_\alpha = 0$ is possible $\Rightarrow \ \mathfrak{R}(\alpha) = \frac{1}{2}$

(iv) $\mathfrak{R}(\alpha) \in (0,1) \ \wedge \ \operatorname{tr} S_\alpha = 0 \ \Rightarrow \ \mathfrak{R}(\alpha) = \frac{1}{2}$

■

CHAPTER 8

Stationary phase approximation

In this chapter, we would like to get to know a little more about the singularity structure of

$$k(x,y)=\int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi,$$

primarily to calculate the integrals

$$\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi).$$

We will prove the following theorem.

THEOREM 8.3 *Let $k(x,y)=\int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi$ be the kernel of a Fourier Integral Operator with poly-*log*-homogeneous amplitude $a=a_0+\sum_{\iota\in I}a_\iota$ and phase function satisfying $\partial_3^2\left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi)\in GL\left(\mathbb{R}^{N-1}\right)$ whenever $\partial_3\vartheta(x,y,\xi)=0$. Let $\tilde{I}:=I\cup\{0\}$ and choose a decomposition $a=a^0+\sum_{s=1}^S a^s$ such that there is no stationary point[1] in the support of $a^0(x,y,\cdot)$ and exactly one stationary point $\hat{\xi}^s(x,y)\in\partial B_{\mathbb{R}^N}$ of $\vartheta(x,y,\cdot)$ in the support of each $a^s(x,y,\cdot)$.*

Let $\hat{\vartheta}^s(x,y)=\vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Theta^s(x,y)=\partial_{\partial B}^2\vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\operatorname{sgn}\Theta^s(x,y)$ the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta^s(x,y)$, $\mathrm{grad}_{\partial B_{\mathbb{R}^N}}=\partial_{\partial B}$ and $\mathrm{div}_{\partial B_{\mathbb{R}^N}}$ are the gradient and divergence operators on the $(N-1)$-sphere $\partial B_{\mathbb{R}^N}$, and

$$\Delta_{\partial B,\Theta^s(x,y)}=\left\langle\Theta^s(x,y)^{-1}\partial_{\partial B},\partial_{\partial B}\right\rangle=-\mathrm{div}_{\partial B_{\mathbb{R}^N}}\,\Theta^s(x,y)^{-1}\,\mathrm{grad}_{\partial B_{\mathbb{R}^N}}.$$

[1] A point $\xi\in\mathbb{R}^N$ is called a stationary point of $\vartheta(x,y,\cdot)$ if and only if $\partial_3\vartheta(x,y,\xi)=0$.

Furthermore, let

$$h^s_{j,\iota}(x,y) := \frac{(2\pi)^{\frac{N-1}{2}}\left|\det\Theta^s(x,y)\right|^{-\frac{1}{2}}e^{\frac{i\pi}{4}\operatorname{sgn}\Theta^s(x,y)}}{j!(2i)^j}\Delta^j_{\partial B,\Theta^s(x,y)}a^s_\iota\left(x,y,\hat{\xi}^s(x,y)\right)$$

and

$$g^s_{j,\iota}(x,y) := \begin{cases} \partial^{l_\iota}\left(z\mapsto\Gamma\left(q+1+z\right)i^{q+1+z}\left(\hat{\vartheta}^s(x,y)+i0\right)^{-q-1-z}\right)(0) & ,\ q\in\mathbb{C}\setminus(-\mathbb{N}_0) \\ \partial^{l_\iota}\left(z\mapsto\frac{-\Gamma(z+1)}{2\pi i\ (-q)!}\int_{c+i\mathbb{R}}\frac{(-\sigma)^{-q}(c_{\ln}+\ln\sigma)}{\left(-i\hat{\vartheta}^s(x,y)+0-\sigma\right)^{z+1}}d\sigma\right)(0) & ,\ q\in-\mathbb{N}_0 \end{cases}$$

with $q := d_\iota + \frac{N+1}{2} - j$*,* $c\in\mathbb{R}_{>0}$*, and some constant* $c_{\ln}\in\mathbb{C}$*.*

Then,

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a^0(x,y,\xi)d\xi + \sum_{\iota\in\tilde{I}}\sum_{s=1}^{S}\sum_{j\in\mathbb{N}_0}h^s_{j,\iota}(x,y)g^s_{j,\iota}(x,y)$$

holds in a neighborhood of the diagonal in X^2*.*

This will yield the following theorems.

Theorem 8.5 *Let* A *be a Fourier Integral Operator with kernel*

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi$$

whose phase function ϑ *satisfies* $\forall x\in X\ \forall\xi\in\mathbb{R}^N:\ \vartheta(x,x,\xi)=0$*, and whose amplitude has an asymptotic expansion* $a\sim\sum_{\iota\in\mathbb{N}}a_\iota$ *where each* a_ι *is* log*-homogeneous with degree of homogeneity* d_ι *and logarithmic order* l_ι*, and* $\Re(d_\iota)\to-\infty$*. Let* $N_0\in\mathbb{N}$ *such that* $\forall\iota\in\mathbb{N}_{>N_0}:\ \Re(d_\iota)<-N$ *and let*

$$k^{\text{sing}}(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}\sum_{\iota=1}^{N_0}a_\iota(x,y,\xi)d\xi$$

denote the singular part of the kernel.

Then, the regularized kernel $k - k^{\text{sing}}$ is continuous along the diagonal and independent of the particular choice of N_0 (along the diagonal). Furthermore, the generalized Kontsevich-Vishik density is given by

$$\left(k - k^{\text{sing}}\right)(x,x)d\mathrm{vol}_X(x) = \int_{\mathbb{R}^N} a(x,x,\xi) - \sum_{\iota=1}^{N_0} a_\iota(x,x,\xi)d\xi d\mathrm{vol}_X(x).$$

THEOREM 8.7 *Let A be a Fourier Integral Operator with phase function ϑ satisfying $\partial_3^2\left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right)$ whenever $\partial_3\vartheta(x,y,\xi) = 0$ and $\hat{\xi}^s$ $\left(s \in \mathbb{N}_{\leq S}\right)$ the stationary points. Furthermore, let*

$$\forall x \in X\ \ \forall s \in \mathbb{N}_{\leq S}:\ \vartheta\left(x,x,\hat{\xi}^s(x,x)\right) \neq 0.$$

Then,

$$\left(\ X \ni x \mapsto k(x,x) \in \mathbb{C}\ \right) \in C(X)$$

and

$$\operatorname{tr} A = \int_X k(x,x)d\mathrm{vol}_X(x)$$

is well-defined, i.e. A is a Hilbert-Schmidt operator. Furthermore, ζ-functions of families of such operators have no poles.

For the remainder of the chapter, let a be log-homogeneous. Then, we obtain

$$\begin{aligned}
k(x,y) &:= \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}a(x,y,\xi)d\xi \\
&= \int_{\mathbb{R}_{>0}}\int_{\partial B_{\mathbb{R}^N}} r^{N-1}e^{ir\vartheta(x,y,\eta)}a(x,y,r\eta)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)dr \\
&= \int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l \underbrace{\int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)}a(x,y,\eta)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)}_{=:I(x,y,r)}dr.
\end{aligned}$$

Let (x,y) be off the critical manifold, i.e. $\forall \eta \in \partial B_{\mathbb{R}^N}:\ \partial_3\vartheta(x,y,\eta) \neq 0$. Then, we observe

$$\partial_3 e^{ir\vartheta(x,y,\eta)} = ire^{ir\vartheta(x,y,\eta)}\partial_3\vartheta(x,y,\eta),$$

i.e.

$$e^{ir\vartheta(x,y,\eta)} = \frac{\left\langle \partial_3 e^{ir\vartheta(x,y,\eta)}, \partial_3\vartheta(x,y,\eta)\right\rangle_{\mathbb{R}^N}}{ir\,\|\partial_3\vartheta(x,y,\eta)\|^2_{\ell_2(N)}},$$

and

$$\begin{aligned}
|I(x,y,r)| &= \left| \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} a(x,y,\eta) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \right| \\
&= \left| \int_{\partial B_{\mathbb{R}^N}} \frac{\left\langle \partial_3 e^{ir\vartheta(x,y,\eta)}, \partial_3\vartheta(x,y,\eta)\right\rangle_{\mathbb{R}^N}}{ir\,\|\partial_3\vartheta(x,y,\eta)\|^2_{\ell_2(N)}} a(x,y,\eta) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \right| \\
&= \left| \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} \partial_3^* \frac{a(x,y,\eta)\partial_3\vartheta(x,y,\eta)}{ir\,\|\partial_3\vartheta(x,y,\eta)\|^2_{\ell_2(N)}} d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \right| \\
&= \frac{1}{r} \left| \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} \partial_3^* \frac{a(x,y,\eta)\partial_3\vartheta(x,y,\eta)}{\|\partial_3\vartheta(x,y,\eta)\|^2_{\ell_2(N)}} d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \right|.
\end{aligned}$$

Using

$$\mathcal{D}a(x,y,\eta) := \partial_3^* \frac{a(x,y,\eta)\partial_3\vartheta(x,y,\eta)}{\|\partial_3\vartheta(x,y,\eta)\|^2_{\ell_2(N)}},$$

we conclude

$$\begin{aligned}
\forall n \in \mathbb{N}: \ |I(x,y,r)| &= \frac{1}{r} \left| \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} \mathcal{D}a(x,y,\eta) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \right| \\
&= \frac{1}{r^n} \left| \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\eta)} \mathcal{D}^n a(x,y,\eta) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \right| \\
&\le \frac{1}{r^n} \|\mathcal{D}^n a\|_{L_\infty(X\times X\times\partial B_{\mathbb{R}^N})},
\end{aligned}$$

i.e.

$$\forall n \in \mathbb{N}\ \exists c \in \mathbb{R}_{>0}: \ |I(x,y,r)| \le c r^{-n}$$

which proves that k is C^∞ away from the critical manifold.

On the critical manifold, we will assume that

$$\partial_3^2 \left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right)$$

if $\partial_3\vartheta(x,y,\xi) = 0$.

EXAMPLE For pseudo-differential operators

$$\vartheta(x,y,\xi) = \langle x-y,\xi\rangle_{\mathbb{R}^{N-1}}.$$

Let $\operatorname{spt} a(x,y,\cdot) \subseteq \partial B_{\mathbb{R}^N} \setminus \{B_0\}$ uniformly (x,y) in some sufficiently small open set and $\sigma :\ \mathbb{R}^{N-1} \to \partial B_{\mathbb{R}^N} \setminus \{B_0\}$ the stereographic projection (or any other nice diffeomorphism). Let

$$\vartheta_\sigma(x,y,\xi) \coloneqq \langle x-y,\sigma(\xi)\rangle_{\mathbb{R}^N}.$$

Then,

$$\begin{aligned} 0 = \partial_3\vartheta_\sigma(x,y,\xi) = (x-y)^T\sigma'(\xi) \quad &\Longleftrightarrow \quad x-y \text{ is normal to } \partial B_{\mathbb{R}^N} \text{ at } \sigma(\xi) \\ &\Longleftrightarrow \quad x-y \in \operatorname{lin}\{\sigma(\xi)\} \\ &\overset{x\neq y}{\Longleftrightarrow} \quad \sigma(\xi) \in \left\{\frac{x-y}{\|x-y\|_{\ell_2(N)}}, -\frac{x-y}{\|x-y\|_{\ell_2(N)}}\right\}, \end{aligned}$$

as well as,

$$\partial_3^2\vartheta_\sigma(x,y,\xi) = (x-y)^T\sigma''(\xi)$$

which is a multiple of the second fundamental form II if $x-y$ is normal to $\partial B_{\mathbb{R}^N}$ in $\sigma(\xi)$. Using the first fundamental form I and the fact that the Gaussian curvature κ of $\partial B_{\mathbb{R}^N}$ is 1, we obtain

$$1 = \kappa = \frac{\det \mathrm{II}}{\det \mathrm{I}},$$

i.e.

$$\partial_3\vartheta_\sigma(x,y,\xi) = 0 \ \Rightarrow\ \partial_3^2\vartheta_\sigma(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right).$$

In other words, pseudo-differential operators can be treated with the stationary phase approximation considered in this chapter.

■

Lemma 8.1 (Morse' Lemma). *Let* $(x_0, y_0, \xi_0) \in X \times X \times \partial B_{\mathbb{R}^N}$ *be stationary (in particular,* $\partial_{\partial B}\vartheta(x_0, y_0, \xi_0) = 0$*) and* $\partial^2_{\partial B}\vartheta(x_0, y_0, \xi_0) \in GL\left(\mathbb{R}^{N-1}\right)$ *where* $\partial_{\partial B}$ *denotes the spherical derivative, i.e. the derivative in* $\partial B_{\mathbb{R}^N}$.

Then, there are neighborhoods $U \subseteq_{\text{open}} X \times X$ *of* (x_0, y_0) *and* $V \subseteq_{\text{open}} \partial B_{\mathbb{R}^N}$ *of* ξ_0 *and a function* $\hat{\xi} \in C^\infty(U, V)$ *such that*

$$\forall (x, y, \xi) \in U \times V : \ \partial_{\partial B}\vartheta(x, y, \xi) = 0 \ \Leftrightarrow \ \xi = \hat{\xi}(x, y).$$

Furthermore, there is a function $\eta \in C^\infty\left(U \times V, \mathbb{R}^N\right)$ *such that*

$$\forall (x, y, \xi) \in U \times V : \ \eta(x, y, \xi) - \left(\xi - \hat{\xi}(x, y)\right) \in O\left(\left\|\xi - \hat{\xi}(x, y)\right\|^2_{\ell_2(N)}\right)$$

and

$$\partial_3\eta\left(x, y, \hat{\xi}(x, y)\right) = 1.$$

Proof. The existence of U, V, and $\hat{\xi}$ is a direct consequence of the (analytic) implicit function theorem. From now on, we may suppress the first two arguments (that is, "x" and "y") for reasons of brevity. Then, using Taylor's theorem with $A := \left\{\alpha \in \mathbb{N}_0^N;\ \|\alpha\|_{\ell_1(N)} = 2\right\}$, we obtain for all $\xi \in V$

$$\begin{aligned}\vartheta(\xi) =& \vartheta\left(\hat{\xi}\right) + \underbrace{\partial_3\vartheta\left(\hat{\xi}\right)}_{=0}\left(\xi - \hat{\xi}\right) + \sum_{\alpha \in A} \frac{2}{\alpha!}\int_0^1 (1-t)\partial_3^\alpha\vartheta\left(\hat{\xi} + t\left(\xi - \hat{\xi}\right)\right) dt \left(\xi - \hat{\xi}\right)^\alpha \\ =& \vartheta\left(\hat{\xi}\right) + \frac{1}{2}\left\langle B(\xi)\left(\xi - \hat{\xi}\right), \left(\xi - \hat{\xi}\right)\right\rangle_{\mathbb{R}^N}\end{aligned}$$

with some appropriate function $B \in C^\infty\left(U \times V, L\left(\mathbb{R}^N\right)\right)$. According to Taylor's theorem, we have

$$\forall (x, y) \in U : \ B\left(x, y, \hat{\xi}(x, y)\right) = \partial_3^2\vartheta\left(x, y, \hat{\xi}(x, y)\right).$$

We are, therefore, looking for a function $R \in C^\infty\left(U \times V, L\left(\mathbb{R}^N\right)\right)$ with

$$\forall (x,y) \in U: \; R\left(x,y,\hat{\xi}(x,y)\right) = 1$$

and

$$\forall (x,y,\xi) \in U \times V: \; B\left(x,y,\xi\right) = R(x,y,\xi)^* \partial_3^2 \vartheta\left(x,y,\hat{\xi}(x,y)\right) R(x,y,\xi).$$

Since the radial derivative $\partial_r \vartheta(\xi)$ is constant, we obtain

$$\begin{aligned}\forall (x,y,\xi) \in U \times V: \; \partial_3^2 \vartheta(x,y,\xi) &= \begin{pmatrix} \partial_r^2 \vartheta(x,y,\xi) & \partial_r \partial_{\partial B} \vartheta(x,y,\xi) \\ \partial_{\partial B} \partial_r \vartheta(x,y,\xi) & \partial_{\partial B}^2 \vartheta(x,y,\xi) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \partial_{\partial B}^2 \vartheta(x,y,\xi) \end{pmatrix},\end{aligned}$$

where $\partial_{\partial B}$ is the spherical derivative $\partial_{\partial B} \vartheta = \partial_3 \vartheta|_{\partial B_{\mathbb{R}^N}}$, which shows that we may assume, without loss of generality,

$$\forall (x,y,\xi) \in U \times V: \; B(x,y,\xi) = \begin{pmatrix} 0 & 0 \\ 0 & C(x,y,\xi) \end{pmatrix}$$

and

$$\forall (x,y,\xi) \in U \times V: \; R(x,y,\xi) = \begin{pmatrix} 1 & 0 \\ 0 & S(x,y,\xi) \end{pmatrix}.$$

This reduces the problem to showing that a solution of

$$\begin{aligned} S(x,y,\xi)^* \partial_{\partial B}^2 \vartheta\left(x,y,\hat{\xi}(x,y)\right) S(x,y,\xi) &= C(x,y,\xi) \\ S\left(x,y,\hat{\xi}(x,y)\right) &= 1 \end{aligned}$$

exists in $U \times V$ (reducing U and V if necessary).

Note that (using the symmetrization operator $\mathrm{sym}: L(H) \to L(H);\ h \mapsto \frac{h+h^*}{2}$)

$$T:\ L\left(\mathbb{R}^{N-1}\right) \to \mathrm{sym}\left[L\left(\mathbb{R}^{N-1}\right)\right];\ s \mapsto s^* \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right)s$$

has surjective Fréchet derivative

$$T'(1):\ L\left(\mathbb{R}^{N-1}\right) \to \mathrm{sym}\left[L\left(\mathbb{R}^{N-1}\right)\right];$$
$$s \mapsto s^* \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right) + \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right)s$$

since $s := \frac{1}{2}\partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right)^{-1} t$ solves

$$s^* \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right) + \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right)s = t$$

for $t \in \mathrm{sym}\left[L\left(\mathbb{R}^{N-1}\right)\right]$. Let $L_\vartheta := L\left(\mathbb{R}^{N-1}\right)/_{[\{0\}]T'(1)}$. Then,

$$\left(T|_{L_\vartheta}\right)'(1):\ L_\vartheta \to \mathrm{sym}\left[L\left(\mathbb{R}^{N-1}\right)\right]$$

is an isomorphism and the implicit function theorem yields a C^∞-solution of

$$s^* \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right)s = C \in L_\vartheta$$
$$s\left(x,y,\hat{\xi}(x,y)\right) = 1$$

in a neighborhood of $\left(x,y,\hat{\xi}(x,y)\right)$. Let S be a C^∞-representative in $L\left(\mathbb{R}^{N-1}\right)$ of the solution. Thence,

$$\forall \xi \in V:\ \vartheta(\xi) = \vartheta\left(\hat{\xi}\right) + \frac{1}{2}\left\langle \partial_3^2\vartheta\left(\hat{\xi}\right)R(\xi)\left(\xi-\hat{\xi}\right), R(\xi)\left(\xi-\hat{\xi}\right)\right\rangle_{\mathbb{R}^N}.$$

Letting

$$\eta(x,y,\xi) := R(x,y,\xi)\left(\xi - \hat{\xi}(x,y)\right)$$

and observing

$$R(x,y,\xi) - 1 \in O\left(\left\|\xi - \hat{\xi}(x,y)\right\|_{\ell_2(N)}\right)$$

shows

$$\eta(x,y,\xi) - \left(\xi - \hat{\xi}(x,y)\right) \in O\left(\left\|\xi - \hat{\xi}(x,y)\right\|_{\ell_2(N)}^2\right).$$

Finally,

$$\partial_3\eta\left(x,y,\xi\right) = \partial_3 R\left(x,y,\xi\right)\left(\xi - \hat{\xi}(x,y)\right) + R\left(x,y,\xi\right)$$

implies

$$\partial_3\eta\left(x,y,\hat{\xi}(x,y)\right) = R\left(x,y,\hat{\xi}(x,y)\right) = 1$$

which completes the proof.

□

COROLLARY 8.2. *Let ϑ be as in Morse' Lemma (Lemma 8.1). Then, stationary points of $\vartheta(x,y,\cdot)$ are isolated in $\partial B_{\mathbb{R}^N}$. In particular, there are only finitely many.*

PROOF. For given stationary (x,y,ξ) we can find a neighborhood $V \subseteq_{\text{open}} \partial B_{\mathbb{R}^N}$ such that $\xi = \hat{\xi}(x,y)$; thus, stationary points are locally unique. By compactness of $\partial B_{\mathbb{R}^N}$ they are isolated and at most finitely many.

□

Hence, we may assume that

$$k(x,y) = \sum_{s=0}^{S} \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a^s(x,y,\xi) d\xi$$

where a^0 has no stationary points in its support and each of the a^s has exactly one branch $(x,y,\hat{\xi}^s(x,y))$ in its support. As we have already treated the a^0 case, we will assume, without loss of generality, that a is of the form of one of the a^s.

Let $\eta_{\partial B}$ be defined as the spherical part of η and

$$\Theta(x,y) := \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right).$$

Then,

$$\left\langle \partial_3^2\vartheta\left(x,y,\hat{\xi}(x,y)\right)\eta(x,y,\xi),\eta(x,y,\xi)\right\rangle_{\mathbb{R}^N} = \langle \Theta(x,y)\eta_{\partial B}(x,y,\xi),\eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}$$

and, defining $\hat{\vartheta} := \vartheta\left(x,y,\hat{\xi}(x,y)\right)$,

$$\begin{aligned} I(x,y,r) =& \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)}a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\ =& e^{ir\hat{\vartheta}}\int_{\partial B_{\mathbb{R}^N}} e^{i\frac{r}{2}\langle\Theta(x,y)\eta_{\partial B}(x,y,\xi),\eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}}a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi). \end{aligned}$$

Let $\sigma:\ \mathbb{R}^{N-1}\to\partial B_{\mathbb{R}^N}$ be a stereographic projection with pole $-\hat{\xi}(x,y)$ (which is assumed to be outside of $\operatorname{spt} a(x,y,\cdot)$),

$$\eta_\sigma(x,y,\xi) := \eta_{\partial B}(x,y,\sigma(\xi)),$$

and

$$a_\sigma(x,y,\xi) := a(x,y,\sigma(\xi))\sqrt{\det\left(\sigma'(\xi)^*\sigma'(\xi)\right)}.$$

Then,

$$\begin{aligned} I(x,y,r) =& e^{ir\hat{\vartheta}}\int_{\partial B_{\mathbb{R}^N}} e^{i\frac{r}{2}\langle\Theta(x,y)\eta_{\partial B}(x,y,\xi),\eta_{\partial B}(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}}a(x,y,\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\ =& e^{ir\hat{\vartheta}}\int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle\Theta(x,y)\eta_\sigma(x,y,\xi),\eta_\sigma(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}}a_\sigma(x,y,\xi)d\xi \end{aligned}$$

and

$$\partial_3\eta_\sigma(x,y,\xi) = \partial_3\eta_{\partial B}(x,y,\sigma(\xi))\sigma'(\xi)$$

combined with the fact that $\partial_3\eta\left(x,y,\hat{\xi}(x,y)\right) = 1$ yields that $\eta_\sigma(x,y,\cdot)$ is invertible in a neighborhood of $\sigma^{-1}\left(\hat{\xi}(x,y)\right) = 0$ (we will also use $\eta_\sigma(x,y)(\cdot)$ for $\eta_\sigma(x,y,\cdot)$).

Without loss of generality, let $a_\sigma(x,y,\cdot)$ have support in such a neighborhood and

$$\tilde{a}(x,y,\xi) \coloneqq a_\sigma(x,y,\eta_\sigma(x,y)^{-1}(\xi))\sqrt{\det\left(\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)^*\left(\eta_\sigma(x,y)^{-1}\right)'(\xi)\right)}.$$

This yields

$$\begin{aligned} I(x,y,r) =& e^{ir\hat{\vartheta}} \int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle \Theta(x,y)\eta_\sigma(x,y,\xi),\eta_\sigma(x,y,\xi)\rangle_{\mathbb{R}^{N-1}}} a_\sigma(x,y,\xi)d\xi \\ =& e^{ir\hat{\vartheta}} \int_{\mathbb{R}^{N-1}} e^{i\frac{r}{2}\langle \Theta(x,y)\xi,\xi\rangle_{\mathbb{R}^{N-1}}} \tilde{a}(x,y,\xi)d\xi. \end{aligned}$$

Using

$$\begin{aligned} &\mathcal{F}\left(z \mapsto e^{i\frac{1}{2}\langle r\Theta(x,y)\xi,\xi\rangle_{\mathbb{R}^{N-1}}}\right)(\xi) \\ =& \left|\det(r\Theta(x,y))\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}(r\Theta(x,y))} e^{-i\frac{1}{2}\left\langle (r\Theta(x,y))^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}} \\ =& \frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}(\Theta(x,y))} e^{-i\frac{1}{2}\left\langle (r\Theta(x,y))^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}} \end{aligned}$$

where $\operatorname{sgn}(\Theta(x,y))$ is the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta(x,y)$ (cf. Lemma 1.2.3 in **[20]** and noting that Duistermaat uses "$\mathcal{F} = \int_{\mathbb{R}^N}$" whereas we are using "$\mathcal{F} = (2\pi)^{-\frac{N}{2}}\int_{\mathbb{R}^N}$"), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{N-1}} e^{i\frac{1}{2}\langle r\Theta\xi,\xi\rangle_{\mathbb{R}^{N-1}}} \tilde{a}(\xi)d\xi \\ =& \frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta} \int_{\mathbb{R}^{N-1}} \mathcal{F}_3^{-1}\left(e^{-i\frac{1}{2}\left\langle (r\Theta)^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}}\right)\tilde{a}(\xi)d\xi \\ =& \frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta} \int_{\mathbb{R}^{N-1}} \mathcal{F}_3^{*}\left(e^{-i\frac{1}{2}\left\langle (r\Theta)^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}}\right)\tilde{a}(\xi)d\xi \\ =& \frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta} \int_{\mathbb{R}^{N-1}} e^{-i\frac{1}{2}\left\langle (r\Theta)^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}} \mathcal{F}_3\tilde{a}(\xi)d\xi \\ =& \frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta} \sum_{j\in\mathbb{N}_0}\frac{r^{-j}}{j!} \int_{\mathbb{R}^{N-1}} \left(\frac{-i}{2}\left\langle \Theta^{-1}\xi,\xi\right\rangle_{\mathbb{R}^{N-1}}\right)^j \mathcal{F}_3\tilde{a}(\xi)d\xi \\ =& \frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta} \sum_{j\in\mathbb{N}_0}\frac{r^{-j}}{j!} \int_{\mathbb{R}^{N-1}} \left(\frac{-i}{2}\left\langle \Theta^{-1}i\xi,i\xi\right\rangle_{\mathbb{R}^{N-1}}\right)^j \mathcal{F}_3\tilde{a}(\xi)d\xi \\ =& \frac{1}{r^{\frac{N-1}{2}}}\left|\det\Theta\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta} \sum_{j\in\mathbb{N}_0}\frac{r^{-j}}{j!} \int_{\mathbb{R}^{N-1}} \mathcal{F}_3\left(\left(\frac{-i}{2}\left\langle \Theta^{-1}\partial_3,\partial_3\right\rangle_{\mathbb{R}^{N-1}}\right)^j\tilde{a}\right)(\xi)d\xi \end{aligned}$$

and investing

$$\int_{\mathbb{R}^n} \mathcal{F}f(\xi)d\xi = \int_{\mathbb{R}^n} e^{i\langle 0,\xi\rangle_{\mathbb{R}^n}} \mathcal{F}f(\xi)d\xi = (2\pi)^{\frac{n}{2}} \mathcal{F}^{-1}\left(\mathcal{F}f\right)(0) = (2\pi)^{\frac{n}{2}} f(0)$$

yields

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} e^{i\frac{1}{2}\langle r\Theta(x,y)\xi,\xi\rangle_{\mathbb{R}^{N-1}}} \tilde{a}(x,y,\xi)d\xi \\ &= \left(\frac{2\pi}{r}\right)^{\frac{N-1}{2}} \left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta(x,y)} \sum_{j\in\mathbb{N}_0} \frac{(-i)^j r^{-j}}{j!2^j} \left\langle \Theta(x,y)^{-1}\partial_3,\partial_3\right\rangle^j_{\mathbb{R}^{N-1}} \tilde{a}(x,y,0).\end{aligned}$$

Remark The evaluation of $\left\langle \Theta(x,y)^{-1}\partial_3,\partial_3\right\rangle^j_{\mathbb{R}^{N-1}} \tilde{a}(x,y,\cdot)$ at zero yields an evaluation at $\hat{\xi}(x,y)$ undoing all the changes of variables.

■

Hence, defining

$$h_j(x,y) := \frac{(2\pi)^{\frac{N-1}{2}} \left|\det\Theta(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta(x,y)}}{j!(2i)^j} \left\langle \Theta(x,y)^{-1}\partial_3,\partial_3\right\rangle^j_{\mathbb{R}^{N-1}} \tilde{a}(x,y,0)$$

we obtain

$$\begin{aligned}&\int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ dr \\ &= \int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l e^{ir\hat{\vartheta}(x,y)} \int_{\mathbb{R}^{N-1}} e^{i\frac{1}{2}\langle r\Theta(x,y)\xi,\xi\rangle_{\mathbb{R}^{N-1}}} \tilde{a}(x,y,\xi)d\xi\ dr \\ &= \int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l e^{ir\hat{\vartheta}(x,y)} r^{-\frac{N-1}{2}} \sum_{j\in\mathbb{N}_0} r^{-j} h_j(x,y)\ dr \\ &= \sum_{j\in\mathbb{N}_0} h_j(x,y) \int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j}(\ln r)^l e^{ir\hat{\vartheta}(x,y)}\ dr,\end{aligned}$$

i.e.

$$\begin{aligned}k(x,y) &= \sum_{s=0}^{S} \int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)} a^s(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ dr \\ &= \int_{\mathbb{R}_{>0}} r^{N+d-1}(\ln r)^l \int_{\partial B_{\mathbb{R}^N}} e^{ir\vartheta(x,y,\xi)} a^0(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)\ dr \\ &\quad + \sum_{s=1}^{S} \sum_{j\in\mathbb{N}_0} h^s_j(x,y) \int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j}(\ln r)^l e^{ir\hat{\vartheta}^s(x,y)}\ dr.\end{aligned}$$

For $l = 0$ we may invest the well-known fact

$$\forall q \in \mathbb{C}_{\Re(\cdot)>-1}\ \forall s \in \mathbb{C}_{\Re(\cdot)>0} : \int_{\mathbb{R}_{>0}} t^q e^{-st} dt = \Gamma(q+1) s^{-q-1}$$

about the Laplace transform to obtain

$$\begin{aligned}\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j} e^{ir\hat{\vartheta}^s(x,y)} dr =& \Gamma\left(d + \frac{N+1}{2} - j\right)\left(-i\hat{\vartheta}^s(x,y) + 0\right)^{-d-\frac{N+1}{2}+j}\\ =& \Gamma\left(d + \frac{N+1}{2} - j\right) i^{d+\frac{N+1}{2}-j}\left(\hat{\vartheta}^s(x,y) + i0\right)^{-d-\frac{N+1}{2}+j}\end{aligned}$$

if $\Re\left(d + \frac{N+1}{2} - j\right) > 0$ where $f(t+i0) \coloneqq \lim_{\varepsilon\searrow 0} f(t+i\varepsilon)$. By meromorphic extension, we obtain

$$\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j} e^{ir\hat{\vartheta}^s(x,y)} dr = \Gamma\left(d + \frac{N+1}{2} - j\right) i^{d+\frac{N+1}{2}-j}\left(\hat{\vartheta}^s(x,y) + i0\right)^{-d-\frac{N+1}{2}+j}$$

whenever $d + \frac{N+1}{2} - j \in \mathbb{C} \setminus (-\mathbb{N}_0)$ and, for $l \in \mathbb{N}_0$,

$$\begin{aligned}\int_{\mathbb{R}_{>0}} r^q (\ln r)^l e^{ir\hat{\vartheta}^s(x,y)} dr =& \partial^l\left(z \mapsto \int_{\mathbb{R}_{>0}} r^{q+z} e^{ir\hat{\vartheta}^s(x,y)} dr\right)(0)\\ =& \partial^l\left(z \mapsto \Gamma(q+1+z)\, i^{q+1+z}\left(\hat{\vartheta}^s(x,y) + i0\right)^{-q-1-z}\right)(0).\end{aligned}$$

If $d + \frac{N+1}{2} - j \in -\mathbb{N}_0$, i.e. $d + \frac{N-1}{2} - j \in -\mathbb{N}$, then we can use the following property

$$\int_{\mathbb{R}_{>0}} \int_0^t f(\tau) d\tau e^{-st} dt = \frac{1}{s}\int_{\mathbb{R}_{>0}} f(t) e^{-st} dt$$

to obtain

$$\forall q, s \in \mathbb{C}_{\Re(\cdot)>0} : \int_{\mathbb{R}_{>0}} t^q e^{-st} dt = \int_{\mathbb{R}_{>0}} \int_0^t q\tau^{q-1} d\tau e^{-st} dt = \frac{q}{s}\int_{\mathbb{R}_{>0}} t^{q-1} e^{-st} dt$$

and, hence,

$$\int_{\mathbb{R}_{>0}} t^q e^{-st} dt = \frac{s}{q+1}\int_{\mathbb{R}_{>0}} t^{q+1} e^{-st} dt = \frac{s^n}{\prod_{p=1}^n (q+p)}\int_{\mathbb{R}_{>0}} t^{q+n} e^{-st} dt$$

by meromorphic extension. Thus, for $q \in -\mathbb{N}$ and $n = -q-1$, we have

$$\int_{\mathbb{R}_{>0}} t^q e^{-st} dt = \frac{(-s)^{-q-1}}{(-q-1)!}\int_{\mathbb{R}_{>0}} t^{-1} e^{-st} dt$$

reducing the problem to finding $\int_{\mathbb{R}_{>0}} t^{-1} e^{-st} dt$. Consider the Borel measure

$$\mu_{q,s}: \mathcal{B}(\mathbb{R}_{>0}) \to \mathbb{C}; \ A \mapsto \int_A t^{q-1} e^{-st} dt$$

on $\mathbb{R}_{>0}$ for $q, s \in \mathbb{C}_{\Re(\cdot)>0}$. Then,

$$\partial \left(\sigma \mapsto \int_{\mathbb{R}_{>0}} f(t) e^{-\sigma t} dt \right)(s) = -\int_{\mathbb{R}_{>0}} t f(t) e^{-st} dt$$

implies

$$\partial(\sigma \mapsto \mu_{q,\sigma})(s) = -\mu_{q+1,s}$$

and, hence,

$$\partial(\sigma \mapsto \mu_{q,\sigma})(s)(\mathbb{R}_{>0}) = -\mu_{q+1,s}(\mathbb{R}_{>0}) = -\frac{\Gamma(q+2)}{s^{q+2}} \to -\frac{1}{s} \quad (q \to -1).$$

In other words, $\int_{\mathbb{R}_{>0}} t^{-1} e^{-st} dt$ is logarithmic (up to a constant) and $\int_{\mathbb{R}_{>0}} t^q e^{-st} dt$ for $q \in -\mathbb{N}$ is log-homogeneous; namely,

$$\int_{\mathbb{R}_{>0}} r^q e^{ir\hat{\vartheta}^s(x,y)} \, dr = -\frac{\left(i\hat{\vartheta}^s(x,y) - 0\right)^{-q-1}}{(-q-1)!} \left(c_{\ln} + \ln\left(-i\hat{\vartheta}^s(x,y) + 0\right)\right)$$

with some constant $c_{\ln}$. Finally, we can add the $\ln r$ terms for $q \in -\mathbb{N}$ by investing the the multiplication property of the Laplace transform

$$\mathcal{L}(fg)(s) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \mathcal{L}(f)(\sigma) \mathcal{L}(g)(s-\sigma) d\sigma$$

where $c \in \mathbb{R}$ such that $c + i\mathbb{R}$ is a subset of the region of convergence for $\mathcal{L}(f) = \left(s \mapsto \int_{\mathbb{R}_{>0}} f(t) e^{-st} dt\right)$. Thence, for $c \in \mathbb{R}_{>0}$, $q \in -\mathbb{N}$, and $l \in \mathbb{N}_0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_{>0}} r^q (\ln r)^l e^{-sr} dr \Big|_{s=-i\hat{\vartheta}^s(x,y)+0} \\
&= \left(z \mapsto \int_{\mathbb{R}_{>0}} r^q r^z (\ln r)^l e^{-sr} dr \right)(0) \Big|_{s=-i\hat{\vartheta}^s(x,y)+0} \\
&= \partial^l \left(z \mapsto \int_{\mathbb{R}_{>0}} r^q r^z e^{-sr} dr \right)(0) \Big|_{s=-i\hat{\vartheta}^s(x,y)+0} \\
&= \partial^l \left(z \mapsto \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \int_{\mathbb{R}_{>0}} r^q e^{-\sigma r} dr \int_{\mathbb{R}_{>0}} r^z e^{-(s-\sigma)r} dr d\sigma \right)(0) \Big|_{s=-i\hat{\vartheta}^s(x,y)+0}
\end{aligned}$$

$$= \partial^l \left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i(-q-1)!} \int_{c+i\mathbb{R}} (-\sigma)^{-q-1} \left(c_{\ln} + \ln\sigma\right) (s-\sigma)^{-z-1} \, d\sigma \right)(0) \bigg|_{s=-i\hat{\vartheta}^s(x,y)+0}.$$

Thus, we have proven the following Theorem.

THEOREM 8.3. *Let $k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi$ be the kernel of a Fourier Integral Operator with poly-*log*-homogeneous amplitude $a = a_0 + \sum_{\iota\in I} a_\iota$ whose phase function satisfies $\partial_3^2 \left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right)$ whenever $\partial_3\vartheta(x,y,\xi) = 0$. Let $\tilde{I} := I \cup \{0\}$ and choose a decomposition $a = a^0 + \sum_{s=1}^S a^s$ such that there is no stationary point in the support of $a^0(x,y,\cdot)$ and exactly one stationary point $\hat{\xi}^s(x,y) \in \partial B_{\mathbb{R}^N}$ of $\vartheta(x,y,\cdot)$ in the support of each $a^s(x,y,\cdot)$.*

Let $\hat{\vartheta}^s(x,y) = \vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Theta^s(x,y) = \partial_{\partial B}^2 \vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\operatorname{sgn}\Theta^s(x,y)$ the number of positive eigenvalues minus the number of negative eigenvalues of $\Theta^s(x,y)$, and $\Delta_{\partial B,\Theta^s(x,y)} = \left\langle \Theta^s(x,y)^{-1}\partial_{\partial B}, \partial_{\partial B}\right\rangle = -\operatorname{div}_{\partial B_{\mathbb{R}^N}} \Theta^s(x,y)^{-1} \operatorname{grad}_{\partial B_{\mathbb{R}^N}}$. Furthermore, let

$$h^s_{j,\iota}(x,y) := \frac{(2\pi)^{\frac{N-1}{2}} \left|\det\Theta^s(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn}\Theta^s(x,y)}}{j!(2i)^j} \Delta^j_{\partial B,\Theta^s} a^s_\iota\left(x,y,\hat{\xi}^s(x,y)\right)$$

and

$$g^s_{j,\iota}(x,y) := \begin{cases} \partial^{l_\iota}\left(z \mapsto \Gamma(q+1+z)\, i^{q+1+z}\left(\hat{\vartheta}^s(x,y) + i0\right)^{-q-1-z}\right)(0) & , \ q \in \mathbb{C}\setminus(-\mathbb{N}_0) \\ \partial^{l_\iota}\left(z \mapsto \frac{-\Gamma(z+1)}{2\pi i\ (-q)!} \int_{c+i\mathbb{R}} \frac{(-\sigma)^{-q}(c_{\ln}+\ln\sigma)}{\left(-i\hat{\vartheta}^s(x,y)+0-\sigma\right)^{z+1}} d\sigma\right)(0) & , \ q \in -\mathbb{N}_0 \end{cases}$$

with $q := d_\iota + \frac{N+1}{2} - j$, $c \in \mathbb{R}_{>0}$, and some constant $c_{\ln} \in \mathbb{C}$.

Then,

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a^0(x,y,\xi) d\xi + \sum_{\iota\in\tilde{I}} \sum_{s=1}^S \sum_{j\in\mathbb{N}_0} h^s_{j,\iota}(x,y) g^s_{j,\iota}(x,y)$$

holds in a neighborhood of the diagonal in X^2.

REMARK Suppose $\partial^2_{\partial B}\vartheta$ is not invertible at some stationary point but we can split the third variable in a pair (ξ,ζ) such that $\partial_4^2\vartheta(x_0,y_0,\xi_0,\zeta_0)$ is invertible at the stationary point. Then, we can find open neighborhoods U of ξ_0 and V of ζ_0 as well as a function $\hat{\zeta}$ such that $\partial_4\vartheta(x,y,\xi,\zeta) = 0$ if and only if $\zeta = \hat{\zeta}(\xi)$. In particular, since $U \times V$ is open in the compact set $\partial B_{\mathbb{R}^N}$, we can use a partition of unity to reduce $I(x,y,r)$ into a sum of integrals of the form

$$\int_U \int_V e^{ir\vartheta(x,y,\xi,\zeta)} a(x,y,\xi,\zeta) d\mathrm{vol}_V(\zeta) d\mathrm{vol}_U(\xi).$$

Using stationary phase with respect to ζ, then, yields

$$\begin{aligned}&\int_U \int_V e^{ir\vartheta(x,y,\xi,\zeta)} a(x,y,\xi,\zeta) d\mathrm{vol}_V(\zeta) d\mathrm{vol}_U(\xi)\\ =&\int_U e^{ir\vartheta(x,y,\xi,\hat{\zeta}(\xi))} \int_V e^{ir\langle \partial_4^2\vartheta(x,y,\xi,\hat{\zeta}(\xi))\eta(\zeta),\eta(\zeta)\rangle_{\mathbb{R}^n}} a(x,y,\xi,\zeta) d\mathrm{vol}_V(\zeta) d\mathrm{vol}_U(\xi)\end{aligned}$$

which, again, yields an expansion of the form above but where the coefficients need to be integrated once more.

■

Example For a pseudo-differential operator, we have

$$\vartheta(x,y,\xi) = (x-y)^T\sigma(\xi).$$

Choosing coordinates such that $(x-y) = -\|x-y\|_{\ell_2(N)} e_N$ and letting e_N be the pole of the stereographic projection, we obtain

$$\sigma(\xi) = \begin{pmatrix} \frac{2\xi}{1+\|\xi\|_{\ell_2(N-1)}} \\ \frac{\|\xi\|_{\ell_2(N-1)}-1}{\|\xi\|_{\ell_2(N-1)}+1} \end{pmatrix}$$

and

$$\tilde{\vartheta}(\xi) := \frac{\vartheta(x,y,\xi)}{\|x-y\|_{\ell_2(N)}} = \frac{1-\|\xi\|_{\ell_2(N-1)}}{1+\|\xi\|_{\ell_2(N-1)}}.$$

Then, we observe

$$\partial_i\tilde{\vartheta}(\xi) = \frac{-2\xi_i}{1+\|\xi\|_{\ell_2(N-1)}} - 2\xi_i \frac{1-\|\xi\|_{\ell_2(N-1)}}{\left(1+\|\xi\|_{\ell_2(N-1)}\right)^2},$$

as well as,

$$\begin{aligned}\partial_j\partial_i\tilde{\vartheta}(\xi) =& \frac{-2\delta_{ij}}{1+\|\xi\|_{\ell_2(N-1)}} + \frac{4\xi_i\xi_j}{\left(1+\|\xi\|_{\ell_2(N-1)}\right)^2} - 2\delta_{ij}\frac{1-\|\xi\|_{\ell_2(N-1)}}{\left(1+\|\xi\|_{\ell_2(N-1)}\right)^2}\\ &- 2\xi_i\left(\frac{-2\xi_j}{1+\|\xi\|_{\ell_2(N-1)}} - 2\xi_j\frac{1-\|\xi\|_{\ell_2(N-1)}}{\left(1+\|\xi\|_{\ell_2(N-1)}\right)^2}\right)\\ =& \frac{-4\delta_{ij}}{\left(1+\|\xi\|_{\ell_2(N-1)}\right)^2} + \frac{12\xi_i\xi_j}{\left(1+\|\xi\|_{\ell_2(N-1)}\right)^2}.\end{aligned}$$

From $\Theta(x,y) \coloneqq \partial^2_{\partial B}\vartheta\left(x,y,\hat{\xi}(x,y)\right)$ and $\hat{\xi}(x,y) = \frac{x-y}{\|x-y\|_{\ell_2(N)}} = \sigma(0)$ in these coordinates, we obtain

$$\Theta(x,y) = \|x-y\|_{\ell_2(N)}\,\tilde{\vartheta}''(0) = -4\,\|x-y\|_{\ell_2(N)}\,.$$

Hence, using $z \coloneqq x-y$,

$$\begin{aligned}h_j(x,y) =& \frac{(2\pi)^{\frac{N-1}{2}}\left|\det\Theta(x,y)\right|^{-\frac{1}{2}}e^{\frac{i\pi}{4}\operatorname{sgn}\Theta(x,y)}}{j!(2i)^j}\left\langle\Theta(x,y)^{-1}\partial_3,\partial_3\right\rangle^j_{\mathbb{R}^{N-1}}\tilde{a}(x,y,0)\\ =& \frac{(2\pi)^{\frac{N-1}{2}}\left(4\,\|z\|_{\ell_2(N)}\right)^{-\frac{N-1}{2}}e^{-\frac{i\pi}{4}(N-1)}}{j!(-8i\,\|z\|_{\ell_2(N)})^j}\left\langle\partial_3,\partial_3\right\rangle^j_{\mathbb{R}^{N-1}}\tilde{a}(x,y,0)\\ =& \frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}\|z\|^{-\frac{N-1}{2}-j}_{\ell_2(N)}e^{-\frac{i\pi}{4}(N-1)}}{j!(-8i)^j}\Delta^j_{\partial B}a\left(x,y,\frac{z}{\|z\|_{\ell_2(N)}}\right).\end{aligned}$$

Let

$$\tilde{h}_j(x,y) \coloneqq \frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}e^{-\frac{i\pi}{4}(N-1)}}{j!(-8i)^j}\Delta^j_{\partial B}a\left(x,y,\frac{z}{\|z\|_{\ell_2(N)}}\right).$$

Then,

$$h_j(x,y) = \tilde{h}_j(x,y)\,\|z\|^{-\frac{N-1}{2}-j}_{\ell_2(N)}$$

and

$$\begin{aligned}&\sum_{j\in\mathbb{N}_0} h_j(x,y)\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j}(\ln r)^l e^{ir\hat{\vartheta}(x,y)}\,dr\\ =& \sum_{j\in\mathbb{N}_0}\tilde{h}_j(x,y)\,\|z\|^{-\frac{N-1}{2}-j}_{\ell_2(N)}\int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j}(\ln r)^l e^{ir\|z\|_{\ell_2(N)}}\,dr.\end{aligned}$$

In particular, for $l = 0$ and $d + \frac{N-1}{2} - j \in \mathbb{C} \setminus (-\mathbb{N})$,

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}_0} h_j(x,y) \int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j} e^{ir\hat{\vartheta}(x,y)}\, dr \\
&= \sum_{j\in\mathbb{N}_0} \tilde{h}_j(x,y)\, \|z\|_{\ell_2(N)}^{-\frac{N-1}{2}-j} \int_{\mathbb{R}_{>0}} r^{d+\frac{N-1}{2}-j} e^{ir\|z\|_{\ell_2(N)}}\, dr \\
&= \sum_{j\in\mathbb{N}_0} \tilde{h}_j(x,y)\, \|z\|_{\ell_2(N)}^{-\frac{N-1}{2}-j}\, \Gamma\left(d + \frac{N+1}{2} - j\right) \left(-i\, \|z\|_{\ell_2(N)} + 0\right)^{-d-\frac{N+1}{2}+j} \\
&= \sum_{j\in\mathbb{N}_0} \tilde{h}_j(x,y) \Gamma\left(d + \frac{N+1}{2} - j\right) (-i)^{-d-\frac{N+1}{2}+j} \left(\|z\|_{\ell_2(N)} + i0\right)^{-d-N}
\end{aligned}
$$

yields the following proposition since, for $k = \delta_{\mathrm{diag}}$, we have $\vartheta(x,y,\xi) = \langle x - y, \xi\rangle$ and $a(x,y,\xi) = \frac{1}{2\pi}$, i.e. $d = 0$ and

$$
\tilde{h}_j(x,y) := \begin{cases} \frac{1}{2\pi} \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)} & , \ j = 0 \\ 0 & , \ j \in \mathbb{N} \end{cases}.
$$

Proposition 8.4.

$$
\begin{aligned}
\delta_{\mathrm{diag}}(x,y) =& \frac{1}{2\pi} \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)} \Gamma\left(\frac{N+1}{2}\right) (-i)^{-\frac{N+1}{2}} \left(\|x-y\|_{\ell_2(N)} + i0\right)^{-N} \\
&+ \frac{1}{2\pi} \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)} \Gamma\left(\frac{N+1}{2}\right) (-i)^{-\frac{N+1}{2}} \left(-\|x-y\|_{\ell_2(N)} + i0\right)^{-N}.
\end{aligned}
$$

In particular, for $N = 1$, we obtain

$$
\delta_{\mathrm{diag}}(x,y) = \frac{i}{2\pi} \left(\left(\|x-y\|_{\ell_2(N)} + i0\right)^{-1} - \left(\|x-y\|_{\ell_2(N)} - i0\right)^{-1} \right).
$$

This is precisely what we expect; cf. end of section 4.4.3.1 in **[67]**.

∎

Remark Note that in the $N = 1$ case everything collapses as there are no spherical derivatives. We will simply obtain

$$
k_d(x,y) = \int_{\mathbb{R}_{>0}} r^d e^{ir\vartheta(x,y,1)} a_d(x,y,1) dr + \int_{\mathbb{R}_{>0}} r^d e^{ir\vartheta(x,y,-1)} a_d(x,y,-1) dr
$$

and

$$\int_{\mathbb{R}_{>0}} r^d e^{ir\vartheta(x,y,\pm 1)} a_d(x,y,\pm 1) dr$$
$$= \begin{cases} c_d a_d(x,y,\pm 1) \left(\vartheta(x,y,\pm 1) + i0\right)^{-d-1} & , \ d \notin -\mathbb{N} \\ a_d(x,y,\pm 1) \frac{(i\vartheta(x,y,\pm 1)-0)^{-d-1}}{(-d-1)!} \left(c_d + \ln\left(-i\vartheta(x,y,\pm 1) + 0\right)\right) & , \ d \in -\mathbb{N} \end{cases}$$

with some constants c_d. Hence, for

$$k(x,y) \sim \sum_{j\in\mathbb{N}_0} \int_{\mathbb{R}} e^{i\vartheta(x,y,\xi)} a_{d-j}(x,y,\xi) d\xi$$

with $d \in \mathbb{Z}$ and a_{d-j} homogeneous of degree $d - j$, the coefficient of the logarithmic terms are

$$\sum_{j\in\mathbb{N}_{\geq d+1}} a_{d-j}(x,y,\pm 1) \frac{(i\vartheta(x,y,\pm 1) - 0)^{j-d-1}}{(j-d-1)!}.$$

In particular, in the critical case where $\vartheta(x,y,\pm 1) = 0$ (in fact, we are only interested in $\vartheta(x,x,\pm 1)$) we are reduced to the known fact (cf. formulae (3) and (4) in [**7**]) that the densities of the residue traces at x (that is, $a_{-1}(x,x,\pm 1)$) coincide with the coefficients of the logarithmic terms (that is, $\ln\left(-i\vartheta(x,x,\pm 1) + 0\right)$) in the singularity structure of k.

Furthermore, we can calculate the generalized Kontsevich-Vishik trace for $a = a_0 + \sum_{\iota\in I} a_\iota$ if $\forall \iota \in I : \ d_\iota \in \mathbb{R} \smallsetminus \{-1\} \ \wedge \ l_\iota = 0$. Then, the kernel k satisfies (note $\vartheta(x,x,r) = 0$ by assumption)

$$k(x,x) = \int_{\mathbb{R}_{>0}} a_0(x,x,r) dr + \sum_{\iota\in I} \int_{\mathbb{R}_{>0}} a_\iota(x,x,r) dr.$$

Since $1_{\mathbb{R}_{>0}} a_\iota(x,x,\cdot)$ is homogeneous of degree d_ι, we obtain that $\int_{\mathbb{R}_{>0}} a_\iota(x,x,r) dr$ vanishes for $d_\iota < -1$ since the Fourier transform $\mathcal{F}(1_{\mathbb{R}_{>0}} a_\iota(x,x,\cdot))$ over $\mathbb{R}$ is a homogeneous distribution of degree $-1 - d_\iota$. For $d_\iota > -1$, we obtain

$$\int_{\mathbb{R}_{>0}} e^{i\vartheta(x,y,r)} a_\iota(x,x,r) dr = c_\iota a_\iota(x,y,1) \left(\vartheta(x,y,1) + i0\right)^{-d_\iota - 1}$$

which is precisely the other singular contribution (that is the $f(x,y)(\varphi+0)^{-N}$ term in equation (3) of [**7**]) to the kernel singularity. In other words, the difference of $k(x,y)$ and its singular part $k^{\mathrm{sing}}(x,y)$ satisfies

$$\left(k - k^{\mathrm{sing}}\right)(x,x) = \int_{\mathbb{R}_{>0}} a_0(x,x,r)dr.$$

In order to use Theorem 4.1, we will have to show that the regularized singular terms vanish. This follows directly from the Laurent expansion with mollification. For $d_\iota > -1$, we have the two terms

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0} \frac{\int_X \int_0^1 e^{i\vartheta(x,x,\xi)} \partial^n a_\iota(0)(x,x,\xi) d\xi d\mathrm{vol}_X(x)}{n!} z^n \\
&+ \sum_{n\in\mathbb{N}_0} \sum_{j=0}^{n} \frac{(-1)^{j+1} j! \int_X e^{i\vartheta(x,x,1)} \partial^n a_\iota(0)(x,x,1) d\mathrm{vol}_X(x)}{n!(1+d_\iota)^{j+1}} z^n
\end{aligned}$$

to evaluate at $z = 0$, i.e.

$$\begin{aligned}
&\lim_{h\searrow 0} \int_X \int_0^1 (h+r)^{d_\iota} a_\iota(x,x,1) dr d\mathrm{vol}_X(x) \\
&= \int_X a_\iota(x,x,1) \lim_{h\searrow 0} \int_h^{1+h} r^{d_\iota} dr d\mathrm{vol}_X(x) \\
&= \int_X a_\iota(x,x,1) \lim_{h\searrow 0} \frac{(1+h)^{d_\iota+1} - h^{d_\iota+1}}{d_\iota + 1} d\mathrm{vol}_X(x) \\
&= \int_X \frac{a_\iota(x,x,1)}{d_\iota + 1} d\mathrm{vol}_X(x)
\end{aligned}$$

and

$$\frac{-\int_X a_\iota(x,x,1) d\mathrm{vol}_X(x)}{1 + d_\iota}.$$

Hence, the generalized Kontsevich-Vishik trace reduces to the pseudo-differential form. Let $a \sim \sum_{j\in\mathbb{N}_0} a_{d-j}$ and N be sufficiently large, then

$$\mathrm{tr}_{KV} A = \int_X \int_{\mathbb{R}_{>0}} a(x,x,r) - \sum_{j=0}^{N} a_{d-j}(x,x,r)\ dr\ d\mathrm{vol}_X(x)$$

which is independent of N.

$\blacksquare$

In fact, we can generalize the case above.

THEOREM 8.5. *Let A be a Fourier Integral Operator with kernel*

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\xi$$

whose phase function ϑ satisfies $\forall x \in X\ \forall \xi \in \mathbb{R}^N :\ \vartheta(x,x,\xi) = 0$ and whose amplitude has an asymptotic expansion $a \sim \sum_{\iota\in\mathbb{N}} a_\iota$ where each a_ι is log*-homogeneous with degree of homogeneity d_ι and logarithmic order l_ι, and $\Re(d_\iota) \to -\infty$. Let $N_0 \in \mathbb{N}$ such that $\forall \iota \in \mathbb{N}_{>N_0} :\ \Re(d_\iota) < -N$ and let*

$$k^{\mathrm{sing}}(x,y) = \int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)} \sum_{\iota=1}^{N_0} a_\iota(x,y,\xi) d\xi$$

denote the singular part of the kernel.

Then, the regularized kernel $k - k^{\mathrm{sing}}$ is continuous along the diagonal and independent of the particular choice of N_0 (along the diagonal). Furthermore, the generalized Kontsevich-Vishik density[2] *is given by*

$$\left(k - k^{\mathrm{sing}}\right)(x,x) d\mathrm{vol}_X(x) = \int_{\mathbb{R}^N} a(x,x,\xi) - \sum_{\iota=1}^{N_0} a_\iota(x,x,\xi) d\xi d\mathrm{vol}_X(x).$$

PROOF. Note that $k - k^{\mathrm{sing}}$ is regular because it has an amplitude in the Hörmander class $S^m(X \times X \times \mathbb{R}^N)$ for some $m \in \mathbb{R}_{<-N}$. Hence, it suffices to show that the ζ-regularized singular contributions of a_ι vanish for $d_\iota \neq -N$. Let $\iota \in \mathbb{N}$ such that $d_\iota \neq -N$. Then, we need to show that

$$\int_X \int_{B_{\mathbb{R}^N}(0,1)} a_\iota(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x)$$
$$+ \frac{(-1)^{l_\iota+1} l_\iota! \int_{X\times\partial B_{\mathbb{R}^N}} \tilde{a}_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}}$$

[2]Mind that this density is only locally defined. It only patches together (modulo pathologies) if we assume the kernel patched together in the first place and the derivatives of terms of critical dimension $d_\iota = -N$ regularize to zero, i.e. if $\zeta(\mathfrak{fp}_0 A)(0)$ is tracial and independent of gauge.

vanishes. Mollifying

$$\begin{aligned}\int_{B_{\mathbb{R}^N}(0,1)} a_\iota(0)(x,x,\xi)d\xi &= \int_0^1 \int_{\partial B_{\mathbb{R}^N}} r^{N-1} a_\iota(0)(x,x,r\nu)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\nu)dr \\ &= \int_0^1 \int_{\partial B_{\mathbb{R}^N}} r^{N+d_\iota-1}(\ln r)^{l_\iota} \tilde{a}_\iota(0)(x,x,\nu)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\nu)dr\end{aligned}$$

yields (note that $f_n \to f$ compactly implies $f_n' \to f'$ compactly for holomorphic functions)

$$\begin{aligned}\lim_{h\searrow 0}\int_0^1 (h+r)^{N+d_\iota-1}(\ln(h+r))^{l_\iota}dr &= \lim_{h\searrow 0}\int_h^{1+h} r^{N+d_\iota-1}(\ln r)^{l_\iota}dr \\ &= \lim_{h\searrow 0}\int_h^{1+h} \partial^{l_\iota}\left(z\mapsto r^{N+d_\iota-1+z}\right)(0)dr \\ &= \lim_{h\searrow 0}\partial^{l_\iota}\left(z\mapsto \frac{(1+h)^{N+d_\iota+z}-h^{N+d_\iota+z}}{N+d_\iota+z}\right)(0) \\ &= \partial^{l_\iota}\left(z\mapsto (N+d_\iota+z)^{-1}\right)(0) \\ &= \left(z\mapsto \frac{(-1)^{l_\iota}l_\iota!}{(N+d_\iota+z)^{l_\iota+1}}\right)(0),\end{aligned}$$

i.e.

$$\begin{aligned}&\int_X\int_{B_{\mathbb{R}^N}(0,1)} a_\iota(0)(x,x,\xi)\ d\xi\ d\mathrm{vol}_X(x) \\ &+\frac{(-1)^{l_\iota+1}l_\iota!\int_{X\times\partial B_{\mathbb{R}^N}}\tilde{a}_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}} \\ =&\frac{(-1)^{l_\iota}l_\iota!\int_{X\times\partial B_{\mathbb{R}^N}}\tilde{a}_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}} \\ &+\frac{(-1)^{l_\iota+1}l_\iota!\int_{X\times\partial B_{\mathbb{R}^N}}\tilde{a}_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{X\times\partial B_{\mathbb{R}^N}}(x,\xi)}{(N+d_\iota)^{l_\iota+1}} \\ =&0.\end{aligned}$$

□

Remark (i) Reduction to the pseudo-differential form is highly non-trivial and, in general, false. Consider, for instance,

$$\int_X\int_{\mathbb{R}} e^{i\Theta(x,x)r}r^{-n}drd\mathrm{vol}_X(x) = \int_X \frac{-i\pi(-2\pi i\Theta(x,x))^{n-1}\operatorname{sgn}(\Theta(x,x))}{(n-1)!}d\mathrm{vol}_X(x).$$

If $\Theta(x,x) = 1$ and $n = 4$, then this term reduces to $\frac{4\pi^4 \mathrm{vol}(X)}{3}$. In other words, such a term would violate independence of N.

(ii) Instead of using mollification directly, we could have used the generalized Mellin transform which yields

$$\int_{\mathbb{R}_{>0}} r^\alpha dr = \mathcal{M}\left(r \mapsto r^\alpha\right)(1) = 0$$

where $\int_{\mathbb{R}_{>0}} r^\alpha dr$ is understood in the regularized sense. However, this does not apply to the critical case $d_\iota = -N$ because the coefficients in the Laurent expansion are integrals over $\tilde{a}_\iota(0)$ on $B_{\mathbb{R}^N}$ and over $\partial^{l_\iota+1}\tilde{a}_\iota(0)$ outside $B_{\mathbb{R}^N}$. Hence, we cannot re-write those integrals such that the generalized Mellin transform appears as a factor and the critical terms will not vanish, in general.

■

At this point, we can return to Proposition 4.5 where we had the formula

$$\mathfrak{fp}_0\zeta(z \mapsto BQ^z) = \int_X \mathrm{tr}_x\left(\mathfrak{fp}_0 B\right) - \frac{1}{q}\mathrm{res}\left(B \ln Q\right)_x d\mathrm{vol}_X(x) - \mathrm{tr}\left(B 1_{\{0\}}(Q)\right)$$

with B and Q polyhomogeneous, Q admitting holomorphic functional calculus and the logarithm, and with finite dimensional kernel (e.g. an elliptic classical pseudo-differential operator on a closed manifold with spectral cut), and q is the order of Q. In **[56]** (equation (2.14)) it was shown that

$$\mathfrak{fp}_0\zeta(z \mapsto BQ^z) = -\frac{1}{q}\mathrm{res}\left(B \ln Q\right) - \mathrm{tr}\left(B 1_{\{0\}}(Q)\right)$$

holds if $(x \mapsto \mathrm{tr}_x\left(\mathfrak{fp}_0 B\right)) = 0$ (e.g. if B is a differential operator) and Sylvie Paycha conjectured that this formula should hold more generally. (Note that we are using a different notation as we might want to assume a global point of view rather than just considering everything a sum of local patches without patching properties. Under these stronger conditions, we cannot simply write

$\int_X \operatorname{tr}_x(\mathfrak{f}\mathfrak{p}_0 B) - \frac{1}{q}\operatorname{res}(B\ln Q)_x \,d\mathrm{vol}_X(x) = \operatorname{tr}_{KV}(B) - \frac{1}{q}\operatorname{res}(B\ln Q)$ since they are not separately globally defined densities.) The following corollary shows an equivalent characterization of Paycha's conjecture for Fourier Integral Operators as in Theorem 8.5 (in particular for pseudo-differential operators) in terms of the regular part of B.

Corollary 8.6. *Let Q be as above and B be a Fourier Integral Operator whose phase function ϑ satisfies $\forall x \in X\ \forall \xi \in \mathbb{R}^N:\ \vartheta(x,x,\xi) = 0$ and whose amplitude has an asymptotic expansion $b \sim \sum_{\iota\in\mathbb{N}} b_\iota$ where each b_ι is homogeneous (on $\mathbb{R}^N \setminus \{0\}$) with degree of homogeneity d_ι and $\Re(d_\iota) \to -\infty$. Furthermore, let $I \subseteq \mathbb{N}$ be such that the amplitude b decomposes into the form $b_0 + \sum_{\iota\in I} b_\iota$ where b_0 is integrable in $\mathbb{R}^N$ (i.e. of Hörmander class $S^m(X\times X\times\mathbb{R}^N)$ with $m < -N$), and let B_0 the part of B corresponding to b_0. Then,*

$$\begin{aligned}\mathfrak{f}\mathfrak{p}_0\zeta(z\mapsto BQ^z) &= \int_X \operatorname{tr}_x(\mathfrak{f}\mathfrak{p}_0 B) - \frac{1}{q}\operatorname{res}(B\ln Q)_x \,d\mathrm{vol}_X(x) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)\\ &= \int_X \operatorname{tr}_x(B_0) - \frac{1}{q}\operatorname{res}(B\ln Q)_x \,d\mathrm{vol}_X(x) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right).\end{aligned}$$

In particular, the following are equivalent.

(i) *Paycha's conjecture: $\mathfrak{f}\mathfrak{p}_0\zeta(z\mapsto BQ^z) = -\frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)$.*

(ii) *$x \mapsto \int_{\mathbb{R}^N} b_0(x,x,\xi)d\xi d\mathrm{vol}_X(x)$ is a globally defined density on X and*

$$\operatorname{tr}(B_0) = \int_X \int_{\mathbb{R}^N} b_0(x,x,\xi)d\xi d\mathrm{vol}_X(x) = 0.$$

Remark If we remove the question of global patching and simply consider sums of local representations, then we obtain

$$\begin{aligned}\mathfrak{f}\mathfrak{p}_0\zeta(z\mapsto BQ^z) &= \operatorname{tr}_{KV}(B) - \frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)\\ &= \operatorname{tr}(B_0) - \frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)\end{aligned}$$

by default. In particular,

(i) Paycha's conjecture: $\mathfrak{fp}_0\zeta(z \mapsto BQ^z) = -\frac{1}{q}\operatorname{res}(B\ln Q) - \operatorname{tr}\left(B1_{\{0\}}(Q)\right)$.

and

(ii') $\operatorname{tr}(B_0) = \int_X \int_{\mathbb{R}^N} b_0(x,x,\xi)d\xi d\mathrm{vol}_X(x) = 0$.

are equivalent.

■

Finally, we will consider an example of linear phase functions which will be generalized to find algebras of Fourier Integral Operators which are Hilbert-Schmidt and whose trace-integrals are regular.

Example Let $\vartheta(x,y,\xi) := \langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N}$ and $\Theta(x_0,y_0) \neq 0$. Then,

$$k(x,y) = \int_{\mathbb{R}^N} e^{i\langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N}} a(x,y,\xi)d\xi = \mathcal{F}\left(a(x,y,\cdot)\right)\left(-\Theta(x,y)\right)$$

is continuous in a sufficiently small neighborhood of (x_0,y_0) for homogeneous a because $\mathcal{F}(a(x,y,\cdot))$ is homogeneous and $\Theta(x,y)$ non-zero. Hence, if Θ does not vanish on the diagonal, then $X \ni x \mapsto k(x,x) \in \mathbb{C}$ is continuous and, by compactness of X, $\int_X k(x,x)d\mathrm{vol}_X(x)$ well-defined.

■

The stationary phase approximation above generalizes this observation (here, $\hat{\xi}(x,y) = \pm\frac{\Theta(x,y)}{\|\Theta(x,y)\|_{\ell_2(N)}}$, i.e. $\hat{\vartheta}^s(x,y) = (-1)^s\|\Theta(x,y)\|_{\ell_2(N)}$ with $s \in \{0,1\}$).

Theorem 8.7. *Let A be a Fourier Integral Operator with phase function ϑ satisfying $\partial_3^2\left(\vartheta|_{X\times X\times\partial B_{\mathbb{R}^N}}\right)(x,y,\xi) \in GL\left(\mathbb{R}^{N-1}\right)$ whenever $\partial_3\vartheta(x,y,\xi) = 0$ and $\left\{\hat{\xi}^s;\ s \in \mathbb{N}_{\leq n}\right\}$ the set of stationary points. Furthermore, let*

$$\forall x \in X\ \ \forall s \in \mathbb{N}_{\leq n}:\ \vartheta\left(x,x,\hat{\xi}^s(x,x)\right) \neq 0.$$

Then,

$$\left(\ X \ni x \mapsto k(x,x) \in \mathbb{C}\ \right) \in C(X)$$

and

$$\operatorname{tr} A = \int_X k(x,x) d\mathrm{vol}_X(x)$$

is well-defined, i.e. A is a Hilbert-Schmidt operator. Furthermore, ζ-functions of families of such operators have no poles.

This yields many algebras $\mathcal{A}$ in which the generalized Kontsevich-Vishik trace is everywhere defined.

EXAMPLE An example for such non-trivial Hilbert-Schmidt operators occurs on quotient manifolds. Let Γ be a co-compact discrete group on M acting continuously[3] and freely[4] on $M/_\Gamma$, $\tilde{k}$ a $\Gamma \times \Gamma$-invariant[5] Schwartz kernel on M, and k its projection to $M/_\Gamma$. Then, $k(x,y) = \sum_{\gamma\in\Gamma} \tilde{k}(x,\gamma y)$ (cf. e.g. equation (3.2.1.3) in [**67**]). Suppose $\tilde{k}$ is pseudo-differential, i.e.

$$\tilde{k}(x,y) = \int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}} a(x,y,\xi) d\xi.$$

Then,

$$k(x,y) = \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^N} e^{i\langle x-\gamma y,\xi\rangle_{\mathbb{R}^N}} a(x,\gamma y,\xi) d\xi.$$

Hence, for $\gamma = \mathrm{id}$ we have a pseudo-differential part and for $\gamma \neq \mathrm{id}$ the phase function $\vartheta_\gamma(x,y,\xi) = \langle x-\gamma y,\xi\rangle_{\mathbb{R}^N}$ has stationary points $\pm\frac{x-\gamma y}{\|x-\gamma y\|_{\ell_2(N)}}$, that is, $\vartheta_\gamma\left(x,y,\hat{\xi}^s(x,y)\right) = (-1)^s \|x-\gamma y\|_{\ell_2(N)}$ does not vanish in a neighborhood of the diagonal.

■

REMARK Note that we may use the stationary phase approximation results to get insights into the Laurent coefficients of the ζ-function without having to consider

[3] $\Gamma \times M/_\Gamma \ni (\gamma,x) \mapsto \gamma x \in M/_\Gamma$ is continuous

[4] $\forall \gamma \in \Gamma: \ (\exists x \in M/_\Gamma : \gamma x = x) \ \Rightarrow \ \gamma = \mathrm{id}$

[5] $\forall \gamma \in \Gamma \ \forall x,y \in M: \ \tilde{k}(x,y) = \tilde{k}(\gamma x,\gamma y)$

all these Laplace transforms because those coefficients are of the form $c \cdot I(x,y,1)$ with some constant $c \in \mathbb{C}$, i.e. we do not need the radial integration and obtain an asymptotic expansion

$$\begin{aligned}
&\int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&= \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a^0(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) + \sum_{s=1}^{S} \sum_{j \in \mathbb{N}_0} e^{i\vartheta\left(x,y,\hat{\xi}(x,y)\right)} h_j^s(x,y) \\
&= \int_{\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,y,\xi)} a^0(x,y,\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi) \\
&\quad + \sum_{s=1}^{S} e^{i\hat{\vartheta}^s(x,y)} \sum_{j \in \mathbb{N}_0} \frac{(2\pi)^{\frac{N-1}{2}} \left|\det \Theta^s(x,y)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4} \operatorname{sgn} \Theta^s(x,y)}}{j!(2i)^j} \Delta_{\partial B, \Theta^s}^j a^s\left(x,y,\hat{\xi}^s(x,y)\right)
\end{aligned}$$

with $\hat{\vartheta}^s(x,y) = \vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Theta^s(x,y) = \partial_{\partial B}^2 \vartheta\left(x,y,\hat{\xi}^s(x,y)\right)$, $\Delta_{\partial B, \Theta^s(x,y)} = \left\langle \Theta^s(x,y)^{-1} \partial_{\partial B}, \partial_{\partial B} \right\rangle = -\operatorname{div}_{\partial B_{\mathbb{R}^N}} \Theta^s(x,y)^{-1} \operatorname{grad}_{\partial B_{\mathbb{R}^N}}$, and $\hat{\xi}^s(x,y)$ is the unique stationary point of $\vartheta(x,y,\cdot)$ in $\partial B_{\mathbb{R}^N} \cap \operatorname{spt} a^s(x,y,\cdot)$ while a^0 has no such point in its support.

■

We will close this chapter by considering two examples of wave traces.

Example Let us consider manifolds with diagonal metric, that is, the metric tensor is given by

$$g^{ij}(x) = \mathfrak{g}(x)^2 \delta^{ij}$$

with some function $\mathfrak{g}$. An example of these are hyperbolic manifolds. Let

$$\mathbb{H}^N := \{x \in \mathbb{R}^N;\ x_N > 0\}$$

with the metric

$$g_{ij}(x) = \mathfrak{g}(x)^{-2} \delta_{ij} = x_N^{-2} \delta_{ij}.$$

Then, $\sqrt{|\det g(x)|} = \mathfrak{g}(x)^{-N}$. The Laplace-Beltrami operator on $\mathbb{H}^N$ is given by

$$\Delta_{\mathbb{H}^N} = \mathfrak{g}(x)^2 \sum_{i=1}^{n} \partial_i^2$$

and the wave operator $\exp(it\sqrt{|\Delta_{\mathbb{H}^N}|})$ has the kernel

$$\kappa_{\mathbb{H}^N}(x,y) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}} e^{it\mathfrak{g}(x)\|\xi\|_{\ell_2(N)}} d\xi.$$

Let Γ be a co-compact, discrete, torsion-free sub-group of the isometries of $\mathbb{H}^N$ such that Γ is a lattice and $X := \mathbb{H}^N/_\Gamma$ can be identified with a fundamental domain in $\mathbb{H}^N$ under action of Γ. Then, we call X a hyperbolic manifold. Since Γ is a subset of the isometries, the metric on X is given by the metric on $\mathbb{H}^N$ taking a representative of the orbit and the wave-operator $\exp(it\sqrt{|\Delta|})$ factors through with the kernel

$$\kappa(x,y) = \sum_{\gamma\in\Gamma} (2\pi)^{-N} \int_{\mathbb{R}^N} e^{i\langle x-\gamma y,\xi\rangle_{\mathbb{R}^N}} e^{it\mathfrak{g}(x)\|\xi\|_{\ell_2(N)}} d\xi.$$

Let A^t be a gauged Fourier Integral Operator with $A^t(0) = \exp(it\sqrt{|\Delta|})$. Then, $A^t(0)$ has the phase function

$$\vartheta_\gamma(x,y,\xi) = \langle x-\gamma y,\xi\rangle_{\mathbb{R}^N} + t\mathfrak{g}(x)\,\|\xi\|_{\ell_2(N)}$$

and amplitude $(x,y,\xi)\mapsto 1$, i.e. each term in the sum $\sum_{\gamma\in\Gamma}$ yields a $\zeta(A_\gamma^t)$ which is holomorphic in a neighborhood of zero. Thus, Lemma 2.6 yields that $\zeta(A_\gamma^t)$ is independent of the gauge and we obtain

$$\zeta(A^t)(0) = \sum_{\gamma\in\Gamma} \zeta(A_\gamma^t)(0) = \sum_{\gamma\in\Gamma} (2\pi)^{-N} \int_X \int_{\mathbb{R}^N} e^{i\langle x-\gamma x,\xi\rangle_{\mathbb{R}^N}} e^{it\mathfrak{g}(x)\|\xi\|_{\ell_2(N)}} d\xi dx.$$

For $\gamma = 1$ we will use the property

$$\forall q\in\mathbb{C}_{\Re(\cdot)>-1}:\ \mathcal{L}\left(r\mapsto r^q\right)(s) = \int_{\mathbb{R}_{>0}} r^q e^{-sr} dr = \Gamma(q+1)s^{-q-1}$$

of the Laplace transform (where Γ is the Γ-function) and obtain

$$\zeta(A_1^t)(0) = (2\pi)^{-N} \int_X \int_{\mathbb{R}^N} e^{it\mathfrak{g}(x)\|\xi\|_{\ell_2(N)}} d\xi dx$$

$$
\begin{aligned}
&=(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}}\int_{\partial B_{\mathbb{R}^N}} r^{N-1}e^{it\mathfrak{g}(x)r}d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)drdx\\
&=\frac{\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})}{(2\pi)^N}\int_X\int_{\mathbb{R}_{>0}} r^{N-1}e^{it\mathfrak{g}(x)r}drdx\\
&=\frac{\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})}{(2\pi)^N}\int_X \mathcal{L}\left(r\mapsto r^{N-1}\right)(-it\mathfrak{g}(x))dx\\
&=\frac{\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})(N-1)!}{(2\pi)^N}\int_X (-it\mathfrak{g}(x))^{-N}dx\\
&=\frac{\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})(N-1)!}{(-2\pi it)^N}\int_X d\mathrm{vol}_X\\
&=\frac{(N-1)!\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})\,\mathrm{vol}_X(X)}{(-2\pi it)^N}.
\end{aligned}
$$

For $\gamma\in\Gamma\smallsetminus\{1\}$ we know $x-\gamma x\neq 0$ and stationary points of $\vartheta_\gamma(x,x,\cdot)$ are $\xi_\pm(x) := \pm\frac{x-\gamma x}{\|x-\gamma x\|_{\ell_2(N)}}$ (since the term $t\mathfrak{g}(x)\|\xi\|_{\ell_2(N)}$ vanishes taking derivatives with respect to $\xi\in\partial B_{\mathbb{R}^N}$) with

$$
\begin{aligned}
\vartheta_\gamma(x,x,\xi_\pm(x)) =&\left\langle x-\gamma x,\pm\frac{x-\gamma x}{\|x-\gamma x\|_{\ell_2(N)}}\right\rangle_{\mathbb{R}^N} + t\mathfrak{g}(x)\left\|\pm\frac{x-\gamma x}{\|x-\gamma x\|_{\ell_2(N)}}\right\|_{\ell_2(N)}\\
=&t\mathfrak{g}(x)\pm\|x-\gamma x\|_{\ell_2(N)}.
\end{aligned}
$$

Since $\mathfrak{g}$ is a positive continuous function and X compact, we obtain that $\mathfrak{g}$ is bounded away from zero and $x\mapsto\|x-\gamma x\|_{\ell_2(N)}$ is bounded, i.e. $\vartheta_\gamma(x,x,\xi_\pm(x))$ has no zeros for t sufficiently large (similarly for sufficiently small t). By Theorem 8.7, we obtain that each $\zeta(A^t_\gamma)(0)$ exists for sufficiently large t (and sufficiently large $-t$, as well).

Hence, we want to evaluate

$$
\begin{aligned}
\zeta(A^t_\gamma)(0) =&(2\pi)^{-N}\int_X\int_{\mathbb{R}^N} e^{i\langle x-\gamma x,\xi\rangle_{\mathbb{R}^N}}e^{it\mathfrak{g}(x)\|\xi\|_{\ell_2(N)}}d\xi dx\\
=&(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} r^{N-1}e^{it\mathfrak{g}(x)r}\int_{\partial B_{\mathbb{R}^N}} e^{ir\langle x-\gamma x,\eta\rangle_{\mathbb{R}^N}}d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)drdx.
\end{aligned}
$$

$\int_{\partial B_{\mathbb{R}^N}} e^{ir\langle x-\gamma x,\eta\rangle_{\mathbb{R}^N}} \mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta)$ can be evaluated using stationary phase approximation. The stationary points are

$$\eta_\pm(x) := \pm \frac{x-\gamma x}{\|x-\gamma x\|_{\ell_2(N)}}$$

and the corresponding phase function $\hat{\vartheta}(x,\eta) := r\langle x-\gamma x,\eta\rangle_{\mathbb{R}^N}$ satisfies

$$\hat{\vartheta}(x,\eta_\pm(x)) = \pm r\|x-\gamma x\|_{\ell_2(N)}.$$

Since the amplitude is the constant function 1, all higher order derivatives in the stationary phase approximation yield zero and we obtain

$$\begin{aligned}\int_{\partial B_{\mathbb{R}^N}} e^{ir\langle x-\gamma x,\eta\rangle_{\mathbb{R}^N}} \mathrm{dvol}_{\partial B_{\mathbb{R}^N}}(\eta) = {}& \|x-\gamma x\|_{\ell_2(N)}^{-\frac{N-1}{2}} \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)} e^{ir\|x-\gamma x\|_{\ell_2(N)}} \\ &+ \|x-\gamma x\|_{\ell_2(N)}^{-\frac{N-1}{2}} \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)} e^{-ir\|x-\gamma x\|_{\ell_2(N)}}\end{aligned}$$

which, in turn, yields

$$\begin{aligned}&\zeta(A_\gamma^t)(0)\\ =&(2\pi)^{-N}\int_X\int_{\mathbb{R}^N} e^{i\langle x-\gamma x,\xi\rangle_{\mathbb{R}^N}} e^{it\mathfrak{g}(x)\|\xi\|_{\ell_2(N)}} d\xi dx\\ =&\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)}}{(2\pi)^N}\int_X \|x-\gamma x\|_{\ell_2(N)}^{-\frac{N-1}{2}} \int_{\mathbb{R}_{>0}} r^{N-1} e^{it\mathfrak{g}(x)r} e^{ir\|x-\gamma x\|_{\ell_2(N)}} drdx\\ &+\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)}}{(2\pi)^N}\int_X \|x-\gamma x\|_{\ell_2(N)}^{-\frac{N-1}{2}} \int_{\mathbb{R}_{>0}} r^{N-1} e^{it\mathfrak{g}(x)r} e^{-ir\|x-\gamma x\|_{\ell_2(N)}} drdx\\ =&\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)}(N-1)!}{(-2\pi i)^N}\int_X \|x-\gamma x\|_{\ell_2(N)}^{-\frac{N-1}{2}} \left(t\mathfrak{g}(x)+\|x-\gamma x\|_{\ell_2(N)}\right)^{-N} dx\\ &+\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)}(N-1)!}{(-2\pi i)^N}\int_X \|x-\gamma x\|_{\ell_2(N)}^{-\frac{N-1}{2}} \left(t\mathfrak{g}(x)-\|x-\gamma x\|_{\ell_2(N)}\right)^{-N} dx.\end{aligned}$$

Let us consider the special case of a flat torus, that is, $\mathfrak{g} = 1$ and $\gamma x = \gamma + x$. Then, the formula collapses to

$$\zeta(A_\gamma^t)(0) = \frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)}(N-1)!}{(-2\pi i)^N}\int_X \|\gamma\|_{\ell_2(N)}^{-\frac{N-1}{2}} \left(t+\|\gamma\|_{\ell_2(N)}\right)^{-N} dx$$

$$+\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)}(N-1)!}{(-2\pi i)^N}\int_X \|\gamma\|_{\ell_2(N)}^{-\frac{N-1}{2}}\left(t-\|\gamma\|_{\ell_2(N)}\right)^{-N}dx$$

$$=\sum_{\pm}\frac{\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} e^{-\frac{i\pi}{4}(N-1)}(N-1)!\mathrm{vol}_X(X)}{(-2\pi i)^N}\|\gamma\|_{\ell_2(N)}^{-\frac{N-1}{2}}\left(t\pm\|\gamma\|_{\ell_2(N)}\right)^{-N}.$$

This shows the well-known result that the ζ-regularized wave trace has a pole if t is equal to the length of a closed geodesic $\|\gamma\|_{\ell_2(N)}$ and for all other t, we obtain

$$\zeta(A^t)(0)=\frac{(N-1)!\mathrm{vol}_X(X)}{(-2\pi i)^N}$$
$$\cdot\left(\frac{\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\partial B_{\mathbb{R}^N})}{t^N}+\sum_{\gamma\in\Gamma}\sum_{\pm}\left(\frac{\pi}{2}\right)^{\frac{N-1}{2}}e^{-\frac{i\pi}{4}(N-1)}\|\gamma\|_{\ell_2(N)}^{-\frac{N-1}{2}}\left(t\pm\|\gamma\|_{\ell_2(N)}\right)^{-N}\right)$$

■

Example In light of the last example, we can even go a step further and consider manifolds where the Laplacian has the symbol $g^{ij}(x)\xi_i\xi_j$, i.e.

$$\zeta(A^t)(0)=\sum_{\gamma\in\Gamma}(2\pi)^{-N}\int_X\int_{\mathbb{R}^N}e^{it\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}e^{i\langle x-\gamma x,\xi\rangle_{\mathbb{R}^N}}d\xi dx.$$

Using Fubini's theorem

Theorem (Fubini) *Let $\Omega\subseteq\mathbb{R}^n$ be open, $\varphi\in C_c(\Omega)$, $f\in C^1(\Omega,\mathbb{R})$, $\forall x\in\Omega$: $\operatorname{grad}f(x)\neq 0$, and $M_r:=[\{r\}]f=\{x\in\Omega;\ f(x)=r\}$. Then,*

$$\int_\Omega\varphi(x)dx=\int_{\mathbb{R}}\int_{M_r}\varphi(\xi)\left\|\operatorname{grad}f(\xi)\right\|_{\ell_2(n)}^{-1}\mathrm{dvol}_{M_r}(\xi)dr.$$

with $f(\xi)=\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}$ on $\mathbb{R}^N\setminus\{0\}$, i.e. $\operatorname{grad}f(\xi)=\frac{G^{-1}(x)\xi}{\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}$, gives rise to the definition

$$\forall x\in X:\ M_x:=\left\{\frac{\xi}{\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}\in\mathbb{R}^N;\ \xi\in\partial B_{\mathbb{R}^N}\right\}$$

and, thus,

$$(2\pi)^{-N}\int_X\int_{\mathbb{R}^N}e^{it\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}e^{i\langle x-\gamma x,\xi\rangle_{\mathbb{R}^N}}d\xi dx$$

$$=\int_X\int_{\mathbb{R}_{>0}}\int_{rM_x}\frac{e^{it\left\|G^{-\frac{1}{2}}(x)\tilde\mu\right\|_{\ell_2(N)}+i\langle x-\gamma x,\tilde\mu\rangle_{\mathbb{R}^N}}\left\|G^{-\frac{1}{2}}(x)\tilde\mu\right\|_{\ell_2(N)}}{(2\pi)^N\left\|G^{-1}(x)\tilde\mu\right\|_{\ell_2(N)}}\mathrm{dvol}_{rM_x}(\tilde\mu)drdx$$

$$=(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}}\int_{M_x} e^{ir\left(t+\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}\right)}\frac{\left\|G^{-\frac{1}{2}}(x)\mu\right\|_{\ell_2(N)}}{\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}}r^{N-1}d\mathrm{vol}_{M_x}(\mu)drdx$$
$$=(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_{M_x} e^{ir\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}^{-1}d\mathrm{vol}_{M_x}(\mu)drdx.$$

Note that integrals similar to $\int_{M_x} e^{ir\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}^{-1}d\mathrm{vol}_{M_x}(\mu)$ also appear if we choose such a decomposition of $\mathbb{R}^N$ and want to calculate Laurent coefficients. Furthermore, note that we can re-write those integrals over M_x into integrals over the sphere[6]; namely,

$$\int_{M_x} f d\mathrm{vol}_{M_x} = \int_{\partial B_{\mathbb{R}^N}} f\circ\Psi\sqrt{\det\left(d\Psi_x^T d\Psi_x\right)}d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}$$

with

$$\Psi_x(\xi) := \frac{\xi}{\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}.$$

For $\gamma = \mathrm{id}$, these integrals simply reduce to

$$(2\pi)^{-N}\int_X\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_{M_x}\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}^{-1}d\mathrm{vol}_{M_x}(\mu)drdx$$
$$=\frac{(N-1)!}{(-2\pi it)^N}\int_X\int_{M_x}\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}^{-1}d\mathrm{vol}_{M_x}(\mu)dx$$
$$=\frac{(N-1)!}{(-2\pi it)^N}\int_X\int_{\partial B_{\mathbb{R}^N}}\left\|\frac{G^{-1}(x)\xi}{\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}\right\|_{\ell_2(N)}^{-1}\eth\Psi_x(\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx$$

---

[6]Let M_1, M_2 be manifolds, $\Phi: T\to M_1$ a parametrization, $\Psi: M_1\to M_2$ a diffeomorphism, and $f\in C_c(M_2)$. Then,

$$\begin{aligned}\int_{M_2} f d\mathrm{vol}_{M_2} &= \int_T (f\circ\Psi\circ\Phi)(t)\sqrt{\det\left((\Psi\circ\Phi)'(t)^T(\Psi\circ\Phi)'(t)\right)}dt\\ &= \int_T (f\circ\Psi\circ\Phi)(t)\sqrt{\det\left(\Phi'(t)^T(d\Psi\circ\Phi)(t)^T(d\Psi\circ\Phi)(t)\Phi'(t)\right)}dt\\ &= \int_T (f\circ\Psi\circ\Phi)(t)\sqrt{\det\left((d\Psi\circ\Phi)(t)^T(d\Psi\circ\Phi)(t)\right)}\sqrt{\det\left(\Phi'(t)^T\Phi'(t)\right)}dt\\ &= \int_{M_1} f\circ\Psi\sqrt{\det\left(d\Psi^T d\Psi\right)}d\mathrm{vol}_{M_1}.\end{aligned}$$

$$=\frac{(N-1)!}{(-2\pi it)^N}\int_X\int_{\partial B_{\mathbb{R}^N}}\frac{\left\|G^{-\frac{1}{2}}(x)\xi\right\|_{\ell_2(N)}}{\|G^{-1}(x)\xi\|_{\ell_2(N)}}\eth\Psi_x(\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx$$

where $\eth\Psi_x(\xi) := \sqrt{\det\left(d\Psi_x(\xi)^T d\Psi_x(\xi)\right)}$.

For $\gamma \neq \mathrm{id}$, we want to evaluate

$$(2\pi)^{-N}\int_{\mathbb{R}_{>0}} e^{irt}r^{N-1}\int_X\int_{M_x} e^{ir\langle x-\gamma x,\mu\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\mu\right\|_{\ell_2(N)}^{-1}d\mathrm{vol}_{M_x}(\mu)dxdr.$$

The stationary points are obviously characterized by $x-\gamma x \perp T_\mu M_x$ and there is always the possibility to change coordinates in the M_x integral to obtain

$$\int_X\int_{\partial B_{\mathbb{R}^N}} e^{ir\langle x-\gamma x,\Psi_x(\xi)\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\Psi_x(\xi)\right\|_{\ell_2(N)}^{-1}\eth\Psi_x(\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx.$$

In particular, for the torus, we have $\gamma x = \gamma + x$ and

$$\int_X\int_{\partial B_{\mathbb{R}^N}} e^{-ir\langle \gamma,\Psi_x(\xi)\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\Psi_x(\xi)\right\|_{\ell_2(N)}^{-1}\eth\Psi_x(\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi)dx$$

can be evaluated applying the stationary phase approximation to

$$\int_{\partial B_{\mathbb{R}^N}} e^{-ir\langle \gamma,\Psi_x(\xi)\rangle_{\mathbb{R}^N}}\left\|G^{-1}(x)\Psi_x(\xi)\right\|_{\ell_2(N)}^{-1}\eth\Psi_x(\xi)d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\xi).$$

■

REMARK Replacing $\partial B_{\mathbb{R}^N}$ by M_x becomes even more interesting if we want to calculate the Laurent coefficients

$$\int_{\Delta(X)\times\partial B_{\mathbb{R}^N}} e^{i\vartheta(x,x,\xi)}\partial^{n+l_\iota+1}\tilde{a}_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{\Delta(X)\times\partial B_{\mathbb{R}^N}}(x,\xi)$$

which are now integrals

$$\int_X\int_{M_x} e^{i\vartheta(x,x,\xi)}\partial^{n+l_\iota+1}\tilde{a}_\iota(0)(x,x,\xi)\ d\mathrm{vol}_{M_x}(\xi)d\mathrm{vol}_X(x).$$

In cases such as the example above, the integration over M_x is now without a phase function because $M_x \ni \xi \mapsto \vartheta(x,x,\xi)$ is a constant ϑ_x, leaving us with integrals of the form

$$e^{i\vartheta_x}\int_{M_x} a_x(\xi)\ d\mathrm{vol}_{M_x}(\xi)$$

where a_x is homogeneous of some degree d. For $M_x = T_x\left[\partial B_{\mathbb{R}^n}\right]$ with $T_x \in GL(\mathbb{R}^n)$, this is equivalent to

$$e^{i\vartheta_x}\int_{M_x} a_x(\xi)\ d\mathrm{vol}_{M_x}(\xi) = e^{i\vartheta_x}\int_{\partial B_{\mathbb{R}^n}} a_x(\xi)\left\|T_x^{-1}\xi\right\|^{-n-d} d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi).$$

In particular, for the case of the residue trace, we have $d = -n$, i.e.

$$e^{i\vartheta_x}\int_{M_x} a_x(\xi)\ d\mathrm{vol}_{M_x}(\xi) = e^{i\vartheta_x}\int_{\partial B_{\mathbb{R}^n}} a_x(\xi) d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi),$$

which shows that we have reduced the pointwise residue of the Fourier Integral Operator to the pointwise residue of a suitably chosen pseudo-differential operator and a rotation in the complex plane ϑ_x. In fact, the symbol of that pseudo-differential operator can be chosen to be the amplitude of the Fourier Integral Operator itself.

■

Part II

Integration in algebras of Fourier Integral Operators

CHAPTER 9

Bochner/Lebesgue integrals in algebras of Fourier Integral Operators

In this chapter, we will lay out the fundamental theorems of integration in topological vector spaces and algebras from our point of view directed to integration of families of Fourier Integral Operators. In particular, we will distinguish between two notions of measurability - those functions that are limits of simple functions and those whose pre-images of measurable sets are measurable. Note that for most Hörmander spaces $\mathcal{D}'_\Gamma$ (the set of distributions whose wave-front set is in the closed cone Γ) those will be different notions.

Definition 9.1. *Let (Ω, Σ, μ) be a measure space and E a topological vector space.*

(i) *A function $f \in E^\Omega$ is called simple if and only if $f[\Omega] \subseteq_{\text{finite}} E$ and $\forall \omega \in f[\Omega] \setminus \{0\} : \ [\{\omega\}]f \in \Sigma \ \wedge \ \mu([\{\omega\}]f) < \infty$ where $[A]f$ denotes the pre-set of A under f. We will use $\mathcal{S}(\mu; E)$ to denote the set of all simple functions.*

(ii) *Let $f := \sum_{\omega \in f[\Omega] \setminus \{0\}} \omega 1_{[\{\omega\}]f} \in \mathcal{S}(\mu; E)$. Then, we define the Bochner integral*

$$\int_\Omega f d\mu := \sum_{\omega \in f[\Omega] \setminus \{0\}} \omega \mu([\{\omega\}]f).$$

(iii) *A function $f \in E^\Omega$ is called measurable if and only if $\forall S \subseteq_{\text{open}} E : \ [S]f \in \Sigma$. We will use $\mathcal{M}(\mu; E)$ to denote the set of all measurable functions.*[1]

[1] $f \in \mathcal{M}(\mu; E)$ is also called Lebesgue measurable.

$f \in E^\Omega$ is called strongly measurable if and only if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions such that $s_n \to f$ μ-almost everywhere. We will use $\mathcal{SM}(\mu;E)$ to denote the set of all strongly measurable functions.[2]

(iv) *Let $f \in \mathcal{M}(\mu,E)$ such that $\forall x' \in E':\ x' \circ f \in L_1(\mu)$ and $I \in (E')^*$ defined by*

$$\forall x' \in E':\ I(x') = \int_\Omega x' \circ f d\mu.$$

Here E' denotes the topological dual of E and $(E')^*$ the algebraic dual of E'. If I is unique, then we will also use the notation $\int_\Omega f d\mu := I$.

f is called μ-Pettis-integrable if and only if I is unique and an element of E. In that case, we call I the Pettis integral of f.[3]

(v) *Let E be locally convex with semi-norms $(p_\iota)_{\iota\in I}$. For $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$, we define*

$$\mathcal{L}_p(\mu;E) := \{f \in \mathcal{M}(\mu;E);\ \forall \iota \in I:\ p_\iota \circ f \in L_p(\mu)\},$$

$$\mathcal{N}_p(\mu;E) := \left\{f \in \mathcal{L}_p(\mu;E);\ \forall \iota \in I:\ \|p_\iota \circ f\|_{L_p(\mu)} = 0\right\},$$

and

$$L_p(\mu;E) := \mathcal{L}_p(\mu;E)/_{\mathcal{N}_p(\mu;E)},$$

as well as the semi-norms

$$p_\iota^{L_p(\mu;E)}:\ \mathcal{L}_p(\mu;E) \to \mathbb{R}_{\geq 0}; f \mapsto \|p_\iota \circ f\|_{L_p(\mu)}.$$

We call $f \in L_p(\mu;E)$ p-integrable or just integrable if $p = 1$.

---

[2] $f \in \mathcal{SM}(\mu;E)$ is also called Bochner measurable.

[3] f is called μ-Dunford-integrable if and only if I is unique in $(E')^*$. In that case, we call I the Dunford integral of f.

REMARK Note that this notion extends the idea of parameter dependent Fourier Integral Operators in the sense of chapters 2.1.2 and 2.2 of **[63]**, as well as families of operators as in the index theorem for families. In both cases, we have a continuous function of operators $b \mapsto D_b$ where b ranges over some interval in the case of **[63]**, and for the family index we have a fibration[4] $M \to B$ and an operator D_b in each fiber M_b. Replacing the manifold/interval B by some more general measure space (Ω, Σ, μ) and relaxing the continuity assumption of $b \mapsto D_b$ to mere measurability (here we will only consider measurability with respect to the Borel σ-algebra in the target space/algebra), we can see that the formalism we are about to develop is a proper extension and we can think of a stochastic version of the index theorem for families, for instance. Furthermore, a stochastic version of the index theorem itself may be interesting because (as we will see) the pointwise index of a measurable functions is only locally constant on a dense set (at best) which does not imply that the function is continuous, let alone constant; in fact, the expectation of the pointwise index might not even be an integer (here, we may think of random manifolds allowing singular deformations like turning a sphere into a torus).

$\blacksquare$

It is obvious that a simple function is Pettis integrable if points in E are separated which itself is a direct consequence of Hahn-Banach's theorem given that E is locally convex (cf. Theorem 2.2 in **[74]** for part (i) and §20.7(2) in **[45]** for part (ii)).

THEOREM 9.2 (Hahn-Banach). *Let E be a topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $A, B \subseteq E$ both convex and non-empty, as well as $A \cap B = \varnothing$.*

[4] A fibration is a continuous map $\pi : X \to Y$ between topological spaces X and Y satisfying the homotopy lifting property for every topological space Z, i.e. for any homotopy $f : Z \times [0,1] \to Y$ and $f_0 : Z \to X$ such that $f(\cdot, 0) = \pi \circ f_0$ there exists a homotopy $\tilde{f} : Z \times [0,1] \to X$ such that $f = \pi \circ \tilde{f}$ and $f_0 = \tilde{f}(\cdot, 0)$.

(i) *If A is open, then there exists an $x' \in E'$ such that*

$$\forall x \in A: \ \Re x'(x) < \inf\{\Re x'(y); \ y \in B\} =: \gamma,$$

i.e. the hyperplane $\{x \in E; \ \Re x'(x) = \gamma\}$ separates A and B.

(ii) *Additionally, let E be locally convex, A compact, and B closed. Then, there exists an $x' \in E'$ such that*

$$\sup\{\Re x'(x); \ x \in A\} < \inf\{\Re x'(y); \ y \in B\}.$$

Since separation of points is a highly important property, we will assume from now on that E is a locally convex topological vector space which is also a Hausdorff space. In other words, the main issue is existence of the Pettis integral (which we will address in chapter 10) since it is the weaker notion of integrability, i.e. the minimum requirement for us to talk about integrals in algebras of Fourier Integral Operators. First, however, we will investigate the L_p spaces and (strong) integrals taking values in the completion of E; existence of the Pettis integral or suitable completeness assumptions on E will, thus, ensure that those (strong) integrals, in fact, take values in E.

Before we can start working with the Lebesgue integrals we should investigate which measurable functions are strongly measurable as many of the proofs for Lebesgue integrals will only work with strongly measurable functions.

LEMMA 9.3. *Let $f \in E^\Omega$ and $s \in \mathcal{S}(\mu; E)^{\mathbb{N}}$ such that $s_n \to f$ μ-almost everywhere. Then, $\forall S \subseteq_{\text{open}} E: \ [S]f \in \Sigma$. In other words, $\mathcal{SM}(\mu; E) \subseteq \mathcal{M}(\mu; E)$.*

PROOF. We observe

$$\begin{aligned} s_n \to f \ \mu\text{-almost everywhere} &\Leftrightarrow \forall \iota \in I: \ p_\iota \circ s_n \to p_\iota \circ f \ \mu\text{-almost everywhere} \\ &\Rightarrow \forall \iota \in I: \ p_\iota \circ f \in \mathcal{M}(\mu; \mathbb{R}) \end{aligned}$$

$$\Leftrightarrow \forall \iota \in I \ \forall S \subseteq_{\text{open}} \mathbb{R} : \ [[S]p_\iota] f = [S](p_\iota \circ f) \in \Sigma.$$

And since $\{[S]p_\iota;\ S \subseteq_{\text{open}} \mathbb{R}\}$ generates the topology in E, the last line is equivalent to

$$\forall S \subseteq_{\text{open}} E : \ [S]f \in \Sigma.$$

□

With the same proof, simply replacing $s_n \in \mathcal{S}(\mu; E)$ by $s_n \in \mathcal{M}(\mu; E)$, we obtain the following corollary.

COROLLARY 9.4. *$\mathcal{M}(\mu; E)$ is sequentially closed with respect to μ-almost everywhere convergence. In other words, let $f \in E^\Omega$ and $s \in \mathcal{M}(\mu; E)^\mathbb{N}$ such that $s_n \to f$ μ-almost everywhere. Then, $f \in \mathcal{M}(\mu; E)$.*

LEMMA 9.5 (Sombrero Lemma). *Let E be metrizable, Ω compact, Σ the Borel σ-algebra, and μ a measure on (Ω, Σ).*

Let $f \in C(\Omega, E)$. Then, there exists $s \in \mathcal{S}(\mu; E)^\mathbb{N}$ such that $s_n \to f$ pointwise, i.e. $C(\Omega, E) \subseteq \mathcal{SM}(\mu; E)$.

PROOF. Let d be a metric on E which generates the topology. For $\varepsilon \in \mathbb{R}_{>0}$, the open balls $B_{(E,d)}(f(\omega), \varepsilon)$ $(\omega \in \Omega)$ are an open cover of $f[\Omega]$ which is a compact subset of E. Hence, there exists a finite set $\Omega_\varepsilon \subseteq_{\text{finite}} \Omega$ such that $f[\Omega] \subseteq \bigcup_{\omega \in \Omega_\varepsilon} B_{(E,d)}(f(\omega), \varepsilon)$. Let $n_\varepsilon := \#\Omega_\varepsilon$ be the cardinality of Ω_ε, $(\omega_{\varepsilon,j})_{j \in \mathbb{N}_{\leq n_\varepsilon}}$ an enumeration of Ω_ε, $A_{\varepsilon,j} := \left[B_{(E,d)}(f(\omega_{\varepsilon,j}), \varepsilon)\right] f \setminus \bigcup_{i=1}^{j-1} A_{\varepsilon,i}$, and

$$s_\varepsilon := \sum_{j=1}^{n_\varepsilon} f(\omega_{\varepsilon,j}) 1_{A_{\varepsilon,j}}.$$

Then, we obtain

$$\forall \omega \in \Omega \ \exists j \in \mathbb{N}_{\leq n_\varepsilon} : \ s_\varepsilon(\omega), f(\omega) \in B_{(E,d)}(f(\omega_{\varepsilon,j}), \varepsilon).$$

In other words,

$$\forall \omega \in \Omega : \; d\left(s_{\varepsilon}(\omega), f(\omega)\right) < 2\varepsilon$$

implies that $\left(s_{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ converges μ-almost everywhere to f.

□

REMARK Note that a Hausdorff topological vector space is metrizable if and only if it is first-countable[5] (cf. §15.11(1) in [**45**]), i.e. replacing the balls $B_{(E,d)}\left(f(\omega), \frac{1}{n}\right)$ by a countable local base will not generalize the lemma.

■

DEFINITION 9.6. *Let* (Ω, Σ, μ) *be a measure space and* E *a topological vector space. Then,* $(\Omega, \Sigma, \mu; E)$ *is called a Sombrero space if and only if* $\mathcal{SM}(\mu; E) = \mathcal{M}(\mu; E)$.

EXAMPLE If $E = \mathbb{R}$, then the usual Sombrero Lemma (cf. e.g. Theorem 8.8 in [**65**]) shows that $(\Omega, \Sigma, \mu; E)$ is a Sombrero space (independent of the choice of (Ω, Σ, μ)).

■

Lusin's measurability theorem (cf [**53**] and Theorem 2B in [**24**]) yields a useful extension of the Sombrero Lemma.

THEOREM 9.7 (Lusin). *Let* (Ω, Σ, μ) *be a Radon measure*[6] *space,* E *a second-countable*[7] *topological space,* $f : \Omega \to E$ *measurable,* $\varepsilon \in \mathbb{R}_{>0}$*, and* $S \in \Sigma$ *with* $\mu(S) < \infty$.

---

[5]Every point has a countable neighborhood basis, that is, for every point x there exists a countable set U of open neighborhoods of x such that for every neighborhood V of x there exists $U_0 \in U$ satisfying $U_0 \subseteq V$.

[6]Radon measures are locally finite (every point has a neighborhood of finite measure) and regular (every Borel sets B satisfies $\mu(B) = \sup_{K \subseteq_{\text{compact}} B} \mu(K) = \inf_{O \supseteq_{\text{open}} B} \mu(O)$) Borel measures.

[7]The topology has a countable base, i.e. there exists a countable set U of open subsets of E such that U contains a neighborhood basis for every point in E.

Then, there exists a closed set $C_\varepsilon \subseteq \Omega$ such that $\mu(S \setminus C_\varepsilon) < \varepsilon$ and $f|_{C_\varepsilon}$ is continuous.

If E is a topological vector space (thus, separable[8] *and metrizable), then we can choose C_ε to be compact.*

LEMMA 9.8 (Generalized Sombrero Lemma). *Let (Ω, Σ, μ) be a Radon measure space and E a separable metric space. Then, $(\Omega, \Sigma, \mu; E)$ is a Sombrero space.*

PROOF. Let $f \in \mathcal{M}(\mu; E)$. For $n \in \mathbb{N}$, Lusin's measurability theorem warrants the existence of compact sets $\Omega_n \subseteq \Omega$ such that $f|_{\Omega_n}$ is continuous and $\mu(\Omega \setminus \Omega_n) < \frac{1}{n}$. Furthermore, we may assume that $\Omega_m \subseteq \Omega_n$ for $m \leq n$. The Sombrero Lemma, then, implies that there exists an $s_n \in \mathcal{S}(\mu; E)$ with $(d \circ (s_n, f))|_{\Omega_n} < \frac{1}{n}$ where d denotes a metric on E generating the topology. In other words, $s_n(\omega) \to f(\omega)$ for every $\omega \in \bigcup_{n \in \mathbb{N}} \Omega_n$ and

$$\forall n \in \mathbb{N}: \ \mu\left(\Omega \setminus \bigcup_{m \in \mathbb{N}} \Omega_m\right) \leq \mu(\Omega \setminus \Omega_n) < \frac{1}{n}$$

shows that $s_n \to f$ μ-almost everywhere.

□

LEMMA 9.9. *$L_p(\mu; E)$ is a Hausdorff space.*

PROOF. Let $x, y \in L_p(\mu; E)$, $x \neq y$. Then, there exists $\iota \in I$ such that

$$p_\iota^{L_p(\mu;E)}(x - y) =: 2\delta > 0.$$

---

[8]For metric spaces separability and second-countability are equivalent. Note that every second-countable space is separable since choosing a countable base $\{U_n;\ n \in \mathbb{N}\}$ of the topology and $x_n \in U_n$ yields a dense sequence, i.e. proves separability of the space. The other implication follows since $\left\{B\left(x_n, \frac{1}{n}\right);\ n \in \mathbb{N}\right\}$ is a countable base of the topology given that $(x_n)_{n \in \mathbb{N}}$ is dense; cf. chapter I.5 in [**70**].

Hence, the neighborhoods

$$U := \left\{z \in L_p(\mu; E);\ p_\iota^{L_p(\mu;E)}(x - z) < \delta\right\}$$

of x and

$$V := \left\{z \in L_p(\mu; E);\ p_\iota^{L_p(\mu;E)}(y - z) < \delta\right\}$$

of y are open and disjoint.

□

DEFINITION 9.10. *Let E and F be locally convex topological vector spaces with semi-norms $(p_\iota^E)_{\iota \in I_E}$ and $(p_\iota^F)_{\iota \in I_F}$, respectively, and $A\colon\ E \to F$ a linear operator. A is said to be continuous if and only if*

$$\forall \iota \in I_F\ \exists \kappa \in I_E\ \exists c \in \mathbb{R}_{\geq 0} \forall x \in E:\ p_\iota^F(Ax) \leq c p_\kappa^E(x).$$

We will denote the set of all continuous linear operators mapping E to F by $L(E, F)$ and the minimal c satisfying the condition by $\|A\|_{\iota\kappa}$.

In an algebra $\mathcal{A}$, we will assume that the composition is a continuous operator, i.e.

$$\forall \iota \in I\ \exists \kappa, \lambda \in I\ \exists c \in \mathbb{R}_{\geq 0}\ \forall A, B \in \mathcal{A}:\ p_\iota(A \circ B) \leq c p_\kappa(A) p_\lambda(B).$$

The minimal constant c will also be denoted by $\|\circ\|_{\iota,\kappa,\lambda}$.

THEOREM 9.11 (Hölder's inequality). *Let $A_i \in L_{p_i}(\mu; \mathcal{A})$ for $i \in \mathbb{N}_{\leq n}$ and $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{r}$. Then, $A_1 \circ A_2 \circ \ldots \circ A_n \in L_r(\mu; \mathcal{A})$ and*

$$\forall \iota \in I\ \exists \kappa \in I^n\ \exists c \in \mathbb{R}_{\geq 0}:\ p_\iota^{L_r(\mu;\mathcal{A})}(A_1 \circ A_2 \circ \ldots \circ A_n) \leq c \prod_{j=1}^n p_{\kappa_j}^{L_{p_j}(\mu;\mathcal{A})}(A_j).$$

PROOF. First, let $n = 2$. We need to prove

$$\forall \iota \in I: \; p_\iota \circ A_1 \circ A_2 \in L_r(\mu).$$

However, we know that $\forall \iota \in I: \; p_\iota \circ A_1 \in L_{p_1}(\mu) \;\wedge\; p_\iota \circ A_2 \in L_{p_2}(\mu)$.

For $p_1 = \infty$ or $p_2 = \infty$, the usual Hölder inequality yields

$$p_\iota \circ A_1 \circ A_2 \le \|\circ\|_{\iota,\kappa,\lambda} \underbrace{(p_\kappa \circ A_1)}_{\in L_{p_1}(\mu)} \underbrace{(p_\lambda \circ A_2)}_{\in L_{p_2}(\mu)} \in L_r(\mu)$$

for some $\kappa, \lambda \in I$.

Let $p_1, p_2 < \infty$, $p := \frac{p_2}{p_2 - r}$, and $q := \frac{p_2}{r}$. Then, $\frac{1}{rp} = \frac{p_2 - r}{p_2 r} = \frac{1}{r} - \frac{1}{p_2} = \frac{1}{p_1}$, i.e. $(p_\iota \circ A_1)^r \in L_p(\mu)$ and $(p_\iota \circ A_2)^r \in L_q(\mu)$, and we obtain

$$\begin{aligned}
\|p_\iota \circ A_1 \circ A_2\|_{L_r(\mu)} &= \left\|(p_\iota \circ A_1 \circ A_2)^r\right\|_{L_1(\mu)}^{\frac{1}{r}} \\
&\le \|\circ\|_{\iota,\kappa,\lambda} \left\|(p_\kappa \circ A_1)^r (p_\lambda \circ A_2)^r\right\|_{L_1(\mu)}^{\frac{1}{r}} \\
&\le \|\circ\|_{\iota,\kappa,\lambda} \left\|(p_\kappa \circ A_1)^r\right\|_{L_p(\mu)}^{\frac{1}{r}} \left\|(p_\lambda \circ A_2)^r\right\|_{L_q(\mu)}^{\frac{1}{r}} \\
&= \|\circ\|_{\iota,\kappa,\lambda} \|p_\kappa \circ A_1\|_{L_{p_1}(\mu)} \|p_\lambda \circ A_2\|_{L_{p_2}(\mu)}
\end{aligned}$$

for some $\kappa, \lambda \in I$. Hence, $A_1 \circ A_2 \in L_r(\mu; \mathcal{A})$.

For more general n, we assume that the assertion holds for $n - 1$. Let $B_1 := A_1 \circ \ldots \circ A_{n-1}$ and $B_2 := A_n$. Let $q_2 := p_n$ and $\frac{1}{q_1} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. Then, $B_1 \in L_{q_1}(\mu; \mathcal{A})$ by the inductive assumption and

$$A_1 \circ \ldots \circ A_n = B_1 \circ B_2 \in L_r(\mu; \mathcal{A})$$

since we have proven the assertion for the $n = 2$ case.

□

LEMMA 9.12.

$$\int : \; \mathcal{S}(\mu; E) \subseteq L_1(\mu; E) \to E; \; f \mapsto \int_\Omega f d\mu$$

is a continuous linear operator; more precisely, we have the triangle-inequalities

$$\forall \iota \in I \ \forall f \in \mathcal{S}(\mu; E) : \ p_\iota\left(\int_\Omega f d\mu\right) \leq \int_\Omega p_\iota \circ f d\mu.$$

Furthermore, if E is separable, (Ω, Σ, μ) is σ-finite, and $p < \infty$, then $\mathcal{S}(\mu; E)$ is dense in $L_p(\mu; E)$. The same holds for $p = \infty$ and $\mu(\Omega) < \infty$. In particular, the integral extends uniquely to a continuous linear operator

$$\int : \ L_1(\mu; E) \to \tilde{E}$$

where $\tilde{E}$ is the completion of E.

Proof. Linearity of $\int$ and

$$\forall \iota \in I \ \forall f \in \mathcal{S}(\mu; E) : \ p_\iota\left(\int_\Omega f d\mu\right) \leq \int_\Omega p_\iota \circ f d\mu$$

are trivial.

Now, let Ω be σ-finite and $f \in L_p(\mu; E)$. We can find $\Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq \Omega$ such that $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ and $\forall n \in \mathbb{N} : \ \mu(\Omega_n) < \infty$ and we obtain $1_{\Omega_n} f \to f$ in $L_p(\mu; E)$ for $p < \infty$, i.e. we may assume without loss of generality that Ω is finite.

Let $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ be a dense sequence and $\mathcal{U}_0$ the neighborhood filter of zero in E. For $n \in \mathbb{N}$ and $U \in \mathcal{U}_0$ let

$$V_n^U := [x_n + U] f \setminus \left(\bigcup_{k=1}^{n-1} V_k^U\right).$$

Then, we define

$$f_U := \sum_{n\in\mathbb{N}} x_n 1_{V_n^U}.$$

f_U is obviously measurable and the net $(f_U(\omega))_{U\in\mathcal{U}_0}$ converges to $f(\omega)$ for μ-almost every $\omega \in \Omega$.

Let $\iota \in I$ and $\varepsilon \in \mathbb{R}_{>0}$. Then, $U_\iota^\varepsilon := \{x \in E;\ p_\iota^E(x) < \varepsilon\}$ is in $\mathcal{U}_0$ and for every $U \in \mathcal{U}_0$ with $U \subseteq U_\iota^\varepsilon$

$$p_\iota^E(f_U(\omega) - f(\omega)) \leq \varepsilon$$

holds for μ-almost every $\omega \in \Omega$. In particular, $p_\iota^E \circ (f_U - f) \in L_p(\mu)$ and

$$p_\iota^{L_p(\mu;E)}(f_U - f) \leq \varepsilon \max\left\{1, \mu(\Omega)^{\frac{1}{p}}\right\}$$

where 1 is the $p = \infty$ case. Hence, $(f_U)_{U \in \mathcal{U}_0}$ converges to f in $L_p(\mu; E)$. Finally,

$$p_\iota^{L_p(\mu;E)}\left(f_U - \sum_{k=1}^{n} x_k 1_{V_k^U}\right) \leq \sum_{k \in \mathbb{N}_{>n}} p_\iota^E(x_k)\, \mu\left(V_k^U\right)^{\frac{1}{p}} \to 0\ (n \to \infty)$$

for $p < \infty$ and

$$p_\iota^{L_p(\mu;E)}\left(f_U - \sum_{k=1}^{n} x_k 1_{V_k^U}\right) \leq \sum_{k \in \mathbb{N}_{>n}} p_\iota^E(x_k) \to 0\ (n \to \infty)$$

for $p = \infty$ show $f_U \in \overline{\mathcal{S}(\mu; E)}^{L_p(\mu;E)}$ and, hence, $f \in \overline{\mathcal{S}(\mu; E)}^{L_p(\mu;E)}$.

The existence of the unique extension of $\int$ follows directly from the fact that any uniformly continuous[9] function $f:\ Y_0 \subseteq Y \to H$ has a unique uniformly continuous extension to the closure of Y_0 in Y, where Y is any topological vector space and Y_0 any subset and H is any complete Hausdorffian topological vector space (cf. Theorem 2.6 in [**1**]). Linearity follows from taking two nets $x_\alpha \to: x$ and $y_\beta \to: y$, as well as $\lambda \in \mathbb{K}$, and observing

$$f(x + \lambda y) \leftarrow f(x_\alpha + \lambda y_\beta) = f(x_\alpha) + \lambda f(y_\beta) \to f(x) + \lambda f(y).$$

□

[9]Let E and F be topological vector spaces. $f:\ E_0 \subseteq E \to F$ is called uniformly continuous if and only if for every open neighborhood V of zero in F there exists an open neighborhood U of zero in E such that $\forall x, y \in E_0:\ (x - y \in U\ \Rightarrow\ f(x) - f(y) \in V)$.

REMARK If $L_1(\mu;E) \subseteq \mathcal{SM}(\mu;E)$ (in particular, if (Ω,Σ,μ) a Sombrero space), then we do not need the separability assumption on E because $f[\Omega]$ is contained in a separable subspace of E.

■

DEFINITION 9.13. *The subspace*

$$\mathcal{S}L_p(\mu;E) := \mathcal{SM}(\mu;E) \cap L_p(\mu;E) \subseteq L_p(\mu;E)$$

is called the strong $L_p(\mu;E)$. *Furthermore, we define*

$$\begin{aligned}\overline{\mathcal{S}}\mathcal{M}(\mu;E) &:= \overline{\mathcal{SM}(\mu;E)}^{\mathcal{M}(\mu;E)\ \mu\text{-almost everywhere}}\\ &= \{f \in \mathcal{M}(\mu;E);\ \exists\,(f_\alpha)_\alpha\ \text{net}:\ f_\alpha \to f\ \mu\text{-almost everywhere}\}\end{aligned}$$

and

$$\overline{\mathcal{S}}L_p(\mu;E) := \overline{\mathcal{S}L_p(\mu;E)}^{L_p(\mu;E)}.$$

In other words, $\mathcal{S}L_p(\mu;E)$ is the sequential closure of the set of simple functions in $L_p(\mu;E)$ and $\overline{\mathcal{S}}L_p(\mu;E)$ is the closure of set of simple functions in $L_p(\mu;E)$.

EXAMPLE Let Ω be a compact space, Σ the induced Borel σ-algebra, and μ a finite measure. Since Σ is the Borel σ-algebra, we obtain $C(\Omega;E) \subseteq \mathcal{M}(\mu;E)$ and Ω being compact implies $C(\Omega;E) \subseteq L_\infty(\mu;E)$. Furthermore, μ being finite implies $L_\infty(\mu;E) \subseteq L_1(\mu;E)$. In other words,

$$C(\Omega;E) \subseteq L_1(\mu;E).$$

If E is metrizable or separable, then $C(\Omega;E) \subseteq \mathcal{S}L_1(\mu;E)$.

■

THEOREM 9.14 (Fischer-Riesz). *Let E be a Fréchet space, (Ω,Σ,μ) be σ-finite, and $p < \infty$. Then, $L_p(\mu;E)$ is complete, i.e. a Fréchet space, and every Cauchy-sequence contains a μ-almost everywhere convergent sub-sequence.*

PROOF. Let $(f_n)_{n\in\mathbb{N}} \in L_p(\mu;E)^{\mathbb{N}}$ be a Cauchy-sequence and $\iota \in I$. Choose a sub-sequence $(f_{n_j})_{j\in\mathbb{N}}$ such that

$$\forall j \in \mathbb{N}: \; p_\iota^{L_p(\mu;E)}\left(f_{n_{j+1}} - f_{n_j}\right) \le 2^{-j}$$

and let $f_{n_0} := 0$. For $j \in \mathbb{N}$, let $g_j := f_{n_j} - f_{n_{j-1}}$. Then, we obtain for $n \in \mathbb{N}$

$$\begin{aligned}\left(\int_\Omega \left(\sum_{k=1}^n p_\iota^E \circ g_k\right)^p d\mu\right)^{\frac{1}{p}} &= \left\|\sum_{k=1}^n p_\iota^E \circ g_k\right\|_{L_p(\mu)} \\ &\le \sum_{k=1}^n \left\|p_\iota^E \circ g_k\right\|_{L_p(\mu)} \\ &= \sum_{k=1}^n p_\iota^{L_p(\mu;E)}(g_k) \\ &< 1\end{aligned}$$

and

$$\left(\sum_{k=1}^n p_\iota^E \circ g_k\right)^p \nearrow \left(\sum_{k\in\mathbb{N}} p_\iota^E \circ g_k\right)^p.$$

Hence,

$$\int_\Omega \left(\sum_{k\in\mathbb{N}} p_\iota^E \circ g_k\right)^p d\mu < \infty.$$

In particular, $g(\omega) := \sum_{k\in\mathbb{N}} p_\iota^E(g_k(\omega)) < \infty$ for μ-almost every $\omega \in \Omega$ and $f_\iota := \sum_{k\in\mathbb{N}} g_k$ converges absolutely with respect to p_ι^E for these ω. Then,

$$\underbrace{p_\iota^E\left(f_\iota(\omega) - \sum_{k=1}^n g_k(\omega)\right)}_{\to 0} \le \sum_{k\in\mathbb{N}_{>n}} p_\iota^E(g_k(\omega)) \le g(\omega)$$

wherever $g(\omega) < \infty$ and the theorem of dominated convergence implies

$$p_\iota^{L_p(\mu;E)}(f_\iota - f_n) = p_\iota^{L_p(\mu;E)}\left(f_\iota - \sum_{k=1}^n g_k\right) \to 0 \quad (n \to \infty).$$

Finally, let $\kappa \in I$. The same argument with p_κ applied to the sub-sequence constructed with p_ι, then, shows that $f_\iota = f_\kappa$ μ-almost everywhere and $f_n \to f_\iota$ in with respect to $p_\iota^{L_p(\mu;E)}$ and $p_\kappa^{L_p(\mu;E)}$. Inductively, we continue thinning the subsequences such that the diagonal sequence converges μ-almost everywhere to some f with respect to all p_ι^E and $f_n \to f$ in $L_p(\mu;E)$.

□

The following lemma aims at the composition of Fourier Integral Operators, i.e. if we have an algebra $\mathcal{A}$, $A \in \mathcal{A}$, and $f:\ \Omega \to \mathcal{A}$, then we would like to obtain

$$A\int_\Omega f d\mu = \int_\Omega A f d\mu.$$

However, we only know that $\int_\Omega f d\mu$ is in the closure of $\mathcal{A}$ (which might be quite bad). Since we assumed that the composition is continuous though, we can extend the operator $A\circ$ to the completion $\tilde{\mathcal{A}}$ of $\mathcal{A}$. The lemma also shows that the Bochner and Lebesgue integrals $\int_\Omega f d\mu$ coincides with the Pettis integral if f is μ-Pettis integrable; thus, legitimizing the clash of notation and ensuring that the integral itself is an element of the algebra, again.

LEMMA 9.15. *Let (Ω,Σ,μ) be σ-finite, F another Hausdorffian locally convex topological vector space, and $f \in \overline{\mathcal{S}}L_1(\mu;E)$.*

(i) *Let $B \in L(\tilde{E},\tilde{F})$. Then, $B\circ f \in \overline{\mathcal{S}}L_1(\mu;F)$ and*

$$B\int_\Omega f d\mu = \int_\Omega B\circ f d\mu.$$

(ii) *Let $E_0 \subseteq E$ be a closed subspace and $f(\omega) \in E_0$ for μ-almost every $\omega \in \Omega$. Then, $\int_\Omega f d\mu \in E_0$.*

PROOF. (i) For $f \in \mathcal{S}(\mu;E)$,

$$B\int_\Omega f d\mu = \int_\Omega B\circ f d\mu$$

is trivial. Furthermore, the functions

$$L_1(\mu; E) \ni f \mapsto B \int_\Omega f d\mu \in \tilde{F}$$

and

$$L_1(\mu; E) \ni f \mapsto \int_\Omega B \circ f d\mu \in \tilde{F}$$

are linear and continuous because of

$$p_\kappa^F \left(B \int_\Omega f d\mu \right) \leq \|B\|_{\kappa,\iota} \, p_\iota^E \left(\int_\Omega f d\mu \right) \leq \|B\|_{\kappa,\iota} \, p_\iota^{L_1(\mu;E)}(f)$$

and

$$\begin{aligned} p_\kappa^F \left(\int_\Omega B \circ f d\mu \right) &\leq \int_\Omega p_\kappa^F (Bf(\omega)) d\mu(\omega) \\ &\leq \int_\Omega \|B\|_{\kappa,\iota} \, p_\iota^E \circ f d\mu \\ &= \|B\|_{\kappa,\iota} \, p_\iota^{L_1(\mu;E)}(f) \end{aligned}$$

for some ι depending on κ. Hence, we obtain

$$B \int_\Omega f d\mu = \int_\Omega B \circ f d\mu$$

on $L_1(\mu; E)$ by the unique extension property.

(ii) Let $\varphi \in E'$ with $\varphi|_{E_0} = 0$. Then,

$$\varphi \left(\int_\Omega f d\mu \right) = \int_\Omega \underbrace{\varphi \circ f}_{=0} \, d\mu = 0.$$

Hence, $\int_\Omega f d\mu \in E_0$ by Hahn-Banach's theorem (otherwise there exists a $\varphi \in E'$ with $\varphi \left(\int_\Omega f d\mu \right) = 1$).

□

Theorem 9.16 (Hille). *Let $f \in \overline{\mathcal{S}} L_1(\mu; E)$, F another Hausdorffian locally convex topological vector space, and $A : D(A) \subseteq E \to F$ a closed linear operator* *(that is, $A \subseteq E \times F$ is a closed subspace). Let $f(\omega) \in D(A)$ for μ-almost every $\omega \in \Omega$ and $A \circ f \in \overline{\mathcal{S}}L_1(\mu; F)$. Then, we obtain $\int_\Omega f d\mu \in D(A)$ and $A \int_\Omega f d\mu = \int_\Omega A \circ f d\mu$.*

Proof. Since the injections

$$i_E:\ E \to E \times F;\ x \mapsto (x, 0)$$

and

$$i_F:\ F \to E \times F;\ y \mapsto (0, y)$$

are continuous, it follows that $\Omega \ni \omega \mapsto (f(\omega), Af(\omega)) = i_E(f(\omega)) + i_F(Af(\omega))$ is in $\overline{\mathcal{S}}L_1(\mu; E \times F)$ and, since A is a closed linear subspace and μ-almost every $\omega \in \Omega$ satisfies $(f(\omega), Af(\omega)) \in A$, we obtain

$$\int_\Omega (f(\omega), Af(\omega)) d\mu(\omega) \in A.$$

Let

$$\mathrm{pr}_E:\ E \times F \to E;\ (x, y) \mapsto x$$

and

$$\mathrm{pr}_F:\ E \times F \to F;\ (x, y) \mapsto y.$$

Then,

$$\mathrm{pr}_E \int_\Omega (f(\omega), Af(\omega)) d\mu(\omega) = \int_\Omega \mathrm{pr}_E (f(\omega), Af(\omega)) d\mu(\omega) = \int_\Omega f d\mu$$

and

$$\mathrm{pr}_F \int_\Omega (f(\omega), Af(\omega)) d\mu(\omega) = \int_\Omega \mathrm{pr}_F (f(\omega), Af(\omega)) d\mu(\omega) = \int_\Omega A \circ f d\mu$$

yield

$$\left(\int_\Omega f d\mu, \int_\Omega A \circ f d\mu \right) = \int_\Omega (f(\omega), Af(\omega)) d\mu(\omega) \in A,$$

i.e.

$$\int_\Omega f d\mu \in D(A) \ \wedge \ A\int_\Omega f d\mu = \int_\Omega A \circ f d\mu.$$

□

COROLLARY 9.17. *Let $f \in \mathcal{S}L_1(\mu;E)$, F another Hausdorffian locally convex topological vector space, and $A\colon\ D(A) \subseteq E \to F$ a sequentially closed linear operator (that is, $A \subseteq E \times F$ is a sequentially closed subspace). Let $f(\omega) \in D(A)$ for μ-almost every $\omega \in \Omega$ and $(s_n)_{n\in\mathbb{N}} \in \mathcal{S}(\mu;D(A))^{\mathbb{N}}$ a sequence of simple functions approximating f μ-almost everywhere such that $A\circ s_n \to A\circ f$ μ-almost everywhere. Then, we obtain $\int_\Omega f d\mu \in D(A)$ and $A\int_\Omega f d\mu = \int_\Omega A\circ f d\mu$.*

PROOF. From Hille's theorem, we directly obtain $\left(\int_\Omega f d\mu, \int_\Omega A\circ f d\mu\right) \in \overline{A}$. However, $\left(\int_\Omega f d\mu, \int_\Omega A\circ f d\mu\right)$ is the limit of the sequence $\left(\int_\Omega s_n d\mu, \int_\Omega A\circ s_n d\mu\right)_{n\in\mathbb{N}}$ in A and A being sequentially closed yields the assertion.

□

For the rest of this chapter, we will develop some fundamental theorems allowing us to actually use this integral.

THEOREM 9.18 (Fundamental Theorem of Calculus). *Let $J \subseteq \mathbb{R}$ be an interval.*

(i) *Let $f \in C^1(J;E)$, $a,b \in J$, $a < b$, and λ the Lebesgue measure. Then,*

$$\int_{[a,b]} f' d\lambda = f(b) - f(a).$$

(ii) *Let $f \in C(J;E)$ and $x \in J$ such that $g\colon\ J \to E;\ t \mapsto \int_x^t f(s)d\lambda(s)$. Then, g is differentiable and $g' = f$.*

PROOF. (i) Let $\varphi \in E'$. Then, $\varphi \circ f \in C^1(J)$ and $(\varphi \circ f)' = \varphi \circ f'$. Hence, the classical fundamental theorem of calculus yields

$$\varphi\left(\int_{[a,b]} f' d\lambda - (f(b) - f(a))\right) = \int_{[a,b]} (\varphi \circ f)' d\lambda - (\varphi \circ f(b) - \varphi \circ f(a)) = 0.$$

(ii) Let $x \in J$ and $h \in \mathbb{R} \setminus \{0\}$ such that $B_{\mathbb{R}}[x, |h|] \subseteq J$, as well as $\iota \in I$. Then, we obtain

$$\begin{aligned}
p_\iota\left(\frac{g(x+h) - g(x)}{h} - f(x)\right) =& p_\iota\left(\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\right) \\
=& p_\iota\left(\frac{1}{h}\int_x^{x+h} f(t) - f(x)dt\right) \\
\leq& \frac{1}{|h|} p_\iota\left(\int_x^{x+h} f(t) - f(x)dt\right) \\
\leq& \frac{1}{|h|} |h| \sup\left\{p_\iota\left(f(t) - f(x)\right);\ t \in B_{\mathbb{R}}[x, |h|]\right\} \\
\to& 0 \quad (h \searrow 0)
\end{aligned}$$

since f is continuous.

□

PROPOSITION 9.19 (Dominated Convergence). *Let $u \in E^\Omega$ be the pointwise limit (μ-almost everywhere) of $(u_j)_{j\in\mathbb{N}} \in L_1(\mu; E)^{\mathbb{N}}$ and*

$$\forall \iota \in I\ \exists v_\iota \in L_1(\mu; E)\ \forall j \in \mathbb{N}:\ p_\iota \circ u_j \leq p_\iota \circ v_\iota.$$

Then, $u \in \mathcal{M}(\mu; E)$, and

$$(*) \qquad \lim_{j\to\infty} \int_\Omega p_\iota \circ (u_j - u) d\mu = 0$$

and

$$(**) \qquad \lim_{j\to\infty} \int_\Omega p_\iota \circ u_j d\mu = \int_\Omega p_\iota \circ u d\mu$$

hold. In particular, $u \in L_1(\mu; E)$.

Proof. Since u is the pointwise limit (μ-almost everywhere) of a sequence of measurable functions, it is measurable, as well. Then, $(*)$ and $(**)$ follow directly from the usual dominated convergence theorem and $(**)$ implies that $u \in L_1(\mu;E)$. □

Lemma 9.20 (L_p-Dominated Convergence). *Let $u \in E^\Omega$ be the pointwise limit (μ-almost everywhere) of $(u_j)_{j\in\mathbb{N}} \in L_p(\mu;E)^{\mathbb{N}}$ with $p \in \mathbb{R}_{\geq 1}$ and*

$$\forall \iota \in I\ \exists v_\iota \in L_1(\mu;E)\ \forall j \in \mathbb{N}:\ p_\iota \circ u_j \leq p_\iota \circ v_\iota.$$

Then, $u \in L_p(\mu;E)$, $u_j \to u$ in $L_p(\mu;E)$, and $\forall \iota \in I:\ p_\iota^{L_p(\mu;E)}(u_j) \to p_\iota^{L_p(\mu;E)}(u)$.

Proof. The assertion $u_j \to u$ μ-almost everywhere implies $u \in \mathcal{M}(\mu;E)$. Then, L_p-dominated convergence theorem in $\mathbb{R}$ yields $u \in L_p(\mu;E)$, $\forall \iota \in I$: $p_\iota^{L_p(\mu;E)}(u_j) \to p_\iota^{L_p(\mu;E)}(u)$, and $\forall \iota \in I:\ p_\iota^E \circ u_j \to p_\iota^E \circ u$ in $L_p(\mu)$. Finally,

$$p_\iota^{L_p(\mu;E)}(u_j - u) = \left\|p_\iota^E \circ (u_j - u)\right\|_{L_p(\mu)} = \left\|\left(p_\iota^E \circ (u_j - u)\right)^p\right\|_{L_1(\mu)}^{\frac{1}{p}}$$

converges to zero because $\left(p_\iota^E \circ (u_j - u)\right)^p \to 0$ μ-almost everywhere and the convergence is dominated by $(2p_\iota \circ v_\iota)^p$. □

Theorem 9.21 (Riesz). *Let $(u_j)_{j\in\mathbb{N}} \in L_p(\mu;E)^{\mathbb{N}}$, $u \in L_p(\mu;E)$, and $u_j \to u$ μ-almost everywhere. Then,*

$$u_j \to u \text{ in } L_p(\mu;E) \iff \forall \iota \in I:\ p_\iota^{L_p(\mu;E)}(u_j) \to p_\iota^{L_p(\mu;E)}(u).$$

Proof. We have $u_j \to u$ in $L_p(\mu;E)$ if and only if $\forall \iota \in I:\ p_\iota^{L_p(\mu;E)}(u_j - u) \to 0$. Thus, "⇒" holds by reversed triangle inequality

$$\left|p_\iota^{L_p(\mu;E)}(u_j) - p_\iota^{L_p(\mu;E)}(u)\right| \leq p_\iota^{L_p(\mu;E)}(u_j - u) \to 0.$$

"$\Leftarrow$" Since $x \mapsto x^p$ is convex for $p \geq 1$ on $\mathbb{R}_{\geq 0}$, we obtain

$$\forall a, b \in \mathbb{R}_{\geq 0} : \left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}$$

and, hence,

$$\left(p_\iota^E \circ (u_j - u)\right)^p \leq \left(p_\iota^E \circ u_j + p_\iota^E \circ u\right)^p \leq 2^{p-1}\left(\left(p_\iota^E \circ u_j\right)^p + \left(p_\iota^E \circ u\right)^p\right)$$

which implies

$$2^{p-1}\left(\left(p_\iota^E \circ u_j\right)^p + \left(p_\iota^E \circ u\right)^p\right) - \left(p_\iota^E \circ (u_j - u)\right)^p \geq 0.$$

Thus, using Fatou's lemma, we obtain

$$\begin{aligned}
&2^p p_\iota^{L_p(\mu;E)}(u)^p \\
&= \int_\Omega \liminf_{j\to\infty} 2^{p-1}\left(\left(p_\iota^E \circ u_j\right)^p + \left(p_\iota^E \circ u\right)^p\right) - \left(p_\iota^E \circ (u_j - u)\right)^p \, d\mu \\
&\leq \liminf_{j\to\infty}\left(\int_\Omega 2^{p-1}\left(p_\iota^E \circ u_j\right)^p d\mu + \int_\Omega 2^{p-1}\left(p_\iota^E \circ u\right)^p d\mu - \int_\Omega \left(p_\iota^E \circ (u_j - u)\right)^p d\mu\right) \\
&= \liminf_{j\to\infty}\left(2^{p-1} p_\iota^{L_p(\mu;E)}(u_j)^p + 2^{p-1} p_\iota^{L_p(\mu;E)}(u)^p - p_\iota^{L_p(\mu;E)}(u_j - u)^p\right) \\
&= 2^{p-1} p_\iota^{L_p(\mu;E)}(u)^p + 2^{p-1} p_\iota^{L_p(\mu;E)}(u)^p - \limsup_{j\to\infty} p_\iota^{L_p(\mu;E)}(u_j - u)^p \\
&= 2^p p_\iota^{L_p(\mu;E)}(u)^p - \limsup_{j\to\infty} p_\iota^{L_p(\mu;E)}(u_j - u)^p,
\end{aligned}$$

i.e.

$$0 \leq \limsup_{j\to\infty} p_\iota^{L_p(\mu;E)}(u_j - u)^p \leq 0$$

and, thus, the $\lim_{j\to\infty} p_\iota^{L_p(\mu;E)}(u_j - u)^p = 0$.

□

LEMMA 9.22 (Continuity Lemma). *Let $J \subseteq \mathbb{R}$ an open interval, and $u : J \times \Omega \to E$ satisfying*

(i) $\forall t \in J : u(t, \cdot) \in L_1(\mu; E)$,

(ii) $\forall \omega \in \Omega : u(\cdot, \omega) \in C(J, E)$,

(iii) $\forall \iota \in I\ \exists v_\iota \in L_1(\mu; E)\ \forall t \in J:\ p_\iota \circ u(t,\cdot) \le p_\iota \circ v_\iota$.

Then,

$$t \mapsto V(t) := \int_\Omega u(t,\cdot) d\mu$$

is continuous.

PROOF. Since J is an interval, continuity of V is equivalent to sequential continuity. In other words, if $t_0 \in J$ and $(t_j)_{j\in\mathbb{N}} \in J^{\mathbb{N}}$ with $t_j \to t_0$, then we need to show that $V(t_j) \to V(t_0)$ in E.

Let $u_j := u(t_j,\cdot)$ for $j \in \mathbb{N}_0$. Then, we have $u_j \in L_1(\mu, E)$ by (i) for all $j \in \mathbb{N}$, $u_j \to u_0$ pointwise by (ii), and

$$\forall \iota \in I\ \exists v_\iota \in L_1(\mu; E)\ \forall j \in \mathbb{N}:\ p_\iota \circ u_j \le p_\iota \circ v_\iota.$$

by (iii). Hence, dominated convergence yields

$$V(t_j) = \int_\Omega u_j d\mu \to \int_\Omega u_0 d\mu = V(t_0) \quad (j \to \infty).$$

□

LEMMA 9.23 (Differentiability Lemma). *Let $J \subseteq \mathbb{R}$ an open interval, and $u : J \times \Omega \to E$ satisfying*

(i) $\forall t \in J:\ u(t,\cdot) \in L_1(\mu; E)$,

(ii) $\forall \omega \in \Omega:\ u(\cdot,\omega)$ *differentiable,*

(iii) $\forall \iota \in I\ \exists v_\iota \in L_1(\mu; E)\ \forall (s,t,\omega) \in J \times J \times \Omega:\ p_\iota\left(\frac{u(s,\omega)-u(t,\omega)}{s-t}\right) \le p_\iota(v_\iota(\omega))$.

Then,

$$V(t) := \int_\Omega u(t,\cdot) d\mu$$

is differentiable and

$$V'(t) = \int_\Omega \partial_1 u(t,\cdot) d\mu$$

holds.

Proof. Let $t_0 \in J$, and $(t_j)_{j\in\mathbb{N}} \in (J \setminus \{t_0\})^{\mathbb{N}}$ with $t_j \to t_0$ and define

$$u_j := \frac{u(t_j,\cdot) - u(t_0,\cdot)}{t_j - t_0}.$$

Then, assumption (ii) implies $u_j \to \partial_1 u(t_0,\cdot)$ pointwise. In particular, $\partial_1 u(t_0,\cdot)$ is measurable. Furthermore, (iii) implies $\forall j \in \mathbb{N}: \; u_j \in L_1(\mu;E)$ and dominated convergence yields

$$\frac{V(t_j) - V(t_0)}{t_j - t_0} = \int_\Omega \frac{u(t_j,\cdot) - u(t_0,\cdot)}{t_j - t_0} d\mu = \int_\Omega u_j d\mu \to \int_\Omega \partial_1 u(t_0,\cdot) d\mu \quad (j \to \infty).$$

$\square$

Theorem 9.24 (Fubini). *Let (Ω,Σ,μ) and $(\tilde\Omega,\tilde\Sigma,\tilde\mu)$ be σ-finite. Let $u \in \mathcal{M}(\mu\times\tilde\mu;E)$ satisfy at least one of the following conditions.*

(a) $\forall \iota \in I: \; \int_\Omega \int_{\tilde\Omega} p_\iota \circ u \; d\tilde\mu \; d\mu < \infty$

(b) $\forall \iota \in I: \; \int_{\tilde\Omega} \int_\Omega p_\iota \circ u \; d\mu \; d\tilde\mu < \infty$

(c) $\forall \iota \in I: \; \int_{\Omega\times\tilde\Omega} p_\iota \circ u \; d(\mu\times\tilde\mu) < \infty$

Then, all of the above are true and we obtain

(i) $u \in L_1(\mu\times\tilde\mu;E)$

(ii) $u(\cdot,\omega) \in L_1(\mu;E)$ *for $\tilde\mu$-almost every $\omega \in \tilde\Omega$*

(iii) $u(\omega,\cdot) \in L_1(\tilde\mu;E)$ *for μ-almost every $\omega \in \Omega$*

(iv) $\int_\Omega u(\omega,\cdot) d\mu(\omega) \in L_1(\tilde\mu;E)$

(v) $\int_{\tilde\Omega} u(\cdot,\omega) d\tilde\mu(\omega) \in L_1(\mu;E)$

Furthermore, if $u \in \overline{\mathcal{S}}L_1(\mu;E)$, then

(vi) $\int_{\Omega\times\tilde\Omega} u \; d(\mu\times\tilde\mu) = \int_\Omega \int_{\tilde\Omega} u \; d\tilde\mu \; d\mu = \int_{\tilde\Omega}\int_\Omega u \; d\mu \; d\tilde\mu$

holds, as well.

PROOF. Everything but (vi) follows directly from Fubini's theorem in $\mathbb{R}$ using composition with p_ι. Then, (vi) follows by approximation with a net of simple functions.

$$\begin{aligned}\int_{\Omega\times\tilde{\Omega}} ud(\mu\times\tilde{\mu}) &\leftarrow \int_{\Omega\times\tilde{\Omega}} s_\nu d(\mu\times\tilde{\mu})\\ &= \sum_{i=1}^{m} y_i(\mu\times\tilde{\mu})(S_i)\\ &= \sum_{i=1}^{m} y_i \underbrace{\int_{\Omega\times\tilde{\Omega}} 1_{S_i} d(\mu\times\tilde{\mu})}_{\text{usual Fubini}}\\ &= \sum_{i=1}^{m} y_i \int_{\tilde{\Omega}}\int_{\Omega} 1_{S_i} d\mu d\tilde{\mu}\\ &= \int_{\tilde{\Omega}}\int_{\Omega} s_\nu d\mu d\tilde{\mu}\\ &\to \int_{\tilde{\Omega}}\int_{\Omega} u d\mu d\tilde{\mu}\end{aligned}$$

where the $s_\nu = \sum_{i=1}^{m} y_i 1_{S_i}$ are simple functions approximating u in L_1.

□

PROPOSITION 9.25 (push-forward measures). *Let $f \in \mathcal{M}(\mu; E)$. Then,*

$$\forall S \subseteq \mathcal{B}(E):\ \nu(S) := \mu([S]f)$$

defines a Borel measure ν on E where $\mathcal{B}(E)$ denotes the Borel σ-algebra on E.

Let F be another Hausdorffian locally convex topological vector space and $u \in \mathcal{M}(\nu; F)$. Then, $u \in L_1(\nu; F)$ if and only if $u \circ f \in L_1(\mu; F)$. Furthermore, for $u \in \overline{\mathcal{S}}L_1(\nu; F)$ we obtain

$$\int_{\Omega} u \circ f d\mu = \int_E u d\nu.$$

PROOF. The equivalence "$u \in L_1(\nu; F)$ if and only if $u \circ f \in L_1(\mu; F)$" follows directly from "$p_\iota^F \circ u \in L_1(\nu)$ if and only if $p_\iota^F \circ u \circ f \in L_1(\mu)$" for every semi-norm

p_ι^F of F. The assertion $\int_\Omega u \circ f d\mu = \int_E u d\nu$ then follows by approximation with a net of simple functions

$$\int_\Omega u \circ f d\mu \leftarrow \int_\Omega s_\nu \circ f d\mu = \sum_{i=1}^m y_i \int_\Omega 1_{S_i} \circ f d\mu = \sum_{i=1}^m y_i \int_E 1_{S_i} d\nu = \int_E s_\nu d\nu \to \int_E u d\nu$$

where the $s_\nu = \sum_{i=1}^m y_i 1_{S_i}$ are simple functions approximating u in $L_1(\nu; F)$.

□

DEFINITION 9.26. *Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mu; E)^{\mathbb{N}}$ and $u \in \mathcal{M}(\mu; E)$. We say $(u_n)_{n\in\mathbb{N}}$ converges to u (globally) in measure if and only if*

$$\forall \iota \in I \ \forall \varepsilon \in \mathbb{R}_{>0}: \ \mu([\mathbb{R}_{>\varepsilon}] p_\iota \circ (u_n - u)) \to 0 \quad (n \to \infty).$$

If $I \subseteq \mathbb{N}$ is countable, then we define the metric

$$d: \ E \times E \to \mathbb{R}; \ (x, y) \mapsto \sum_{\iota \in I} 2^{-\iota} \frac{p_\iota(x-y)}{1 + p_\iota(x-y)}$$

and say that $(u_n)_{n\in\mathbb{N}}$ converges metrically to A (globally) in measure if and only if

$$\forall \varepsilon \in \mathbb{R}_{>0}: \ \mu([\mathbb{R}_{>\varepsilon}] d \circ (u_n, u)) \to 0 \quad (n \to \infty).$$

COROLLARY 9.27. *Global metric convergence in measure implies global convergence in measure.*

PROOF. Let $I \subseteq \mathbb{N}$ and $u_n \to u$ (globally) metrically in measure. Then, for $\iota \in I$ and $\varepsilon \in \mathbb{R}_{>0}$,

$$\begin{aligned}
\mu([\mathbb{R}_{>\varepsilon}] d \circ (u_n, u)) =& \mu\left(\left\{\omega \in \Omega; \ \sum_{\kappa \in I} 2^{-\kappa} \frac{p_\kappa(u_n(\omega) - u(\omega))}{1 + p_\kappa(u_n(\omega) - u(\omega))} > \varepsilon\right\}\right) \\
\geq& \mu\left(\left\{\omega \in \Omega; \ 2^{-\iota} \frac{p_\iota(u_n(\omega) - u(\omega))}{1 + p_\iota(u_n(\omega) - u(\omega))} > \varepsilon\right\}\right) \\
=& \mu\left(\left\{\omega \in \Omega; \ p_\iota(u_n(\omega) - u(\omega)) > \frac{2^\iota \varepsilon}{1 - 2^\iota \varepsilon}\right\}\right)
\end{aligned}$$

shows

$$\forall \iota \in I \ \forall \varepsilon \in \mathbb{R}_{>0}: \ \mu([\mathbb{R}_{>\varepsilon}] p_\iota \circ (u_n - u)) \leq \mu\left(\left[\mathbb{R}_{>\frac{\varepsilon}{2^\iota(1+\varepsilon)}}\right] d \circ (u_n, u)\right)$$

which converges to zero.

□

COROLLARY 9.28. *Let* $(u_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mu;E)^{\mathbb{N}}$ *converge to* $u \in \mathcal{M}(\mu;E)$ μ*-almost everywhere. Then,* $(u_n)_{n\in\mathbb{N}}$ *converges to* u *(globally) in measure.*

If E *is metrizable, then* $(u_n)_{n\in\mathbb{N}}$ *converges metrically to* u *(globally) in measure.*

PROOF. $u_n \to u$ μ-almost everywhere implies $\forall \iota \in I :\ p_\iota \circ (u_n - u) \to 0$ μ-almost everywhere. Since the assertion is known for real random variables (cf. Lemma 16.4 in **[65]**), we directly obtain

$$\forall \varepsilon \in \mathbb{R}_{>0} :\ \mu\left([\mathbb{R}_{>\varepsilon}]\, p_\iota \circ (u_n - u)\right) \to 0 \quad (n \to \infty)$$

for each of the semi-norms; thus, the assertion. Similarly, the "metrizable" assertion follows from the fact that $d \circ (u_n, u) \to 0$ μ-almost everywhere for the real random variables $d \circ (u_n, u)$.

□

COROLLARY 9.29. *Let* $(u_n)_{n\in\mathbb{N}} \in L_p(\mu;E)^{\mathbb{N}}$ *converge to* $u \in L_p(\mu;E)$ μ*-almost everywhere. Then,* $(u_n)_{n\in\mathbb{N}}$ *converges to* u *(globally) in measure.*

If E *is metrizable, then* $(u_n)_{n\in\mathbb{N}}$ *converges metrically to* u *(globally) in measure.*

PROOF. $u_n \to u$ in $L_p(\mu;E)$ implies $\forall \iota \in I :\ p_\iota \circ (u_n - u) \to 0$ in $L_p(\mu)$. Since the assertion is known for real random variables (cf. Lemma 16.4 in **[65]**), we directly obtain

$$\forall \varepsilon \in \mathbb{R}_{>0} :\ \mu\left([\mathbb{R}_{>\varepsilon}]\, p_\iota \circ (u_n - u)\right) \to 0 \quad (n \to \infty)$$

for each of the semi-norms; thus, the assertion. Similarly, the assertion follows from the fact that $d\circ(u_n,u)\to 0$ in $L_p(\mu)$ for the real random variables $d\circ(u_n,u)$.

□

THEOREM 9.30. *Let E be a Fréchet space and $(u_n)_{n\in\mathbb{N}}\in\mathcal{M}(\mu;E)^{\mathbb{N}}$. Then, the following are equivalent.*

(i) $\exists u\in\mathcal{M}(\mu;E):\ u_n\to u$ *(globally) metrically in measure.*

(ii) $\exists u\in\mathcal{M}(\mu;E):\ u_n\to u$ *(globally) in measure.*

(iii) $\forall\iota\in I\ \forall\varepsilon\in\mathbb{R}_{>0}:\ \lim_{n\to\infty}\sup_{m\in\mathbb{N}_{\geq n}}\mu\left([\mathbb{R}_{>\varepsilon}]\,p_\iota\circ(u_n-u_m)\right)=0$

(iv) *There exists $u\in\mathcal{M}(\mu;E)$ such that every sub-sequence of $(u_n)_{n\in\mathbb{N}}$ contains a sub-sequence which converges μ-almost everywhere to u.*

If E is not a Fréchet space, then we still obtain (iv)⇒(ii)⇒(iii).

PROOF. “(i)⇒(ii)” Corollary 9.27.

“(ii)⇒(iii)” $u_n\to u$ (globally) in measure means that

$$\forall\iota\in I\ \forall\delta,\varepsilon\in\mathbb{R}_{>0}\ \exists N_\varepsilon(\delta)\in\mathbb{N}\ \forall n\in\mathbb{N}_{\geq N_\varepsilon(\delta)}:\ \mu\left([\mathbb{R}_{>\varepsilon}]\,p_\iota\circ(u_n-u)\right)<\delta.$$

Let $\delta,\varepsilon\in\mathbb{R}_{>0}$. Then, we obtain for $m,n\in\mathbb{N}_{\geq N_\varepsilon(\delta)}$

$$\begin{aligned}\mu\left([\mathbb{R}_{>2\varepsilon}]\,p_\iota\circ(u_n-u_m)\right)\leq&\mu\left([\mathbb{R}_{>2\varepsilon}]\left(p_\iota\circ(u_n-u)+p_\iota\circ(u_m-u)\right)\right)\\ \leq&\mu\left([\mathbb{R}_{>\varepsilon}]\,p_\iota\circ(u_n-u)\right)+\mu\left([\mathbb{R}_{>\varepsilon}]\,p_\iota\circ(u_m-u)\right)\\ \leq&2\delta,\end{aligned}$$

i.e.

$$\lim_{n\to\infty}\sup_{m\in\mathbb{N}_{\geq n}}\mu\left([\mathbb{R}_{>2\varepsilon}]\,p_\iota\circ(u_n-u_m)\right)\leq 2\delta$$

for every $\delta \in \mathbb{R}_{>0}$. Hence,

$$\lim_{n\to\infty} \sup_{m\in\mathbb{N}_{\geq n}} \mu\left([\mathbb{R}_{>2\varepsilon}]\, p_\iota \circ (u_n - u_m)\right) = 0.$$

"(iii)⇒(iv)" Let $(u'_n)_{n\in\mathbb{N}}$ be a sub-sequence of $(u_n)_{n\in\mathbb{N}}$. For every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\forall m, n \in \mathbb{N}_{\geq n_k}:\ \mu\left([\mathbb{R}_{>2^{-k}}]\, p_\iota \circ (u_n - u_m)\right) < 2^{-k}.$$

Without loss of generality, let $\forall k, m \in \mathbb{N}:\ (\ k < m\ \Rightarrow\ n_k < n_m\ )$. Let

$$u^\iota_k := u'_{n_k}$$

and

$$\Omega^\iota_k := [\mathbb{R}_{>2^{-k}}]\, p_\iota \circ (u^\iota_{k+1} - u^\iota_k).$$

Then, $\mu(\Omega^\iota_k) < \frac{1}{2^k}$, i.e. $\sum_{k\in\mathbb{N}} \mu(\Omega^\iota_k) < \infty$. Borel-Cantelli[10], thus, implies

$$\mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}_{\geq n}} \Omega^\iota_k\right) = 0,$$

that is, for μ-almost every $\omega \in \Omega$ there exists $k^\iota_\omega \in \mathbb{N}$ such that for every $k \in \mathbb{N}_{\geq k^\iota_\omega}$

$$p_\iota\left(u^\iota_{k+1}(\omega) - u^\iota_k(\omega)\right) \leq \frac{1}{2^k}.$$

Thus, for $n \in \mathbb{N}_{\geq k^\iota_\omega}$,

$$\sup_{m\in\mathbb{N}_{\geq n}} p_\iota\left(u^\iota_m(\omega) - u^\iota_n(\omega)\right) \leq \sum_{k\in\mathbb{N}_{\geq n}} p_\iota\left(u^\iota_{k+1}(\omega) - u^\iota_k(\omega)\right) \leq \sum_{k\in\mathbb{N}_{\geq n}} \frac{1}{2^k} \to 0 \quad (n \to \infty).$$

[10] cf. Theorem 18.9 in **[65]**

Theorem (Borel-Cantelli). *Let (Ω, Σ, μ) be a probability space and $(S_j)_{j\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$. Then,*

$$\sum_{j\in\mathbb{N}} \mu(S_j) < \infty\ \Rightarrow\ \mu\left(\bigcap_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}_{\geq k}} S_j\right) = 0.$$

If the sets S_j are pairwise independent, i.e. $\forall j, k \in \mathbb{N}:\ \mu(S_j \cap S_k) = \mu(S_j)\mu(S_k)$, then

$$\sum_{j\in\mathbb{N}} \mu(S_j) = \infty\ \Rightarrow\ \mu\left(\bigcap_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}_{\geq k}} S_j\right) = 1.$$

Furthermore, $u_\iota(\omega) \coloneqq u_1^\iota + \sum_{k\in\mathbb{N}} (u_{k+1}^\iota - u_k^\iota)$ converges μ-almost everywhere absolutely with respect to p_ι. Now, we can find a p_κ-pointwise limit u_κ of a sub-sequence of $(u_k^\iota)_{k\in\mathbb{N}}$ so that the resulting sub-sequence converges μ-almost everywhere to some u_κ with respect to p_κ and p_ι. Inductively, reducing to sub-sequences, the diagonal sequence converges μ-almost everywhere with respect to all p_ι.

"(iv)⇒(i)" If $(u_n)_{n\in\mathbb{N}}$ does not converge (globally) metrically in measure, then there is a sub-sequence $(u_n')_{n\in\mathbb{N}}$, as well as $\delta, \varepsilon \in \mathbb{R}_{>0}$, such that

$$\mu\left([\mathbb{R}_{>\varepsilon}]\, d\circ(u_n', u)\right) > \delta.$$

However, this sub-sequence has no sub-sequence $(u_n'')_{n\in\mathbb{N}}$ which converges μ-almost everywhere to u. This is a contradiction.

If E is not metrizable, then the same contradiction holds for at least one of the $\mu\left([\mathbb{R}_{>\varepsilon}]\, p_\iota\circ(u_n' - u)\right) > \delta$.

□

Remark Note that this theorem implies that, in general, there exists no topology of μ-almost everywhere convergence in a Fréchet space because convergence of a sequence in a topological space is equivalent to the face that every sub-sequence has a convergent sub-sub-sequence. In other words, if there were a topology of μ-almost everywhere convergence, then condition (iv) would show equivalence of μ-almost everywhere convergence and convergence in measure. However, we know this to be false in $\mathbb{R}$.

■

CHAPTER 10

The Pettis integral

Now we shall be interested in the existence of Pettis integrals. Often the natural assumption is to require that E is quasi-complete, i.e. all bounded and closed sets are complete. For topological vector spaces, quasi-completeness is (usually) the appropriate general completeness notion and, as such, Hilbert, Banach, Fréchet, and LF-spaces are all quasi-complete, as well as their weak-$*$-duals and many spaces of operators, e.g. the bounded linear operators on a Hilbert space with the weak and strong operator topologies. In particular, the fact that weak-$*$-duals of LF spaces (e.g. C_c^∞) are quasi-complete is of prime importance for the integration theory of distribution valued functions. Another very compelling argument for assuming quasi-completeness is

$$\left(\ \forall \varphi \in E' :\ \varphi \circ f \in C^m(\Omega, \mathbb{C})\ \right) \ \Rightarrow\ f \in C^{m-1}(\Omega, E)$$

where $\Omega \subseteq_{\text{open}} \mathbb{R}^n$.[1]

However, as we are interested in algebras of Fourier Integral Operators we might not have the luxury of working in a quasi-complete algebra. Luckily, the Hörmander space $\mathcal{D}'_\Gamma$, the set of distributions with wave front set in the closed cone Γ, is a nuclear, semi-reflexive, semi-Montel, complete normal space of distributions in its normal topology and quasi-complete in the Hörmander topology (cf. **[17]**), i.e. the canonical examples are still "nice". It should also be noted that the topological properties of Hörmander spaces and generalized Hörmander spaces[2] are still actively

[1] cf. Theorem 3.7 in **[11]**

[2] cf. **[15]**; we are not going to discuss them here.

under investigation (cf. [**15**–**17**]). Especially the topological properties of subspaces of Hörmander spaces are interesting, keeping in mind that considering subspaces of nice spaces can mean that we lose a lot of nice properties even if the spaces are reasonable. For instance, if we look at the space of compact operators between two Banach spaces with the strong operator topology, then we have a space that is not even sequentially complete (in particular, not quasi-complete). However, the technical condition we need for Pettis integration, the convex compactness property, is still satisfied (cf. [**75**]).

DEFINITION 10.1. *Let E be a locally convex topological vector space and a Hausdorff space. Then, E has the convex compactness property if and only if*

$$\forall C \subseteq_{\text{compact}} E: \ \overline{\operatorname{conv} C} \subseteq_{\text{compact}} E.$$

Here, $\operatorname{conv} C$ *denotes the convex hull of C.*

Furthermore, E has the metric convex compactness property if and only if

$$\forall C \subseteq_{\text{compact,metrizable}} E: \ \overline{\operatorname{conv} C} \subseteq_{\text{compact}} E.$$

The following observation by Pfister (1981) is stated as Theorem 0.1 in [**75**].

THEOREM 10.2. *Let E be a locally convex topological vector space and a Hausdorff space. Then, the following are equivalent.*

(i) *E has the (metric) convex compactness property.*

(ii) *Let Ω be a compact (metric) space, μ a (positive) Borel measure on Ω, and $f \in C(\Omega, E)$. Then, f is μ-Pettis integrable.*

In [**75**], we can also find the following remarks.

- The metric convex compactness property is equivalent to the fact that every continuous function $f: [0,1] \to E$ is Pettis-integrable with respect to the Lebesgue measure. In other words, the metric convex compactness property is a natural property to consider if we want to extend ideas from algebras with a continuous functional calculus to those without.

- All the implications are strict:

$$
\begin{array}{ccccc}
 & & \text{complete} & & \\
 & & \Downarrow & & \\
 & & \text{quasi-complete} & & \\
 & \swarrow & & \searrow & \\
\begin{array}{c}\text{sequentially}\\ \text{complete}\end{array} & & & & \begin{array}{c}\text{convex compactness}\\ \text{property}\end{array} \\
 & \searrow & & \swarrow & \\
 & & \begin{array}{c}\text{metric convex}\\ \text{compactness property}\end{array} & & \\
 & & \Downarrow & & \\
 & & \begin{array}{c}\text{Mackey complete}\\ (=\text{locally complete})\end{array} & &
\end{array}
$$

where we note (as in **[75]**) that Mackey completeness is equivalent to compactness of the closed convex hull of any convergent sequence.

At this point, we would also like to remark that condition (ii) can be applied to measurable functions, as well, by virtue of Lusin's measurability theorem.

THEOREM 10.3 (Lusin). *Let (Ω, Σ, μ) be a Radon measure space, E a second-countable topological space, $f: \Omega \to E$ measurable, $\varepsilon \in \mathbb{R}_{>0}$, and $S \in \Sigma$.*

Then, there exists a closed set $C_\varepsilon \subseteq \Omega$ such that $\mu(S \smallsetminus C_\varepsilon) < \varepsilon$ and $f|_{C_\varepsilon}$ is continuous.

If E is a topological vector space, then we can choose C_ε to be compact.

In the light of the Schwartz kernel theorem, we are considering algebras which are endowed with the weak-$*$-topology (or finer topologies). If we integrate a function A with pointwise kernel κ, then $\int_\Omega A d\mu$ should satisfy for $\varphi, \psi \in C_c^\infty(X)$ with sufficiently small support

$$\begin{aligned}
\left\langle \int_\Omega A(\omega) d\mu(\omega)\ \varphi, \psi \right\rangle &= \int_X \psi(x) \int_\Omega \left(A(\omega)\varphi\right)(x) d\mu(\omega) d\mathrm{vol}_X(x) \\
&= \int_X \int_\Omega \int_X \kappa(\omega)(x,y)\varphi(y)\psi(x)\ d\mathrm{vol}_X(y)\ d\mu(\omega)\ d\mathrm{vol}_X(x) \\
&= \int_{X^2} \int_\Omega \kappa(\omega)(x,y) d\mu(\omega)\ \varphi(y)\psi(x)\ d\mathrm{vol}_{X^2}(x,y),
\end{aligned}$$

i.e. $\int_\Omega A d\mu$ ought to be the operator with the kernel $\int_\Omega \kappa d\mu$. If we assume that $\int_\Omega A d\mu$ is a Pettis integral, then we need to find conditions relating it to the integral $\int_\Omega \kappa d\mu$ which itself can be defined as a Pettis integral in a subspace $\mathcal{D}'_{\mathcal{A}}$ of $C_c^\infty(X^2)'$. In particular, if $\mathcal{D}'_{\mathcal{A}}$ has a convex compactness property, then Theorem 10.2 tells us about the existence of the integral $\int_\Omega \kappa d\mu$.

Proposition 10.4. *Let $\mathcal{D}'_{\mathcal{A}}$ be sequentially complete and with convex compactness property, (Ω, Σ, μ) a Radon measure space, and $\kappa \in \mathcal{S}L_1(\mu; \mathcal{D}'_{\mathcal{A}})$. Then, $\int_\Omega \kappa d\mu \in \mathcal{D}'_{\mathcal{A}}$.*

Proof. Since κ is strongly measurable, there is a separable subspace $E \subseteq \mathcal{D}'_{\mathcal{A}}$ such that $\kappa(\omega) \in E$ for μ-almost every $\omega \in \Omega$. For $\varepsilon \in \mathbb{R}_{>0}$, Lusin's measurability theorem implies the existence of an $\Omega_\varepsilon \subseteq_{\text{compact}} \Omega$ such that $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and $\kappa|_{\Omega_\varepsilon}$ is continuous. Thus, by the convex compactness property,

$$\int_{\Omega_\varepsilon} \kappa d\mu \in \mathcal{D}'_{\mathcal{A}}.$$

Since $\kappa \in L_1(\mu; \mathcal{D}'_{\mathcal{A}})$,

$$\left(\int_{\Omega_{\frac{1}{n}}} \kappa d\mu \right)_{n \in \mathbb{N}} \in \left(\mathcal{D}'_{\mathcal{A}}\right)^{\mathbb{N}}$$

is a Cauchy sequence and we obtain

$$\int_{\Omega_{\frac{1}{n}}} \kappa d\mu \to \int_{\Omega} \kappa d\mu \in \mathcal{D}'_{\mathcal{A}}$$

by sequential completeness of $\mathcal{D}'_{\mathcal{A}}$.

□

For applications of Cauchy's Integral Theorem (from complex analysis), we only need to integrate with respect to metric spaces. Hence, we can choose slightly weaker assumptions which yields the following version of the proposition above (using the same proof because now we only need the metric convex compactness property which follows from sequential completeness).

Proposition 10.5. *Let $\mathcal{D}'_{\mathcal{A}}$ be sequentially complete, Ω a metric space, μ a positive Radon measure, and $\kappa \in \mathcal{S}L_1(\mu;\mathcal{D}'_{\mathcal{A}})$. Then, $\int_\Omega \kappa d\mu \in \mathcal{D}'_{\mathcal{A}}$.*

Remark Note that all closed and bounded sets in a Hörmander space $\mathcal{D}'_\Gamma$ are compact, complete, and metrizable (cf. Proposition 1 in [**17**]). Hence every bounded continuous function on a compact space with values in $\mathcal{D}'_\Gamma$ is strongly measurable by the Sombrero lemma.

■

Remark If we want to consider the algebra $\mathcal{A}$ directly, then there are a couple of interesting topologies. For instance, we may want to consider the integrals $\int_\Omega A d\mu$ with respect to the strong operator topology, that is, $\int_\Omega A d\mu$ is defined by

$$\int_\Omega A d\mu\ \varphi := \int_\Omega A\varphi d\mu$$

where $\int_\Omega A\varphi d\mu$ is a Pettis integral in $C_c^\infty(X)'$, i.e.

$$\forall \varphi,\psi \in C_c^\infty(X):\ \left\langle \int_\Omega A d\mu\ \varphi, \psi \right\rangle = \int_\Omega \langle A\varphi,\psi\rangle d\mu.$$

Another interesting topology would be the gap topology (cf. appendix B). A particularly interesting case arises if the algebra $\mathcal{A}$ is a closed (in the norm topology) subspace of $L(B,C)$ where B is a separable Banach space and C any Banach space. Then, as pointed out in Remark 3.1 in [**75**], the convex compactness property and the metric convex compactness property are equivalent even with respect to the strong operator topology.

■

The Pettis integral also allows generalizations of some of the theorems in the previous chapter.

LEMMA 10.6. *Let* (Ω,Σ,μ) *be* σ*-finite,* F *a Hausdorffian locally convex topological vector space with separating dual and* $f \in L_1(\mu;E)$.

(i) *Let* $B \in L(E,F)$*. Then,* $B \circ f$ *is* μ*-Pettis integrable and*

$$B\int_\Omega f d\mu = \int_\Omega B \circ f d\mu.$$

(ii) *Let* $E_0 \subseteq E$ *be a closed subspace and* $f(\omega) \in E_0$ *for* μ*-almost every* $\omega \in \Omega$*. Then,* $\int_\Omega f d\mu \in E_0$.

PROOF. The assertion (i) follows directly from the fact that for every $\varphi \in F'$, $\varphi \circ B \in E'$ and

$$\varphi B \int_\Omega f d\mu = \int_\Omega \varphi \circ B \circ f d\mu = \varphi \int_\Omega B \circ f d\mu.$$

The proof of assertion (ii) is unchanged; namely, for $\varphi \in E'$ with $\varphi|_{E_0} = 0$, we obtain

$$\varphi\left(\int_\Omega f d\mu\right) = \int_\Omega \underbrace{\varphi \circ f}_{=0} d\mu = 0.$$

□

This extends to Hille's theorem (same proof as in Theorem 9.16).

THEOREM 10.7 (Hille). *Let $f \in L_1(\mu; E)$ be μ-Pettis integrable, F a Hausdorffian locally convex topological vector space with separating dual, and $A: D(A) \subseteq E \to F$ a closed linear operator (that is, $A \subseteq E \times F$ is a closed subspace). Let $f(\omega) \in D(A)$ for μ-almost every $\omega \in \Omega$ and $A \circ f \in L_1(\mu; F)$ μ-Pettis integrable. Then, we obtain $\int_\Omega f d\mu \in D(A)$ and $A \int_\Omega f d\mu = \int_\Omega A \circ f d\mu$.*

Furthermore, we obtain Fubini's theorem and the theorem of push-forward measures.

THEOREM 10.8 (Fubini). *Let (Ω, Σ, μ) and $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ be σ-finite. Let $u \in \mathcal{M}(\mu \times \tilde{\mu}; E)$ be μ-Pettis integrable and satisfy at least one of the following conditions.*

(a) $\forall \iota \in I: \int_\Omega \int_{\tilde{\Omega}} p_\iota \circ u \ d\tilde{\mu} \ d\mu < \infty$

(b) $\forall \iota \in I: \int_{\tilde{\Omega}} \int_\Omega p_\iota \circ u \ d\mu \ d\tilde{\mu} < \infty$

(c) $\forall \iota \in I: \int_{\Omega \times \tilde{\Omega}} p_\iota \circ u \ d(\mu \times \tilde{\mu}) < \infty$

Then, all of the above are true and we obtain

(i) $u \in L_1(\mu \times \tilde{\mu}; E)$

(ii) $u(\cdot, \omega) \in L_1(\mu; E)$ *for $\tilde{\mu}$-almost every $\omega \in \tilde{\Omega}$*

(iii) $u(\omega, \cdot) \in L_1(\tilde{\mu}; E)$ *for μ-almost every $\omega \in \Omega$*

(iv) $\int_\Omega u(\omega, \cdot) d\mu(\omega) \in L_1(\tilde{\mu}; E)$

(v) $\int_{\tilde{\Omega}} u(\cdot, \omega) d\tilde{\mu}(\omega) \in L_1(\mu; E)$

(vi) $\int_{\Omega \times \tilde{\Omega}} u \ d(\mu \times \tilde{\mu}) = \int_\Omega \int_{\tilde{\Omega}} u \ d\tilde{\mu} \ d\mu = \int_{\tilde{\Omega}} \int_\Omega u \ d\mu \ d\tilde{\mu}$

holds, as well.

PROOF. (i-v) are unchanged. (vi) follows from Fubini's theorem in $\mathbb{R}$ since

$$\forall \varphi \in E': \int_{\Omega \times \tilde{\Omega}} \varphi \circ u \ d(\mu \times \tilde{\mu}) = \int_\Omega \int_{\tilde{\Omega}} \varphi \circ u \ d\tilde{\mu} \ d\mu = \int_{\tilde{\Omega}} \int_\Omega \varphi \circ u \ d\mu \ d\tilde{\mu}$$

□

PROPOSITION 10.9 (push-forward measures). *Let F another Hausdorffian locally convex topological vector space and $f \in \mathcal{M}(\mu; E)$ μ-Pettis integrable. Then,*

$$\forall S \subseteq \mathcal{B}(E) : \ \nu(S) := \mu([S]f)$$

defines a Borel measure ν on E where $\mathcal{B}(E)$ denotes the Borel σ-algebra on E.

Let $u \in \mathcal{M}(\nu; F)$. Then, $u \in L_1(\nu; F)$ if and only if $u \circ f \in L_1(\mu; F)$. In that case we obtain

$$\int_\Omega u \circ f d\mu = \int_E u d\nu.$$

PROOF. Here, the only change is that we lost a restriction on u in

$$\int_\Omega u \circ f d\mu = \int_E u d\nu.$$

However, since we are using Pettis integrals, we observe

$$\forall \varphi \in F' : \ \int_\Omega \varphi \circ u \circ f d\mu = \int_E \varphi \circ u d\nu.$$

□

CHAPTER 11

The index bundle

In this chapter, we want to consider measurable index bundles, i.e. we want to show that the theory above extends the continuous case of the Atiyah-Jänich index bundles (cf. e.g. [**4**]). In order to do that, we will have to define a topology in a suitable space the index bundle maps into. Then, we can define Borel sets and, thus, measurability of the index bundle. Similar considerations for continuous families can be found in [**6**] and (very extensively) in chapter 6 of [**76**].

The index bundle of a family of operators $(f(\omega))_{\omega\in\Omega}$ is given by

$$\mathrm{IND}(f)(\omega) = \ker f(\omega) - \ker f(\omega)^*$$

as interpreted in the K-theory of vector bundles with the direct sum where

$$\ker f(\omega) = N(f(\omega)) = [\{0\}]f(\omega)$$

is the kernel (null space) of $f(\omega)$.

Here, we will consider the following construction. Let S be an abelian monoid. Then, we define

$$K(S) := S^2/_{\{(x,y)\in S^2;\ x=y\}}$$

with the canonical injection $S \ni s \mapsto (s,0) \in K(S)$ and $\forall s \in S:\ -s = (0,s)$.

Hence,

$$\begin{aligned}\mathrm{IND}(f)(\omega) =& \ker f(\omega) - \ker f(\omega)^* \\ =& (\ker f(\omega), 0) - (\ker f(\omega)^*, 0)\end{aligned}$$

$$=(\ker f(\omega), 0) + (0, \ker f(\omega)^*)$$

$$= (\ker f(\omega), \ker f(\omega)^*)$$

can be interpreted as $\ker f(\omega) \oplus \ker f(\omega)^*$ and, if each $f(\omega)$ is a closed linear operator between Hilbert spaces H_0 and H_1, we obtain

$$\mathrm{IND}(f)(\omega) = \ker f(\omega) \oplus \ker f(\omega)^* \subseteq H_0 \oplus H_1.$$

In particular, $\mathrm{IND}(f)(\omega)$ is a closed linear relation in $H_0 \oplus H_1$. Since the space of non-empty closed linear relations $\mathrm{CLR}(H_0, H_1)$ in $H_0 \oplus H_1$ is a complete metric space, we have found a space and topology we could consider; namely the gap-topology $\hat{\delta}$ (cf. appendix B). However, we cannot use this topology directly because the function

$$\mathrm{CLR}(H_0, H_1) \ni f \mapsto \ker f \oplus \ker f^* \in \mathrm{CLR}(H_0, H_1)$$

is not continuous. If we assume that $f, g \in \mathrm{CLR}(H_0, H_1)$ are Fredholm operators and g a small perturbation of f (in the gap-topology), then it is well known that $\dim\ker g < \dim\ker f$ and $\dim\ker g^* < \dim\ker f^*$ are possible[1] which implies $\dim\ker g \oplus \ker g^* < \dim\ker f \oplus \ker f^*$, i.e.

$$\hat{\delta}\,(\ker g \oplus \ker g^*, \ker f \oplus \ker f^*) = 1$$

no matter how small $\hat{\delta}(f, g)$ is (cf. Theorem B.21 and the following discussion). Luckily, the index of Fredholm operators is locally constant in the gap-topology (cf. Theorem IV.5.17 in [**44**]), i.e. for $\hat{\delta}(f, g)$ sufficiently small

$$\dim\ker f - \dim\ker f^* = \dim\ker g - \dim\ker g^*$$

[1]Theorems IV.5.17 and IV.5.22 in [**44**] and the fact that $[0,1] \ni t \mapsto A(t) := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ satisfies $\forall t \in (0,1):\ 1 = \dim\ker A(0) > \dim\ker A(t) = 0.$

or equivalently

$$\dim\ker f - \dim\ker g = \dim\ker f^* - \dim\ker g^*,$$

and, since we are interested in the K-theory, we are allowed to consider

$$(\ker g + V_0) \oplus (\ker g^* + V_1)$$

with $V_0 \subseteq (\ker g)^\perp$, $V_1 \subseteq (\ker g^*)^\perp$ and

$$\dim V_0 = \dim V_1 \in \mathbb{N}_0$$

instead of $\ker g \oplus \ker g^*$. Similarly, we may add finite dimensional subspaces of $(\ker f)^\perp$ and $(\ker f^*)^\perp$ of the same dimension to $\ker f \oplus \ker f^*$.

The following is close to Atiyah's construction in [**3**]. For Hilbert spaces H_0 and H_1, we define the set of Fredholm operators

$$F(H_0, H_1) := \{f \in \mathrm{CLR}(H_0, H_1);\ f \text{ Fredholm operator}\}$$

endowed with the metric $\hat{\delta}$ (cf. appendix B). Let Ω be a topological space, $F \in C(\Omega, F(H_0, H_1))$, and $\omega_0 \in \Omega$.

Let $(e_{ij})_{j\in\mathbb{N}_0}$ be an orthonormal basis of H_i such that $(e_{0j})_{j\in\mathbb{N}_{0,<\dim\ker F(\omega_0)}}$ is an orthonormal basis of $\ker F(\omega_0)$ and $(e_{1j})_{j\in\mathbb{N}_{0,<\dim\ker F(\omega_0)^*}}$ is an orthonormal basis of $\ker F(\omega_0)^*$. Furthermore, for $n \in \mathbb{N}$, let

$$H_{in} := \overline{\mathrm{lin}\{e_{ij};\ j \in \mathbb{N}_{\geq n}\}}^{H_i}$$

and $\mathrm{pr}_{H_{in}} :\ H_i \to H_i$ the orthoprojection onto H_{in}. Then, all $\mathrm{pr}_{H_{in}}$ are self-adjoint Fredholm operators, i.e. they have vanishing index, and the operators

$$F_n(\omega) := \mathrm{pr}_{H_{1n}} \circ F(\omega)$$

satisfy

$$\operatorname{ind} F_n(\omega) = \operatorname{ind} \operatorname{pr}_{H_{1n}} + \operatorname{ind} F(\omega) = \operatorname{ind} F(\omega).$$

For $n \geq \dim \ker F(\omega_0)^*$, we obtain $F(\omega_0)[H_0]^\perp = \ker F(\omega_0)^* \subseteq H_{1n}^\perp$, i.e. $H_{1n} \subseteq F(\omega_0)[H_0]$. In other words, $F_n(\omega_0)[H_0] = H_{1n}$ and $\ker F_n(\omega_0)^* = H_{1n}^\perp$. Let

$$\begin{aligned} G(\omega):\ F_n(\omega_0) \to & H_{1n} \oplus \ker F_n(\omega_0); \\ & x \mapsto \left(\operatorname{pr}_{H_1} \left(\operatorname{pr}_{F_n(\omega_0)}\big|_{F_n(\omega)}^{F_n(\omega_0)} \right)^{-1} x\ ,\ \operatorname{pr}_{\ker F_n(\omega_0)} \operatorname{pr}_{H_0} x \right). \end{aligned}$$

Then, G is well-defined and continuous in ω (cf. Lemma B.17) and we observe for $(x,y) \in F_n(\omega_0)$

$$\begin{aligned} G(\omega_0)(x,y) &= \left(\operatorname{pr}_{H_1} \left(\operatorname{pr}_{F_n(\omega_0)}\big|_{F_n(\omega_0)}^{F_n(\omega_0)} \right)^{-1} (x,y)\ ,\ \operatorname{pr}_{\ker F_n(\omega_0)} \operatorname{pr}_{H_0} (x,y) \right) \\ &= \left(\operatorname{pr}_{H_1} (x,y)\ ,\ \operatorname{pr}_{\ker F_n(\omega_0)} x \right) \\ &= \left(y, \operatorname{pr}_{\ker F_n(\omega_0)} x \right) \\ &= \left(F_n(\omega_0)x, \operatorname{pr}_{\ker F_n(\omega_0)} x \right). \end{aligned}$$

Hence, $G(\omega_0)$ is an isomorphism and there exists and open neighborhood $\Omega_0 \subseteq \Omega$ of ω_0 such that each $G(\omega)$ is an isomorphism for $\omega \in \Omega_0$. This also implies that

$$\tilde{G}(\omega) := G(\omega)\ \operatorname{pr}_{F_n(\omega_0)}\big|_{F_n(\omega)}^{F_n(\omega_0)} :\ F_n(\omega) \to H_{1n} \oplus \ker F_n(\omega_0)$$

is an isomorphism for every $\omega \in \Omega_0$. Let $(e_j)_{j \in \mathbb{N}_{\leq d_0}}$ be a basis of $\ker F_n(\omega_0)$. Then,

$$\begin{aligned} \tilde{G}(\omega)(x,y) =& G(\omega)\ \operatorname{pr}_{F_n(\omega_0)}\big|_{F_n(\omega)}^{F_n(\omega_0)} (x,y) \\ &= \left(\operatorname{pr}_{H_1} (x,y)\ ,\ \operatorname{pr}_{\ker F_n(\omega_0)} \operatorname{pr}_{H_0}\ \operatorname{pr}_{F_n(\omega_0)}\big|_{F_n(\omega)}^{F_n(\omega_0)} (x,y) \right) \\ &= \left(y\ ,\ \operatorname{pr}_{\ker F_n(\omega_0)} \operatorname{pr}_{H_0}\ \operatorname{pr}_{F_n(\omega_0)}\big|_{F_n(\omega)}^{F_n(\omega_0)} (x,y) \right) \\ &= \left(F_n(\omega)x\ ,\ \operatorname{pr}_{\ker F_n(\omega_0)} \operatorname{pr}_{H_0}\ \operatorname{pr}_{F_n(\omega_0)}\big|_{F_n(\omega)}^{F_n(\omega_0)} (x,y) \right) \end{aligned}$$

for $(x,y) \in F_n(\omega)$ shows that

$$\forall j \in \mathbb{N}_{\leq d_0} : \; s_j(\omega) := \mathrm{pr}_{H_0} \tilde{G}(\omega)^{-1} (0, e_j)$$

defines a basis of $\ker F_n(\omega)$ and

$$\forall j \in \mathbb{N}_{\leq d_0} : \; s_j \in C(\Omega_0, H_0).$$

In [**3**], Atiyah defined the index bundle for bounded Fredholm operators on a Hilbert space H as $\ker F_n - \ker F_n^* = \ker F_n - \Omega \times H_{1n}^{\perp}$. If we use this representative of $\mathrm{IND}(F)$, then it suffices to show that continuity of the s_j implies gap-continuity of $\ker F_n$.

However, for $m \geq \dim \ker F(\omega_0)$ we can define

$$F_m^*(\omega) := \mathrm{pr}_{H_{0m}} \circ F(\omega)^*$$

and the same construction yields $t_j \in C(\Omega_1, H_1)$ for $j \in \mathbb{N}_{\leq d_1}$ such that each $(t_j(\omega))_{j \in \mathbb{N}_{\leq d_1}}$ is a basis of $\ker F_m^*(\omega)$. Furthermore, we have

$$\forall \omega \in \Omega_1 : \; \ker F(\omega)^* \subseteq \ker F_m^*(\omega).$$

Let $\hat{\Omega} := \Omega_0 \cap \Omega_1$. Then, we have

$$\forall \omega \in \hat{\Omega} : \; \ker F(\omega) \subseteq \ker F_n(\omega) \;\wedge\; \ker F(\omega)^* \subseteq \ker F_m^*(\omega).$$

Furthermore, the co-dimension of H_{1n} increases by one if n is replaced by $n+1$. Since the index of $F_n(\omega)$ is constant with respect to n, this means that the dimension of $\ker F_n(\omega)$ must increase by one, as well. Hence, it is possible to choose $m \geq \dim \ker F(\omega_0)$ and $n \geq \dim \ker F(\omega_0)^*$ such that

$$\dim \ker F_n(\omega) - \dim \ker F(\omega) = \dim \ker F_m^*(\omega) - \dim \ker F(\omega)^*,$$

i.e.

$$\dim \ker F_n(\omega) - \dim \ker F_m^*(\omega) = \dim \ker F(\omega) - \dim \ker F(\omega)^*$$

$$= \operatorname{ind} F(\omega)$$

$$= \operatorname{ind} F(\omega_0),$$

e.g. by setting

$$m = \dim\ker F(\omega_0)$$

$$n = \dim\ker F(\omega_0)^* + \operatorname{ind} F(\omega_0) + \dim\ker F^*_{\dim\ker F(\omega_0)}(\omega)$$

$$- \dim\ker F_{\dim\ker F(\omega_0)^*}(\omega)$$

for $\dim\ker F_{\dim\ker F(\omega_0)^*}(\omega) - \dim\ker F^*_{\dim\ker F(\omega_0)}(\omega) \le \operatorname{ind} F(\omega_0)$ and

$$m = \dim\ker F(\omega_0) + \dim\ker F_{\dim\ker F(\omega_0)^*}(\omega) - \dim\ker F^*_{\dim\ker F(\omega_0)}(\omega)$$

$$- \operatorname{ind} F(\omega_0)$$

$$n = \dim\ker F(\omega_0)^*$$

for $\dim\ker F_{\dim\ker F(\omega_0)^*}(\omega) - \dim\ker F^*_{\dim\ker F(\omega_0)}(\omega) > \operatorname{ind} F(\omega_0)$.

Definition 11.1. *Let* $\mathcal{P}\left(\mathrm{CLR}(H_0, H_1)\right) \coloneqq \{A; A \subseteq \mathrm{CLR}(H_0, H_1)\}$ *be the power set of* $\mathrm{CLR}(H_0, H_1)$ *and let*

$$\mathrm{IND}\colon\ F(H_0, H_1) \to \mathcal{P}\left(\mathrm{CLR}(H_0, H_1)\right)$$

be defined such that, for $f \in F(H_0, H_1)$*,* $\mathrm{IND}(f)$ *is the set of all* $\ker f_n \oplus \ker f_m^*$ *satisfying* $m \in \mathbb{N}_{\ge \dim\ker f}$*,* $n \in \mathbb{N}_{\ge \dim\ker f^*}$*, and* $\dim\ker f_n - \dim\ker f_m^* = \operatorname{ind} f$*.*

Furthermore, the sets

$$B_{\mathrm{IND}}(f, \varepsilon) \coloneqq \Big\{ g \in \mathrm{IND}[F(H_0, H_1)];\ \exists x \in f\ \exists y \in g:\ \hat{\delta}(x, y) < \varepsilon$$
$$\wedge \left(\dim x = \min_{x' \in f} \dim x' \ \vee\ \dim y = \min_{y' \in g} \dim y' \right) \Big\}$$

for $\varepsilon \in \mathbb{R}_{>0}$ *and* $f \in \mathrm{IND}[F(H_0, H_1)]$ *define a subbasis of the topology* $\mathcal{T}_{\mathrm{IND}}$ *in* $\mathrm{IND}[F(H_0, H_1)]$*.*

This topologizes "$(\ker g + V_0) \oplus (\ker g^* + V_1)$" in $H_0 \oplus H_1$ with minimal $\dim V_0$ and still it suffices to show that continuity of the s_j implies gap-continuity of $\ker F_n$ in order to show that "$g \mapsto (\ker g + V_0) \oplus (\ker g^* + V_1)$" is continuous.

Proposition 11.2. *Let H_0 and H_1 be Hilbert spaces. Then,*

$$\mathrm{IND} \in C(F(H_0, H_1), \mathrm{IND}[F(H_0, H_1)]).$$

Proof. Let $\varepsilon \in \mathbb{R}_{>0}$ and $A \in F(H_0, H_1)$. Then, we define for $B \in B_{\hat{\delta}}\left(A, \frac{1}{3}\right)$

$$G(B): \ A_n \to H_{1n} \oplus \ker A_n;$$

$$x \mapsto \left(\mathrm{pr}_{H_1}\left(\mathrm{pr}_{A_n}\big|_{B_n}^{A_n}\right)^{-1} x \ , \ \mathrm{pr}_{\ker A_n} \mathrm{pr}_{H_0} x\right)$$

and

$$\tilde{G}(B) := G(B)\ \mathrm{pr}_{A_n}\big|_{B_n}^{A_n}: \ B_n \to H_{1n} \oplus \ker A_n$$

similar to $G(\omega)$ and $\tilde{G}(\omega)$ above where A takes the role of $F(\omega_0)$ and B the role of $F(\omega)$, and the constructions of A_n and B_n are as above. Then, $G(A_n)$ is an isomorphism, again. Let $\varepsilon_0 \in \mathbb{R}_{>0}$ such that for all $B \in B_{\hat{\delta}}(A, \varepsilon_0)$ the map $\tilde{G}(B)$ is an isomorphism. Since the same holds for the similar construction with respect to A^*, let ε_0 be sufficiently small such that $\tilde{G}(B^*)$ is an isomorphism, as well.

Let $s_j^A := \mathrm{pr}_{H_0} \tilde{G}(A)^{-1}(0, e_j)$ and $s_j^B := \mathrm{pr}_{H_0} \tilde{G}(B)^{-1}(0, e_j)$ for an orthonormal basis $(e_j)_j$ of $\ker A_n$, and $t_j^A := \mathrm{pr}_{H_1} \tilde{G}(A^*)^{-1}(0, e'_j)$ and $t_j^B := \mathrm{pr}_{H_1} \tilde{G}(B^*)^{-1}(0, e'_j)$ for an orthonormal basis $(e'_j)_j$ of $\ker A_m^*$ accordingly. Without loss of generality, let $\varepsilon_0 \in (0, \varepsilon)$ be sufficiently small such that each of the following conditions holds for every $B \in B_{\hat{\delta}}(A, \varepsilon_0)$.[2]

- $\sup_{\alpha_j \in \mathbb{C}} \left\{ \sum_j |\alpha_j| \left\| \tilde{G}(A) - \tilde{G}(B) \right\|_{\mathrm{Lip}} ; \ \sum_j \alpha_j s_j^A \in \partial B_{H_0} \right\} < \frac{\varepsilon}{\sqrt{2}}$

[2]Note that the first and third point are merely a matter of choosing ε_0 sufficiently small. However, we can also satisfy the second and fourth point since $s_j^B \to s_j^A$ and $t_j^B \to t_j^A$ as $B \to A$.

- $\sup_{\alpha_j \in \mathbb{C}} \left\{ \sum_j |\alpha_j| \left\| \tilde{G}(A) - \tilde{G}(B) \right\|_{\mathrm{Lip}} ;\ \sum_j \alpha_j s_j^B \in \partial B_{H_0} \right\} < \frac{\varepsilon}{\sqrt{2}}$
- $\sup_{\alpha_j \in \mathbb{C}} \left\{ \sum_j |\alpha_j| \left\| \tilde{G}(A^*) - \tilde{G}(B^*) \right\|_{\mathrm{Lip}} ;\ \sum_j \alpha_j t_j^A \in \partial B_{H_1} \right\} < \frac{\varepsilon}{\sqrt{2}}$
- $\sup_{\alpha_j \in \mathbb{C}} \left\{ \sum_j |\alpha_j| \left\| \tilde{G}(A^*) - \tilde{G}(B^*) \right\|_{\mathrm{Lip}} ;\ \sum_j \alpha_j t_j^B \in \partial B_{H_1} \right\} < \frac{\varepsilon}{\sqrt{2}}$

Then,

$$\begin{aligned}
&\left\| \left(\sum_j \alpha_j (s_j^A - s_j^B), \sum_j \beta_j (t_j^A - t_j^B) \right) \right\|_{H_0 \oplus H_1} \\
\leq &\sqrt{2} \max \left\{ \left\| \sum_j \alpha_j (s_j^A - s_j^B) \right\|_{H_0}, \left\| \sum_j \beta_j (t_j^A - t_j^B) \right\|_{H_1} \right\} \\
\leq &\sqrt{2} \max \left\{ \sum_j |\alpha_j| \left\| \tilde{G}(A) - \tilde{G}(B) \right\|_{\mathrm{Lip}}, \sum_j |\beta_j| \left\| \tilde{G}(A^*) - \tilde{G}(B^*) \right\|_{\mathrm{Lip}} \right\} \\
< &\varepsilon
\end{aligned}$$

implies $\mathrm{IND}(B) \in B_{\mathrm{IND}}(\mathrm{IND}(A), \varepsilon)$ whenever $B \in B_{\hat{\delta}}(A, \varepsilon_0)$.

□

Corollary 11.3. *Let H_0 and H_1 be Hilbert spaces, Ω a topological space, μ a Borel measure on Ω, $F \in C(\Omega, F(H_0, H_1))$, and $G \in \mathcal{M}(\mu, F(H_0, H_1))$. Then,*

$$\mathrm{IND} \circ F \in C(\Omega, \mathrm{IND}[F(H_0, H_1)])$$

and

$$\mathrm{IND} \circ G \in \mathcal{M}(\mu, \mathrm{IND}[F(H_0, H_1)]).$$

Note that the function $\mathrm{DIM} : \mathrm{IND}[F(H_0, H_1)] \to \mathbb{Z}$ defined as

$$\mathrm{DIM}(f) = \dim \ker f_n - \dim \ker f_m^*$$

for any $\ker f_n \oplus \ker f_m^* \in f$ is locally constant with respect to $\mathcal{T}_{\mathrm{IND}}$; in particular, it is continuous, i.e.

$$\mathrm{ind} = \mathrm{DIM} \circ \mathrm{IND} \in C(F(H_0, H_1), \mathbb{Z}).$$

Similarly, we may consider other functions than DIM, e.g. the odd first Chern character c_1 to obtain a measurable version of the spectral flow (cf. Proposition 7.3.1 in [**76**]).

Hence, we are able to consider measurable index bundles, that is, the integration theory extends. The next chapter will consider an example of "holomorphic functional calculus" in algebras, that do not have a holomorphic functional calculus, by means of a replaced phase function. Afterwards, we shall apply the integration theory to ζ-functions.

CHAPTER 12

"Holomorphic functional calculus" in algebras without holomorphic functional calculus via replacement of phase functions

As an example, we are now able to calculate the spectral invariants of the heat trace from this generalized point of view. Luckily, the algebra of pseudo-differential operators allows us to use the functional calculus which makes the calculations a lot easier. A more in-depth account of the calculations in the pseudo-differential case can be found in chapter 3 and the appendices A and B of [**31**]. For the purpose of this chapter, however, the extension to the Fourier Integral Operator case is the vital observation. In other words, this chapter is all about using the integration techniques above and applying them to formally use the idea of functional calculus with Fourier Integral Operators.

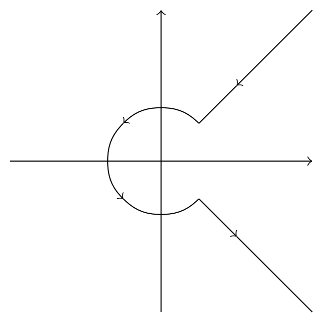
Example Let (X,g) be a compact Riemannian C^∞-manifold of dimension[1] $N \in 2\mathbb{N}$ without boundary. Let $|g|$ be the determinant of the metric tensor G and write $d\mathrm{vol}_X = \sqrt{|g|}dx$ with the Lebesgue measure dx in the parameter space. Then, the Laplace-Beltrami operator is given by

$$\Delta = -\frac{1}{\sqrt{|g|}}\partial_j g^{jk}\sqrt{|g|}\partial_k$$

[1]Note that $N \in 2\mathbb{N}$ has very far reaching implications; compare with stationary phase approximation and the problem $d + \frac{N-1}{2} - j \in -\mathbb{N}$ which cannot happen if $d \in \mathbb{Z}$ and $N \in 2\mathbb{N}$.

where g^{jk} are the coefficients of the inverse of the metric tensor G^{-1}. Let γ be the positively oriented contour

$$\left\{re^{i\frac{\pi}{4}};\ r\in\mathbb{R}_{\geq c}\right\}\cup\left\{ce^{i\varphi};\ \varphi\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]\right\}\cup\left\{re^{-i\frac{\pi}{4}};\ r\in\mathbb{R}_{\geq c}\right\}$$



with $c\in\mathbb{R}_{>0}$ and consider the integral

$$\frac{i}{2\pi}\int_\gamma e^{-\lambda t}(\Delta-\lambda)^{-1}d\lambda$$

which has the kernel

$$(2\pi)^{-N}\int_{\mathbb{R}^N}e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}}\frac{i}{2\pi}\int_\gamma e^{-\lambda t}\sigma\left((\Delta-\lambda)^{-1}\right)(x,\xi)d\lambda d\xi.$$

For now, we will ignore that we already know the existence of these integral since it is simply an application of the holomorphic functional calculus. Instead, we will use that $\sigma\left((\Delta-\lambda)^{-1}\right)$ has an asymptotic expansion

$$\sigma\left((\Delta-\lambda)^{-1}\right)(x,\xi)\sim\sum_{j\in\mathbb{N}_0}r_{-2-j}(x,\xi,\lambda)$$

with

$$r_{-2-j}(x,t\xi,t^2\lambda)=t^{-2-j}r_{-2-j}(x,\xi,\lambda)$$

whenever $t>0$ and $\|\xi\|_{\ell_2(N)}+|\lambda|^{\frac{1}{2}}\geq 1$. From

$$\begin{aligned}\Delta&=-\frac{1}{\sqrt{|g|}}\partial_j g^{jk}\sqrt{|g|}\partial_k\\&=-g^{jk}\partial_j\partial_k-(\partial_j g^{jk})\partial_k-\frac{1}{2|g|}g^{jk}(\partial_j|g|)\partial_k\end{aligned}$$

$$=g^{jk}(-i\partial_j)(-i\partial_k) + (-i\partial_j g^{jk})(-i\partial_k) + \frac{1}{2|g|}g^{jk}(-i\partial_j |g|)(-i\partial_k)$$

we obtain

$$\sigma(\Delta) = a_2(x,\xi) + a_1(x,\xi)$$

with

$$a_2(x,\xi) = g^{jk}(x)\xi_j\xi_k$$

and

$$a_1(x,\xi) = \left(\frac{1}{2|g|}g^{jk}(D_j |g|) + D_j g^{jk}\right)\xi_k$$

where $D_j := -i\partial_j$. Furthermore, we have the recursion (which follows from the formula of the symbol of the composition of pseudo-differential operators)

$$\begin{aligned} r_{-2}(x,\xi,\lambda) &= (a_2(x,\xi) - \lambda)^{-1} \\ r_{-2-j}(x,\xi,\lambda) &= -r_{-2}(x,\xi,\lambda) \sum_{(\mu,k,l)\in I_j} \frac{1}{\mu!}\left(\partial_2^\mu a_{2-k}(x,\xi)\right)\left(D_1^\mu r_{-2-l}(x,\xi,\lambda)\right) \end{aligned}$$

where

$$I_j := \left\{(\mu,k,l) \in \mathbb{N}_0^N \times \{0,1\} \times \mathbb{N}_{0,<j};\ \|\mu\|_{\ell_1(N)} + k + l = j\right\}.$$

To obtain the asymptotic expansion, it suffices to consider the integrals

$$(2\pi)^{-N}\int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}} \frac{i}{2\pi}\int_\gamma e^{-\lambda t} r_{-2-j}(x,\xi,\lambda)d\lambda d\xi.$$

Let $j = 0$. It is easy to see that $\lambda \mapsto e^{-t\lambda}(a_2 - \lambda)^{-1}$ is integrable taking values in the Hörmander class S^{-2} (a Fréchet space). Hence,

$$\frac{i}{2\pi}\int_\gamma e^{-\lambda t}(a_2 - \lambda)^{-1} d\lambda = e^{-ta_2}$$

is well-defined and the top-order contribution of $\operatorname{tr}\exp(-t\Delta)$ evaluates to

$$\begin{aligned}
&\int_X \left((2\pi)^{-N}\int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}} \frac{i}{2\pi}\int_\gamma e^{-\lambda t}(a_2(x,\xi)-\lambda)^{-1} d\lambda d\xi\right)\bigg|_{y=x} dx\\
=&\int_X (2\pi)^{-N}\int_{\mathbb{R}^N} \frac{i}{2\pi}\int_\gamma e^{-\lambda t}(a_2(x,\xi)-\lambda)^{-1} d\lambda d\xi dx\\
=&\int_X (2\pi)^{-N}\int_{\mathbb{R}^N} e^{-ta_2(x,\xi)} d\xi dx\\
=&\int_X (2\pi)^{-N}\int_{\mathbb{R}^N} \exp\left(-tg^{jk}(x)\xi_j\xi_k\right) d\xi dx\\
=&\int_X (2\pi)^{-N}\int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\left\langle tG^{-1}(x)\xi,\xi\right\rangle_{\mathbb{R}^N}\right) d\xi dx\\
=&\int_X (2\pi)^{-N}(2\pi)^{\frac{N}{2}}\left(\det\left((2t)^{-1}G\right)\right)^{\frac{1}{2}} dx\\
=&\int_X (4\pi t)^{-\frac{N}{2}}\sqrt{|g|}dx\\
=&\frac{\operatorname{vol}_X(X)}{(4\pi t)^{\frac{N}{2}}}.
\end{aligned}$$

It is interesting to note that this extends the highest order pole coefficient of our previous observation

$$\zeta(T(t))(0) = \frac{\operatorname{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi t)^{\frac{N}{2}}}\sum_{\gamma\in\Gamma}\exp\left(-\frac{\|\gamma\|^2_{\ell_2(N)}}{4t}\right)$$

for the heat semi-group on a flat torus to a significantly larger class of (even-dimensional) manifolds.

For $j = 1$, the recursion yields

$$\begin{aligned}
I_1 :=& \left\{(\mu,k,l)\in\mathbb{N}_0^N\times\{0,1\}\times\mathbb{N}_{0,<1};\ \|\mu\|_{\ell_1(N)} + k + l = 1\right\}\\
=&\left\{(\mu,0,0)\in\mathbb{N}_0^N\times\mathbb{N}\times\mathbb{N};\ \|\mu\|_{\ell_1(N)} = 1\right\}\cup\{(0,1,0)\}
\end{aligned}$$

and

$$
\begin{aligned}
r_{-3} &= -r_{-2} \sum_{(\mu,k,l)\in I_j} \frac{1}{\mu!} (\partial_2^\mu a_{2-k}) (D_1^\mu r_{-2-l}) \\
&= -r_{-2}^2 a_1 - r_{-2} \sum_{\|\mu\|_{\ell_1(N)}=1} (\partial_2^\mu a_2) (D_1^\mu r_{-2}) \\
&= -r_{-2}^2 a_1 + r_{-2} \sum_{\|\mu\|_{\ell_1(N)}=1} (\partial_2^\mu a_2) (D_1^\mu a_2) r_{-2}^2 \\
&= r_{-2}^3 \underbrace{\sum_{\|\mu\|_{\ell_1(N)}=1} (\partial_2^\mu a_2) (D_1^\mu a_2)}_{=:b} - r_{-2}^2 a_1 .
\end{aligned}
$$

Again, it is easy to see that $\lambda \mapsto e^{-t\lambda} r_{-3}(\cdot_1, \cdot_2, \lambda)$ is integrable with values in the Hörmander class S^{-3}. Cauchy's integral formula

$$
\partial^n f(z_0) = \frac{n!}{2\pi i} \int_{\tilde\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz,
$$

where $\tilde\gamma$ is a cycle around z_0 with winding number one, allows us to calculate the next coefficient

$$
\begin{aligned}
&\int_X (2\pi)^{-N} \int_{\mathbb{R}^N} \frac{i}{2\pi} \int_\gamma e^{-\lambda t} \left(r_{-2}(x,\xi,\lambda)^3 b(x,\xi) - r_{-2}(x,\xi,\lambda)^2 a_1(x,\xi) \right) d\lambda d\xi dx \\
&= \int_X (2\pi)^{-N} \int_{\mathbb{R}^N} b(x,\xi) \frac{i}{2\pi} \int_\gamma e^{-\lambda t} r_{-2}(x,\xi,\lambda)^3 d\lambda d\xi dx \\
&\quad - \int_X (2\pi)^{-N} \int_{\mathbb{R}^N} a_1(x,\xi) \frac{i}{2\pi} \int_\gamma e^{-\lambda t} r_{-2}(x,\xi,\lambda)^2 d\lambda d\xi dx \\
&= \int_X (2\pi)^{-N} \int_{\mathbb{R}^N} b(x,\xi) \frac{i}{2\pi} \int_\gamma \frac{e^{-\lambda t}}{(a_2(x,\xi) - \lambda)^3} d\lambda d\xi dx \\
&\quad - \int_X (2\pi)^{-N} \int_{\mathbb{R}^N} a_1(x,\xi) \frac{i}{2\pi} \int_\gamma \frac{e^{-\lambda t}}{(a_2(x,\xi) - \lambda)^2} d\lambda d\xi dx \\
&= \int_X (2\pi)^{-N} \int_{\mathbb{R}^N} b(x,\xi) \frac{t^2}{2} \exp(-t a_2(x,\xi)) \, d\xi dx \\
&\quad - \int_X (2\pi)^{-N} \int_{\mathbb{R}^N} a_1(x,\xi) t \exp(-t a_2(x,\xi)) \, d\xi dx.
\end{aligned}
$$

Since $\xi \mapsto b(x,\xi)\frac{t^2}{2}$ and $\xi \mapsto a_1(x,\xi)t$ are polynomials where each monomial $c_\alpha \xi^\alpha$ has an odd number of variables, that is, $\|\alpha\|_{\ell_1(N)} \in 2\mathbb{N} - 1$, it follows (cf. equation

(3.27) in [**31**]) that both inner integrals vanish, i.e.

$$\int_X (2\pi)^{-N} \int_{\mathbb{R}^N} \frac{i}{2\pi} \int_\gamma e^{-\lambda t} \left(r_{-2}(x,\xi,\lambda)^3 b(x,\xi) - r_{-2}(x,\xi,\lambda)^2 a_1(x,\xi) \right) d\lambda d\xi dx = 0.$$

For $j \in \mathbb{N}_{\geq 2}$, the recursion

$$r_{-2}(x,\xi,\lambda) = (a_2(x,\xi) - \lambda)^{-1}$$

$$r_{-2-j}(x,\xi,\lambda) = -\, r_{-2}(x,\xi,\lambda) \sum_{(\mu,k,l)\in I_j} \frac{1}{\mu!} \left(\partial_2^\mu a_{2-k}(x,\xi)\right) \left(D_1^\mu r_{-2-l}(x,\xi,\lambda)\right)$$

yields that each $\lambda \mapsto r_{-2-j}(\cdot_1,\cdot_2,\lambda)$ takes values in S^{-2-j} ($\partial_2^\mu a_{2-k} \in S^{2-k-\|\mu\|_{\ell_1(N)}}$, $r_{-2}(\cdot_1,\cdot_2,\lambda) \in S^{-2}$, and $D_1^\mu r_{-2-l}(\cdot_1,\cdot_2,\lambda) \in S^{-2-l}$). Furthermore, note that a_2 and a_1 can be written as sums $a_2(x,\xi) = \sum_j \alpha_{2,j}(x)\sigma_{2,j}(\xi)$ and $a_1(x,\xi) = \sum_j \alpha_{1,j}(x)\sigma_{1,j}(\xi)$ where the $\sigma_{i,j}$ are monomials of degree i. Assuming

$$r_{-2-l}(x,\xi,\lambda) = \sum_{k=1}^{n_l} r_{-2}(x,\xi,\lambda)^k b_{l,k}(x) s_{l,k}(\xi)$$

(which holds for $l = 0$ with $n_0 = 1$, $b_{0,1} = 1$, and $s_{0,1} = 1$) for all $l \in \mathbb{N}_{0,<j}$ implies there are functions β_κ which are sums of products $\sigma_{2,k} D^\nu \alpha_{2,j}$ such that

$$\begin{aligned}
r_{-2-j} &= -\, r_{-2} \sum_{(\mu,k,l)\in I_j} \frac{1}{\mu!} \left(\partial_2^\mu a_{2-k}\right) \left(D_1^\mu \left(\sum_{k=1}^{n_l} r_{-2}^k b_{l,k} s_{l,k} \right) \right) \\
&= -\, r_{-2} \sum_{(\mu,k,l)\in I_j} \frac{1}{\mu!} \left(\partial_2^\mu a_{2-k}\right) \sum_{\nu\leq\mu} \binom{\mu}{\nu} \sum_{k=1}^{n_l} D_1^\nu \left(r_{-2}^k\right) D^{\mu-\nu} b_{l,k} s_{l,k} \\
&= -\, r_{-2} \sum_{(\mu,k,l)\in I_j} \frac{1}{\mu!} \left(\partial_2^\mu a_{2-k}\right) \sum_{\nu\leq\mu} \binom{\mu}{\nu} \sum_{k=1}^{n_l} D_1^\nu \left((a_2-\lambda)^{-k}\right) D^{\mu-\nu} b_{l,k} s_{l,k} \\
&= -\, r_{-2} \sum_{(\mu,k,l)\in I_j} \frac{1}{\mu!} \sum_m \sigma_{2-k,m} \partial^\mu \alpha_{2-k,m} \sum_{\nu\leq\mu} \binom{\mu}{\nu} \sum_{k=1}^{n_l} \sum_{\kappa\leq\nu} \beta_\kappa r_{-2}^{k+\|\kappa\|_{\ell_1(N)}} D^{\mu-\nu} b_{l,k} s_{l,k}
\end{aligned}$$

holds. In other words (inductively), all r_{-2-j} are sums of terms of the form $r_{-2}(x,\xi,\lambda)^k s(x,\xi)$ where the $s(x,\xi)$ are polynomials in ξ. Hence, the j^{th} coefficient is given by a sum of integrals

$$\int_X (2\pi)^{-N} \int_{\mathbb{R}^N} \frac{i}{2\pi} \int_\gamma e^{-\lambda t} r_{-2}(x,\xi,\lambda)^k s(x,\xi) d\lambda d\xi dx.$$

Again, the functions $\lambda \mapsto e^{-\lambda t} r_{-2}(\cdot_1, \cdot_2, \lambda)^k s(x,\xi)$ are integrable with values in some Hörmander class (making the integrals well-defined) and we obtain

$$\begin{aligned}
&\int_X (2\pi)^{-N} \int_{\mathbb{R}^N} \frac{i}{2\pi} \int_\gamma e^{-\lambda t} r_{-2}(x,\xi,\lambda)^k s(x,\xi) d\lambda d\xi dx \\
=&\int_X (2\pi)^{-N} \int_{\mathbb{R}^N} s(x,\xi) \frac{i}{2\pi} \int_\gamma e^{-\lambda t} r_{-2}(x,\xi,\lambda)^k d\lambda d\xi dx \\
=&\int_X (2\pi)^{-N} \int_{\mathbb{R}^N} s(x,\xi) \frac{t^{k-1}}{(k-1)!} \exp\left(-ta_2(x,\xi)\right) d\xi dx.
\end{aligned}$$

As explained in [**31**], the inner integrals can be evaluated

$$\begin{aligned}
&(2\pi)^{-N} \int_{\mathbb{R}^N} s(x,\xi) \frac{t^{k-1}}{(k-1)!} \exp\left(-ta_2(x,\xi)\right) d\xi \\
=&(2\pi)^{-N} \left(\frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det\left(2tG(x)^{-1}\right)}} \exp\left(-\frac{1}{2} \operatorname{div}_2 G(x) \operatorname{grad}_2\right) s(x,\cdot) \frac{t^{k-1}}{(k-1)!} \right)(0) \\
=&\left(\frac{\sqrt{|g(x)|}}{(4\pi t)^{\frac{N}{2}}} \exp\left(-\frac{1}{2} \operatorname{div}_2 G(x) \operatorname{grad}_2\right) s(x,\cdot) \frac{t^{k-1}}{(k-1)!} \right)(0) \\
=&\frac{\sqrt{|g(x)|} t^{k-1-\frac{N}{2}}}{(4\pi)^{\frac{N}{2}} (k-1)!} \left(\exp\left(-\frac{1}{2} \operatorname{div}_2 G(x) \operatorname{grad}_2\right) s(x,\cdot) \right)(0).
\end{aligned}$$

We shall not include higher order calculations here as these get rather lengthy very soon. However, in [**31**] (equation 3.64) the explicit calculation for the $j = 2$ term can be found which produces

$$\frac{t^{1-\frac{N}{2}}}{3(4\pi)^{\frac{N}{2}}} \text{total curvature}(X).$$

■

Example Using our general integration theory, we obtain that replacing the phase function $\langle x-y,\xi\rangle_{\mathbb{R}^N}$ by $\vartheta(x,y,\xi)$ in

$$(2\pi)^{-N} \int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle_{\mathbb{R}^N}} \frac{i}{2\pi} \int_\gamma e^{-\lambda t} r_{-2-j}(x,\xi,\lambda) d\lambda d\xi$$

is perfectly fine (because $\mathcal{D}'_\Gamma$ is at least quasi-complete if you choose the Hörmander or any finer topology - Proposition 29 in [**17**]; hence, the integrals all converge in

$\mathcal{D}'_\Gamma$) and we obtain, for instance,

$$(2\pi)^{-N}\int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}\frac{i}{2\pi}\int_\gamma e^{-\lambda t}r_{-2}(x,\xi,\lambda)d\lambda d\xi = (2\pi)^{-N}\int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}e^{-ta_2(x,\xi)}d\xi.$$

Considering a linear phase function $\vartheta(x,y,\xi) = \langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N}$ the integrand becomes

$$\exp\left(i\langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N} - \frac{1}{2}\left\langle\xi, 2tG(x)^{-1}\xi\right\rangle_{\mathbb{R}^N}\right)$$

which is the characteristic function φ_Y of a normally distributed random variable Y with mean $\Theta(x,y)$ and covariance $2tG(x)^{-1}$. Since $Z \in \mathcal{N}(\mu,\sigma)$ (that is, a normally distributed random variable with mean μ and standard deviation σ) has the density

$$f_Z(\eta) = \frac{\exp\left(-\frac{\langle\eta-\mu,\sigma^{-1}(\eta-\mu)\rangle_{\mathbb{R}^N}}{2}\right)}{\sqrt{(2\pi)^N\det\sigma}} = (2\pi)^{-N}\int_{\mathbb{R}^N} e^{-i\langle t,\eta\rangle_{\mathbb{R}^N}}\varphi_Z(t)dt,$$

we conclude

$$\begin{aligned} f_Y(0) &= (2\pi)^{-N}\int_{\mathbb{R}^N} e^{i\vartheta(x,y,\xi)}e^{-ta_2(x,\xi)}d\xi \\ &= \left((2\pi)^N\det\left(2tG(x)^{-1}\right)\right)^{-\frac{1}{2}}\exp\left(-\frac{\langle\Theta(x,y),G(x)\Theta(x,y)\rangle_{\mathbb{R}^N}}{4t}\right) \\ &= \frac{\sqrt{|g|}}{(4\pi t)^{\frac{N}{2}}}\exp\left(-\frac{\langle\Theta(x,y),G(x)\Theta(x,y)\rangle_{\mathbb{R}^N}}{4t}\right). \end{aligned}$$

In other words, the first coefficient in the trace expansion is given by

$$\frac{1}{(4\pi t)^{\frac{N}{2}}}\int_X \exp\left(-\frac{\|\Theta(x,x)\|_g^2}{4t}\right)d\mathrm{vol}_X(x)$$

where $\|\Theta(x,x)\|_g^2 = \langle\Theta(x,x),G(x)\Theta(x,x)\rangle_{\mathbb{R}^N}$. In particular, if $\forall x\in X:\ \Theta(x,x) = 0$, then we are reduced to the example above. If we have a pseudo-differential operator on the quotient $\mathbb{R}^N/\Gamma$, we obtain $\vartheta_\gamma(x,y,\xi) = \langle x-y-\gamma,\xi\rangle_{\mathbb{R}^N}$, and have to sum over $\gamma\in\Gamma$, i.e.

$$\sum_{\gamma\in\Gamma}\frac{1}{(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N/\Gamma}\exp\left(-\frac{\|\gamma\|_g^2}{4t}\right)d\mathrm{vol}_{\mathbb{R}^N/\Gamma}(x) = \frac{\mathrm{vol}_{\mathbb{R}^N/\Gamma}\left(\mathbb{R}^N/\Gamma\right)}{(4\pi t)^{\frac{N}{2}}}\sum_{\gamma\in\Gamma}\exp\left(-\frac{\|\gamma\|_{\ell_2(N)}^2}{4t}\right).$$

For $j > 0$ we also have polynomial factors to consider, that is, we have integrals of the form

$$(2\pi)^{-N}\int_{\mathbb{R}^N} p(x,\xi,t)e^{i\langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N}-\frac{1}{2}\langle\xi,2tG(x)^{-1}\xi\rangle_{\mathbb{R}^N}}d\xi$$

where p is a polynomial in ξ. For any monomial ξ^α, we obtain

$$\begin{aligned}
&(2\pi)^{-N}\int_{\mathbb{R}^N}\xi^\alpha e^{i\langle\Theta(x,y),\xi\rangle_{\mathbb{R}^N}-\frac{1}{2}\langle\xi,2tG(x)^{-1}\xi\rangle_{\mathbb{R}^N}}d\xi\\
=&(2\pi)^{-\frac{N}{2}}\mathcal{F}\left(\xi\mapsto\xi^\alpha e^{-\frac{1}{2}\langle\xi,2tG(x)^{-1}\xi\rangle_{\mathbb{R}^N}}\right)(-\Theta(x,y))\\
=&(2\pi)^{-\frac{N}{2}}\left((i\partial)^\alpha\mathcal{F}\left(\xi\mapsto e^{-\frac{1}{2}\langle\xi,2tG(x)^{-1}\xi\rangle_{\mathbb{R}^N}}\right)\right)(-\Theta(x,y))\\
=&(2\pi)^{-\frac{N}{2}}\left((i\partial)^\alpha\left(\eta\mapsto\sqrt{\det\left((2t)^{-1}G(x)\right)}e^{-\frac{1}{2}\langle\eta,(2t)^{-1}G(x)\eta\rangle_{\mathbb{R}^N}}\right)\right)(-\Theta(x,y))\\
=&\frac{\sqrt{|g|}}{(4\pi t)^{\frac{N}{2}}}\left((i\partial)^\alpha\left(\eta\mapsto e^{-\frac{1}{4t}\langle\eta,G(x)\eta\rangle_{\mathbb{R}^N}}\right)\right)(-\Theta(x,y))
\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. Let $f(\eta) := e^{-\frac{1}{4t}\langle\eta,G(x)\eta\rangle_{\mathbb{R}^N}}$. Then, the j^{th} coefficient is given by a sum of integrals

$$\int_X\frac{s(x,t)}{(4\pi t)^{\frac{N}{2}}}\left((i\partial)^\alpha f\right)(-\Theta(x,x))d\mathrm{vol}_X(x)$$

where the s are polynomials in t.

For even more general phase functions, we will introduce polar coordinates. Then, coefficients are sums of integrals of the form

$$\int_{\partial B_{\mathbb{R}^N}}\int_{\mathbb{R}_{>0}}p(x,y,\eta)r^k e^{i\vartheta(x,y,\eta)r}e^{-t\langle\eta,G(x)^{-1}\eta\rangle_{\mathbb{R}^N}r^2}\ dr\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta),$$

that is,

$$\int_{\partial B_{\mathbb{R}^N}}p(x,y,\eta)\ \underbrace{\mathcal{L}\left(r\mapsto r^k e^{-t\langle\eta,G(x)^{-1}\eta\rangle_{\mathbb{R}^N}r^2}\right)}_{=(-1)^k\partial^k\mathcal{L}\left(r\mapsto e^{-t\langle\eta,G(x)^{-1}\eta\rangle_{\mathbb{R}^N}r^2}\right)}(-i\vartheta(x,y,\eta))\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)$$

where $\mathcal{L}$ denotes the Laplace transform. These can (in principle) be evaluated since

$$\mathcal{L}\left(r\mapsto e^{-ar^2}\right)(s)=\int_{\mathbb{R}_{>0}}e^{-sr}e^{-ar^2}dr$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{a}} \int_{\mathbb{R}_{>0}} e^{-r^2 - \frac{s}{\sqrt{a}} r} \, dr \\
&= \frac{1}{\sqrt{a}} \int_{\mathbb{R}_{>0}} e^{-\left(r + \frac{s}{2\sqrt{a}}\right)^2} e^{\frac{s^2}{4a}} \, dr \\
&= \frac{e^{\frac{s^2}{4a}}}{\sqrt{a}} \int_{\mathbb{R}_{>\frac{s}{2\sqrt{a}}}} e^{-r^2} \, dr \\
&= \frac{\sqrt{\pi} e^{\frac{s^2}{4a}}}{2\sqrt{a}} \operatorname{erfc}\left(\frac{s}{2\sqrt{a}}\right)
\end{aligned}
$$

where erfc denotes the complementary error function (an entire function) which is defined by the holomorphic extension of

$$
\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}_{>z}} e^{-r^2} \, dr
$$

for $z \in \mathbb{R}$. Let

$$
f(s) := \frac{\sqrt{\pi}}{2\sqrt{t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} e^{\frac{s^2}{4t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} \operatorname{erfc}\left(\frac{s}{2\sqrt{t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}}\right).
$$

Since

$$
\partial \operatorname{erfc}(z) = -\frac{2}{\sqrt{\pi}} e^{-z^2},
$$

it follows that

$$
\partial\left(s \mapsto \operatorname{erfc}\left(\frac{s}{2\sqrt{t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}}\right)\right)(z) = -\frac{1}{\sqrt{t\pi\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} e^{-\frac{z^2}{4t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}},
$$

i.e.

$$
\begin{aligned}
\partial f(s) =& \frac{\sqrt{\pi}}{2\sqrt{t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} \frac{s}{2t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}} e^{\frac{s^2}{4t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} \operatorname{erfc}\left(\frac{s}{2\sqrt{t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}}\right) \\
&- \frac{\sqrt{\pi}}{2\sqrt{t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} e^{\frac{s^2}{4t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} \frac{1}{\sqrt{t\pi\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} e^{-\frac{s^2}{4t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} \\
=& \frac{s\sqrt{\pi}}{4\left(t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}\right)^{\frac{3}{2}}} e^{\frac{s^2}{4t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}} \operatorname{erfc}\left(\frac{s}{2\sqrt{t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}}\right) - \frac{1}{2t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}} \\
=& \frac{s}{2t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}} f(s) - \frac{1}{2t\langle \eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}
\end{aligned}
$$

and we obtain, inductively,

$$\partial^n f(s) = p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right) f(s) + q_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right)$$

with $p_0(x,y) = 1$, $q_0(x,y) = 0$. Furthermore,

$$\begin{aligned}
&\partial\left(\sigma \mapsto p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, \sigma\right) f(s) + q_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, \sigma\right)\right)(s) \\
=&\partial_2 p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right) f(s) + p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right) f'(s) \\
&+ \partial_2 q_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right) \\
=&\partial_2 p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right) f(s) \\
&+ p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right)\left(\frac{s}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}} f(s) - \frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}\right) \\
&+ \partial_2 q_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right) \\
=&\left(\partial_2 p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right) + p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right)\frac{s}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}\right) f(s) \\
&- p_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right)\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}} + \partial_2 q_n\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, s\right)
\end{aligned}$$

implies

$$p_n(x,y) = \partial_2 p_{n-1}(x,y) + xy p_{n-1}(x,y),$$

and

$$q_n(x,y) = \partial_2 q_{n-1}(x,y) - x p_{n-1}(x,y).$$

In particular, p_n is a polynomial of degree n in both arguments, as is q_n in the first argument, whereas q_n is a polynomial of degree $n-1$ in the second argument.

Let $\vartheta(x,y,\xi) := \langle x-y, \xi\rangle_{\mathbb{R}^N} \pm \|\xi\|_{\ell_2(N)}$. For $x = y$ and $j = 0$, we have $p = 1$, i.e.

$$\begin{aligned}
&\int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}\partial^{N-1}\mathcal{L}\left(r \mapsto e^{-t\langle\eta, G(x)^{-1}\eta\rangle_{\mathbb{R}^N} r^2}\right)(\mp i)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \\
=&\int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}\partial^{N-1} f(\mp i)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)
\end{aligned}$$

to evaluate. For $N = 1$, this collapses to

$$\begin{aligned} f(\mp i)\big|_{\eta=1} + f(\mp i)\big|_{\eta=-1} =& 2\frac{\sqrt{\pi}}{2\sqrt{tG^{-1}}}e^{\frac{(\mp i)^2}{4tG^{-1}}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{tG^{-1}}}\right) \\ =& \frac{\sqrt{\pi}}{\sqrt{tG^{-1}}}e^{-\frac{1}{4tG^{-1}}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{tG^{-1}}}\right). \end{aligned}$$

Thus, the leading coefficient is given by

$$\int_X \frac{\sqrt{\pi}}{\sqrt{tG^{-1}(x)}}e^{-\frac{1}{4tG^{-1}(x)}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{tG^{-1}(x)}}\right)dx = \int_X \frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{|g|}{4t}}\operatorname{erfc}\left(\frac{\mp i\sqrt{|g|}}{2\sqrt{t}}\right)d\mathrm{vol}_X.$$

For $N \geq 2$

$$\int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}\partial^{N-1}f(\mp i)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta)$$

becomes

$$\begin{aligned} &\int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}p_{N-1}\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, \mp i\right)f(\mp i)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \\ &+ \int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}q_{N-1}\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, \mp i\right)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \\ =& \int_{\partial B_{\mathbb{R}^N}} \tilde{p}_{N-1}\left(\frac{1}{\sqrt{2t\langle\eta, G^{-1}\eta\rangle}}, \mp i\right)e^{\frac{-1}{4t\langle\eta, G^{-1}\eta\rangle}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t\langle\eta, G^{-1}\eta\rangle}}\right)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \\ &+ \int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}q_{N-1}\left(\frac{1}{2t\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N}}, \mp i\right)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \end{aligned}$$

where

$$\tilde{p}_{N-1}(x,y) := \frac{(-1)^{N-1}\sqrt{\pi}}{\sqrt{2}}xp_{N-1}\left(x^2, y\right).$$

Supposing we have a flat manifold with $G^{-1}(x) = 1$, i.e. $\langle\eta, G^{-1}\eta\rangle_{\mathbb{R}^N} = 1$, we observe

$$\begin{aligned} &\int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}\partial^{N-1}f(\mp i)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \\ =& \int_{\partial B_{\mathbb{R}^N}} \tilde{p}_{N-1}\left(\frac{1}{\sqrt{2t}}, \mp i\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \\ &+ \int_{\partial B_{\mathbb{R}^N}} (-1)^{N-1}q_{N-1}\left(\frac{1}{2t}, \mp i\right)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^N}}(\eta) \\ =& (-1)^{N-1}\mathrm{vol}_{\partial B_{\mathbb{R}^N}}\left(\partial B_{\mathbb{R}^N}\right)\left(\sqrt{\frac{\pi}{4t}}p_{N-1}\left(\frac{1}{2t}, \mp i\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right) + q_{N-1}\left(\frac{1}{2t}, \mp i\right)\right) \end{aligned}$$

$$=(-1)^{N-1}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left(\sqrt{\frac{\pi}{4t}}p_{N-1}\left(\frac{1}{2t},\mp i\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)+q_{N-1}\left(\frac{1}{2t},\mp i\right)\right),$$

i.e. the leading coefficient becomes

$$\frac{(-1)^{N-1}2\pi^{\frac{N}{2}}\operatorname{vol}_X(X)}{\Gamma\left(\frac{N}{2}\right)}\left(\sqrt{\frac{\pi}{4t}}p_{N-1}\left(\frac{1}{2t},\mp i\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)+q_{N-1}\left(\frac{1}{2t},\mp i\right)\right).$$

Since $p_0(x,y)=1$, $q_0(x,y)=0$,

$$p_n(x,y)=\partial_2 p_{n-1}(x,y)+xyp_{n-1}(x,y),$$

and

$$q_n(x,y)=\partial_2 q_{n-1}(x,y)-xp_{n-1}(x,y)$$

hold, we obtain

n	$p_n(x,y)$	$p_n\left(\frac{1}{2t},\mp i\right)$	$q_n(x,y)$	$q_n\left(\frac{1}{2t},\mp i\right)$
0	1	1	0	0
1	xy	$\mp i\frac{1}{2t}$	$-x$	$-\frac{1}{2t}$
2	$x+x^2y^2$	$\frac{1}{2t}-\frac{1}{4t^2}$	$-x^2y$	$\pm i\frac{1}{4t^2}$
3	$3x^2y+x^2y^2$	$\frac{1}{4t^2}\left(\mp 3i-1\right)$	$-2x^2-x^3y^2$	$\frac{-1}{2t^2}+\frac{1}{8t^3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

which yields

N	$\frac{(-1)^{N-1}2\pi^{\frac{N}{2}}\mathrm{vol}_X(X)}{\Gamma\left(\frac{N}{2}\right)}\left(\sqrt{\frac{\pi}{4t}}p_{N-1}\left(\frac{1}{2t},\mp i\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)+q_{N-1}\left(\frac{1}{2t},\mp i\right)\right)$
1	$2\mathrm{vol}_X(X)\left(\sqrt{\frac{\pi}{4t}}e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)\right)$
2	$-2\pi\mathrm{vol}_X(X)\left(\sqrt{\frac{\pi}{4t}}\left(\mp i\frac{1}{2t}\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)-\frac{1}{2t}\right)$
3	$4\pi\mathrm{vol}_X(X)\left(\sqrt{\frac{\pi}{4t}}\left(\frac{1}{2t}-\frac{1}{4t^2}\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)\pm i\frac{1}{4t^2}\right)$
4	$-\pi^2\mathrm{vol}_X(X)\left(\sqrt{\frac{\pi}{4t}}\frac{1}{4t^2}\left(\mp 3i-1\right)e^{\frac{-1}{4t}}\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)+\frac{1}{8t^3}-\frac{1}{2t^2}\right)$
$\vdots$	$\vdots$

where we used $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ and $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$. The complementary error function can be evaluated using the upper incomplete Γ-function Γ_{ui} which satisfies $\operatorname{erfc}(z)=\frac{1}{\sqrt{\pi}}\Gamma_{ui}\left(\frac{1}{2},z^2\right)$ or

$$\operatorname{erfc}(z)=1-\frac{2}{\sqrt{\pi}}\sum_{k\in\mathbb{N}_0}\frac{(-1)^k z^{2k+1}}{k!(2k+1)},$$

i.e.

$$\operatorname{erfc}\left(\frac{\mp i}{2\sqrt{t}}\right)=1\pm\frac{i}{\sqrt{\pi t}}\sum_{k\in\mathbb{N}_0}\frac{(4t)^{-k}}{k!(2k+1)}.$$

∎

CHAPTER 13

The ζ-function on Hörmander spaces $\mathcal{D}'_\Gamma$

Since Radzikowski [**57, 58**] showed the importance of the wave front set in quantum field theories on curved space-time, the Hörmander spaces $\mathcal{D}'_\Gamma$ (set of distributions with wave front set in the closed cone Γ such that the semi-norms $\|\cdot\|_{N,V,\chi}$ in Definition 13.1 are finite) have become very important in the re-formulation of quantum field theories. In this chapter, we want to return to the ζ-function and study it on those spaces $\mathcal{D}'_\Gamma$ whose topological properties were studied in [**15–17**]. There are multiple canonical[1] topologies on $\mathcal{D}'_\Gamma$; most notably, the normal topology (which is the Arens topology as introduced by Arens in [**2**]; the topology of uniform convergence on absolutely convex[2] compact sets) and the coarser Hörmander topology (defined in [**39**] on p. 125) which is given by the following semi-norms (Definition 8.2.2 in [**38**]).

Definition 13.1. *Let $U \subseteq \mathbb{R}^n$ be open, Γ a closed cone in the co-tangent bundle of U without the zero section, and $\mathcal{D}'_\Gamma \subseteq C_c^\infty(U)'$ the set of distributions with wave front set in Γ such that the following semi-norms are finite.*

(i) *For $f \in C_c^\infty(U)$ we define*

$$p_f:\ \mathcal{D}'_\Gamma \to \mathbb{R};\ u \mapsto |\langle u, f\rangle|.$$

[1]Hörmander initially defined the topology as a pseudo-topology, that is, he defined what convergent sequences and their limits are. It should be noted that not every pseudo-topology defines a topology; for instance, there is no topology of almost everywhere convergence. In Hörmander's case, however, there are multiple different topologies which induce his pseudo-topology.

[2]A subset A of a topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is called absolutely convex if and only if $\forall x, y \in A\ \forall \lambda, \mu \in \mathbb{K}:\ (\ |\lambda| + |\mu| \leq 1\ \Rightarrow\ \lambda x + \mu y \in A\ )$.

(ii) *For $N \in \mathbb{N}$, a closed cone $V \subseteq \mathbb{R}^n$, and $\chi \in C_c^\infty(U)$ with $(\operatorname{spt}\chi \times V) \cap \Gamma = \varnothing$ (*spt *denotes the support), we define*

$$\|\cdot\|_{N,V,\chi} :\ \mathcal{D}'_\Gamma \to \mathbb{R};\ u \mapsto \sup_{k \in V} \left(1 + \|k\|_{\ell_2(n)}\right)^N |\mathcal{F}(\chi u)(k)|$$

where $\mathcal{F}$ denotes the Fourier transform which exists because χu is a compactly supported distribution.

In the light of Corollary 9.17, we will want to show that the ζ-function on gauged poly-log-homogeneous elements in $\mathcal{D}'_\Gamma$ defines a sequentially closed linear operator in a certain sense, i.e. it suffices to consider the coarser Hörmander topology which makes $\mathcal{D}'_\Gamma$ quasi-complete (cf. Proposition 29 in [**17**]). The topology on the set of gauged distributions in $\mathcal{D}'_\Gamma$ will be the induced topology of compact convergence in $C^\omega(\Omega, \mathcal{D}'_\Gamma)$ where $\Omega \subseteq \mathbb{C}$ is an open and connected set and C^ω denotes the set of analytic functions.

Definition 13.2. *Let E be a locally convex topological vector space with seminorms $(p_\iota^E)_{\iota \in I}$ and $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$. Then, we endow $C^\omega(\Omega, E)$ with the seminorms*

$$p_{\iota,K}^{C^\omega(\Omega,E)} :\ C^\omega(\Omega, E) \to \mathbb{R};\ f \mapsto \|p_\iota \circ f\|_{L_\infty(K)}$$

for every $\iota \in I$ and $K \subseteq_{\text{compact}} \Omega$.

Definition 13.3. *For $R \in \mathbb{R}$ and $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$ such that $\forall r \in \mathbb{R} :\ \{z \in \Omega;\ \Re(z) < r\} \neq \varnothing$, we define $\mathcal{D}'_{\Gamma,R,\Omega,\text{plh}} \subseteq C^\omega(\mathbb{C}, \mathcal{D}'_\Gamma)$ to be the set of gauged poly-*log*-homogeneous distributions in $\mathcal{D}'_\Gamma$ whose ζ-functions are holomorphic in Ω and none of the degrees of homogeneity at zero have real part greater than R. Furthermore, we define $\mathcal{D}'_{\Gamma,R,\Omega,\text{ph}} := \left\{u \in \mathcal{D}'_{\Gamma,R,\Omega,\text{plh}};\ u \text{ polyhomogeneous}\right\}$.*

With this prelude, we can state the following theorem.

THEOREM 13.4. *Let $R \in \mathbb{R}$ and $\Omega \subseteq \mathbb{C}$ be open and connected such that $\forall r \in \mathbb{R}: \{z \in \Omega;\ \Re(z) < r\} \neq \varnothing$. Then, $\zeta|_{\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}}}: \mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}} \to C^\omega(\Omega)$ has a quasi-complete extension*[3] $\zeta_{R,\Omega}$.

PROOF. Let $(v_\alpha, \zeta(v_\alpha))_{\alpha\in A}$ be a bounded net in $\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}} \oplus C^\omega(\Omega)$ with $v_\alpha \to 0$ and $\zeta(v_\alpha) \to: v \in C^\omega(\Omega)$. Then, we need to show $v = 0$. In fact, it suffices to show $[\{0\}]v$ has an accumulation point in Ω.

Let $z \in \Omega$. Then, $(v_\alpha(z))_{\alpha\in A}$ is a bounded net in $\mathcal{D}'_\Gamma$ and $(\zeta(v_\alpha)(z))_{\alpha\in A}$ is a bounded net in $\mathbb{C}$. In particular, $V := \{v_\alpha(z);\ \alpha \in A\}$ is metrizable (cf. Proposition 1 and Theorem 33 in [**17**]), as is $Z := \{\zeta(v_\alpha)(z);\ \alpha \in A\} \cup \{v(z)\}$. Hence, $\{(v_\alpha(z), \zeta(v_\alpha)(z));\ \alpha \in A\}$ is contained in the metrizable set $V \times Z$, i.e. $(0, v(z))$ can be calculated using sequences.

Let $(u_n(z))_{n\in\mathbb{N}} \in V^{\mathbb{N}}$ be such that $u_n(z) \to 0$ and $\zeta(u_n)(z) \to v(z)$. Note that $\zeta(u_n)(z)$ is the regularized dual pair $\langle u_n(z), \delta_{\mathrm{diag}}\rangle$. Let $(f_m)_{m\in\mathbb{N}}$ be a "δ-sequence" approximating δ_{diag}. Then,

$$\forall m \in \mathbb{N}:\ \langle u_n, f_m\rangle \to 0 \qquad (n \to \infty) \qquad \text{compactly}$$

holds by assumption and implies

$$\forall m \in \mathbb{N}:\ \langle u_n(z), f_m\rangle \to 0 \qquad (n \to \infty)$$

Furthermore, there exists $r \in \mathbb{R}$ such that

$$\forall z \in \Omega \cap \mathbb{C}_{\Re(\cdot)<r}:\ \langle u_n(z), \delta_{\mathrm{diag}}\rangle \text{ need not be regularized,}$$

[3]Just as the completion can be constructed by adding all limits of nets in $\zeta|_{\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}}}$, we can construct the quasi-complete "closure" by taking all bounded nets in $\zeta|_{\mathcal{D}'_{\Gamma,R,\Omega,\mathrm{plh}}}$ and add their limits if they converge in $\mathcal{D}'_\Gamma \oplus C^\omega(\Omega)$

i.e.

$$\langle u_n(z), f_m\rangle \to \zeta(u_n)(z) \qquad (m \to \infty).$$

Let $\varepsilon \in \mathbb{R}_{>0}$, $z \in \Omega$ with $\Re(z) < r$, $n \in \mathbb{N}$ such that $|v(z) - \zeta(u_n)(z)| < \frac{\varepsilon}{3}$, and $m \in \mathbb{N}$ such that $|\zeta(u_n)(z) - \langle u_n(z), f_m\rangle| < \frac{\varepsilon}{3}$ as well as $|\langle u_n(z), f_m\rangle - 0| < \frac{\varepsilon}{3}$. Then,

$$|v(z)| \le |v(z) - \zeta(u_n)(z)| + |\zeta(u_n)(z) - \langle u_n(z), f_m\rangle| + |\langle u_n(z), f_m\rangle| < \varepsilon$$

shows $\forall z \in \Omega \cap \mathbb{C}_{\Re(\cdot)<r}:\ v(z) = 0$, i.e. the assertion.

□

This theorem has a couple of very important consequences. On one hand, it allows us to extend the ζ-function to elements of $D(\zeta_{R,\Omega})$ which may very well include distributions that are not poly-log-homogeneous. On the other hand, and much more importantly, $\zeta_{R,\Omega}$ has the convex compactness property, i.e. we can calculate Pettis integral of continuous functions f on compact Borel spaces (K, Σ, μ) with values in $\zeta_{R,\Omega}$. In other words,

$$\int_K (f(x), \zeta_{R,\Omega}(f(x)))\, d\mu = \left(\int_K f(x)d\mu, \int_K \zeta_{R,\Omega}(f(x))d\mu\right) \in \zeta_{R,\Omega}$$

exists and implies

$$\int_K f(x)d\mu \in D(\zeta_{R,\Omega}) \ \wedge\ \zeta_{R,\Omega}\left(\int_K f(x)d\mu\right) = \int_K \zeta_{R,\Omega}(f(x))d\mu.$$

Remark So far, we only had the fundamental theorem of calculus which allowed the following. Let $a, b, c \in \mathbb{R}$, $a < b$, $c \in [a, b]$, $f \in C([a, b], C^\omega(\mathbb{C}, \mathcal{D}'_\Gamma))$, and

$$g:\ [a, b] \to C^\omega(\mathbb{C}, \mathcal{D}'_\Gamma);\ x \mapsto \int_c^x f(s)ds.$$

Then, g is differentiable with $g' = f$, i.e.

$$\int_a^b f(s)ds = \int_a^b g'(s)ds = g(b) - g(a).$$

Then, we obtain

$$\begin{aligned}\zeta\left(\int_a^b f(s)ds\right) &= \zeta(g(b)-g(a))\\ &= \zeta(g(b))-\zeta(g(a))\\ &= \int_a^b \partial_s\zeta(g(s))ds\\ &= \int_a^b \zeta(g'(s))ds\\ &= \int_a^b \zeta(f(s))ds.\end{aligned}$$

■

Since the restriction of considering only ζ-functions on a shared holomorphic domain is quite technical, it would seem more natural to consider ζ as a map from $C^\omega(\mathbb{C},\mathcal{D}'_\Gamma)$ to the set of meromorphic functions $\mathrm{Mer}(\mathbb{C})$. Furthermore, we would like to still have compact convergence on holomorphic domains; i.e. we are looking for a locally convex Hausdorff topology on $\mathrm{Mer}(\mathbb{C})$ which extends the topology of compact convergence. This, however, is a rather delicate problem.

The probably most natural way of topologizing $\mathrm{Mer}(\Omega)$ for $\Omega \subseteq_{\mathrm{open,connected}} \mathbb{C}$ non-empty was introduced by Ostrowski [**54**] and regards $\mathrm{Mer}(\Omega)$ as a subspace of $C(\Omega,\hat{\mathbb{C}})$ where $\hat{\mathbb{C}}$ is the extended complex plane with the chordal metric (that is, identification with the Riemann sphere and using the induced $\ell_2(3)$ metric of $\mathbb{R}^3$). $C(\Omega,\hat{\mathbb{C}})$ is then endowed with the topology of compact convergence and the induced topology τ_c on $\mathrm{Mer}(\Omega)$ makes $\mathrm{Mer}(\Omega)$ a metric space which is complete if we add the constant function ∞; cf. chapter VII.3 in [**14**]. Unfortunately, this topology is not linear. In fact, Cima and Schober showed (Proposition 4 in [**12**]) that there is no locally convex vector space topology comparable with τ_c.

In **[72]** Tietz introduced a locally convex topology based on the Mittag-Leffler theorem which states that every meromorphic function f in Ω can be decomposed as

$$f(z) = g(z) + \sum_k (h^{\alpha_k}(z) - r_k(z))$$

where h^{α_k} are the principal parts of f at its singularities α_k, and g and the r_k are holomorphic in Ω. Tietz also posed the problem of finding a locally convex topology on $\mathrm{Mer}(\Omega)$ which satisfies a certain duality relation appearing in §5 of **[72]**. This problem was solved in **[32]** studying a topology introduced by Holdgrün in **[40]** and paralleling methods used by Golovin in **[29, 30]** who studied a slightly different topology. Any of these topologies can be considered natural from a certain point of view. However, Tietz's, Holdgrün's, and Golovin's topologies are too strong for our purposes here.

If, for instance, we consider the operator $H := \sqrt{|\Delta|}$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$ where Δ is the Laplacian, then we may think of the continuous function

$$f:\ [0,1] \to C^\omega(\mathbb{C}, \Psi);\ x \mapsto (\mathbb{C} \ni z \mapsto H^{x+z} \in \Psi)$$

where Ψ denotes the set of pseudo-differential operators. Then, we obtain

$$\zeta(f(x))(z) = 2\zeta_R(-z - x)$$

where ζ_R is the Riemann-ζ-function. Hence, $\zeta(f(x))$ has a pole at $-1 - x$ and we would most certainly like

$$[0,1] \ni x \mapsto \zeta(f(x)) \in \mathrm{Mer}(\mathbb{C})$$

to be continuous. In Tietz's and Golovin's topologies, however, this is not the case and, since Holdgrün's topology is strictly stronger than Golovin's, neither of them is adequate for our purposes.

In **[12]** Cima and Schober defined a locally convex topology on $\mathrm{Mer}(\Omega)$ which does allow singularities to converge (and makes $\mathrm{Mer}(\Omega)$ metric, in fact). Unfortunately, these topologies depend on a previously chosen exhaustion of Ω and, depending on the exhaustion, it is possible to construct a sequence $(p_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ with $p_n \to: p$ such that the meromorphic functions $z \mapsto \frac{1}{z-p_n}$ do not converge to $z \mapsto \frac{1}{z-p}$ even though they usually do. Hence, even though this topology looks much more promising, it exhibits properties that are wholly undesirable. Furthermore, these properties are deeply linked to the construction of the topology making it inherently difficult to get rid of with only minor changes to the construction.

In other words, if we want to consider ζ as a function taking values in $\mathrm{Mer}(\mathbb{C})$ as a locally convex Hausdorff space, then we will have to define yet another "natural" topology on $\mathrm{Mer}(\Omega)$ or, at least, $\mathrm{Mer}(\mathbb{C})$ which reduces to the topology of compact convergence on the subspace of holomorphic functions. However, we were not able to find any such topology.

Luckily, any ζ-function of a gauged poly-log-homogeneous distribution is holomorphic on some half-plane $\Re(z) < r \in \mathbb{R}$. Hence, we can consider the subspace

$$M_\zeta := \{f:\ \mathbb{C} \to \mathbb{C} \text{ measurable};\ \exists r \in \mathbb{R}:\ f|_{\mathbb{C}_{\Re(\cdot)<r}} \text{ holomorphic}\}$$

of the measurable functions (note that we need to use the complete[4] Lebesgue measure here, so that almost everywhere continuous functions are measurable). Let

$$D := \{\Omega \subseteq_{\text{open,connected}} \mathbb{C};\ \exists r \in \mathbb{R}:\ \mathbb{C}_{\Re(\cdot)<r} \subseteq \Omega\}$$

[4]A measure is called complete if and only if every subset of a null set is measurable and a null set itself.

and

$$H_\zeta(\Omega) := \{f \in M_\zeta;\ f|_\Omega \text{ holomorphic}\}.$$

Then, $(D, \supseteq)$ is directed[5] and $M_\zeta = \bigcup_{\Omega\in D} H_\zeta(\Omega)$. On $H_\zeta(\Omega)$ we will want to have compact convergence and the plan is to endow M_ζ with the corresponding final topology. This can be done but the inductive limit will not be strict[6], i.e. there is very little we know about that topology. Instead, we will define a slightly different topology on $H_\zeta(\Omega)$.

Let $d_{H(\Omega)}$ be a metric defining compact convergence on the set of holomorphic functions in Ω. We will extend $d_{H(\Omega)}$ to the semi-metric

$$d_\Omega :\ H_\zeta(\Omega)^2 \to \mathbb{R}_{\geq 0};\ (f,g) \mapsto d_{H(\Omega)}\left(f|_\Omega, g|_\Omega\right).$$

Furthermore, let d_μ be the metric of local convergence in measure[7] on $\mathbb{C}$ (cf. 245A and 245E in [**25**]), that is, $f_n \to f$ locally in measure if and only if

$$\forall \varepsilon \in \mathbb{R}_{>0}\ \forall B \in \mathcal{B}(\mathbb{C}) :\ \Big(\ \lambda(B) < \infty\ \Rightarrow\ \lim_{n\to\infty} \lambda\left(\{z \in B;\ |f_n(z) - f(z)| \geq \varepsilon\}\right) = 0\ \Big)$$

---

[5]Let A be a set and $\leq$ a pre-order on A, that is, a reflexive and transitive binary relation. Then, we call $(A, \leq)$ directed if and only if $\forall a, b \in A\ \exists c \in A :\ a \leq c\ \wedge\ b \leq c$.

[6]This is a consequence of the fact that compact convergence in an open set does not imply convergence anywhere else; e.g. $(z \mapsto e^{nz})_{n\in\mathbb{N}}$ converges compactly to zero on $\mathbb{C}_{\Re(\cdot)<0}$ but there is no compact convergence anywhere else. Hence, if we consider $\Omega, \Omega' \in D$ with $\Omega \subseteq \Omega'$ then the topology of compact convergence on Ω for holomorphic functions on Ω' is strictly weaker than compact convergence on Ω', i.e. the inductive limit is not strict (to be strict the topologies need to coincide).

[7]To construct such a metric, choose an increasing and exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of sets of finite measure (e.g. compacta) and consider $\varrho_n(f,g) := \int_{K_n} \min\{|f(z) - g(z)|, 1\} dz$. Then, $\varrho(f,g) := \sum_{n\in\mathbb{N}} \frac{\varrho_n(f,g)}{1+2^n\lambda(K_n)}$ is a metric that induces the topology of local convergence in measure; cf. 245E in [**25**].

holds where λ is the Lebesgue measure in $\mathbb{C}$ and $\mathcal{B}(\mathbb{C})$ is the Borel σ-algebra in $\mathbb{C}$. Note that d_μ is strictly weaker than d_Ω on Ω. We will now endow $H_\zeta(\Omega)$ with the metric

$$d_{H_\zeta(\Omega)} := d_\Omega + d_\mu.$$

LEMMA 13.5. *$H_\zeta(\Omega)$ is complete.*

PROOF. Let $(f_n)_{n\in\mathbb{N}} \in H_\zeta(\Omega)^{\mathbb{N}}$ be a Cauchy sequence. Since the set of measurable functions with the topology of local convergence in measure is complete, we have $f_n \to: f$ with respect to d_μ. Furthermore, $(f_n|_\Omega)_{n\in\mathbb{N}}$ is Cauchy with respect to compact convergence, i.e. f is holomorphic in Ω, that is, $f \in H_\zeta(\Omega)$.

□

In order to prove the following lemma, we will quickly recall Vitali's theorem (cf. e.g. chapter 1 in [**42**]).

THEOREM 13.6 (Vitali). *Let $\Omega \subseteq_{\text{open,connected}} \mathbb{C}$, $f \in C^\omega(\Omega)^{\mathbb{N}}$ locally bounded, and let*

$$\{z \in \Omega;\ (f_n(z))_{n\in\mathbb{N}} \text{ converges}\}$$

have an accumulation point in Ω. Then, f is compactly convergent.

LEMMA 13.7. *Let $\Omega_0, \Omega_1 \in D$ and $\Omega_0 \supseteq \Omega_1$. Then, $H_\zeta(\Omega_0) \subseteq H_\zeta(\Omega_1)$ and the topology induced by $H_\zeta(\Omega_1)$ coincides with the topology of $H_\zeta(\Omega_0)$.*

Furthermore, $H_\zeta(\Omega_0)$ is closed in $H_\zeta(\Omega_1)$.

PROOF. $H_\zeta(\Omega_0) \subseteq H_\zeta(\Omega_1)$ is trivial and since every compact set in Ω_1 is a compact set in Ω_0 we obtain that every semi-norm of $H_\zeta(\Omega_1)$ is a semi-norm of $H_\zeta(\Omega_0)$, i.e. $H_\zeta(\Omega_0) \hookrightarrow H_\zeta(\Omega_1)$ is continuous. It remains to show that any sequence $(f_n)_{n\in\mathbb{N}} \in H_\zeta(\Omega_0)^{\mathbb{N}}$ which converges to $f \in H_\zeta(\Omega_0)$ with respect to the topology of $H_\zeta(\Omega_1)$ implies convergence $f_n \to f$ in $H_\zeta(\Omega_0)$. In other words, we need to show that f_n converges to f compactly in Ω_0. By Vitali's theorem (since we have pointwise convergence in Ω_1), it suffices to show that $(f_n)_{n\in\mathbb{N}}$ is locally bounded in Ω_0.

Suppose $(f_n)_{n\in\mathbb{N}}$ were not locally bounded in Ω_0. Then,

$$\exists z_0 \in \Omega_0\ \forall \varepsilon, M \in \mathbb{R}_{>0}\ \exists n \in \mathbb{N}\ \forall z \in B(z_0,\varepsilon):\ |f_n(z) - f(z)| > M.$$

In particular, there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ such that

$$\exists z_0 \in \Omega_0\ \exists \varepsilon \in \mathbb{R}_{>0}\ \forall j \in \mathbb{N}\ \forall z \in B(z_0,\varepsilon):\ \left|f_{n_j}(z) - f(z)\right| > j.$$

However, this violates local convergence in measure. Hence, $(f_n)_{n\in\mathbb{N}}$ is locally bounded in Ω_0 and the first assertion holds true.

In order to show that $H_\zeta(\Omega_0)$ is closed in $H_\zeta(\Omega_1)$, let $(f_n)_{n\in\mathbb{N}} \in H_\zeta(\Omega_0)^{\mathbb{N}}$ be convergent to $f \in H_\zeta(\Omega_1)$ in $H_\zeta(\Omega_1)$. Then, we need to show that $f \in H_\zeta(\Omega_0)$. However, we already know that $(f_n)_{n\in\mathbb{N}}$ converges compactly in Ω_0 by the previous part of the proof, i.e. the limit is holomorphic in Ω_0.

□

Since each $H_\zeta(\Omega)$ is contained in at least one $H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ for some $n \in \mathbb{N}$ and its topology is given by the induced topology, we may endow

$$M_\zeta = \bigcup_{n\in\mathbb{N}} H_\zeta(\mathbb{C}_{\Re(\cdot)<-n})$$

with the strict inductive limit topology, that is, the finest topology that renders all $H_\zeta(\mathbb{C}_{\Re(\cdot)<-n}) \hookrightarrow M_\zeta$ $(n \in \mathbb{N})$ continuous, i.e. the finest topology rendering all $H_\zeta(\Omega) \hookrightarrow M_\zeta$ $(\Omega \in D)$ continuous.

Theorem 13.8. (i) *M_ζ is a Hausdorff LF-space.*[8]

[8]LF-spaces are countable inductive limits of Fréchet spaces.

(ii) *The topology of* $H_\zeta(\Omega)$ *coincides with the topology induced by* M_ζ.

(iii) $B \subseteq M_\zeta$ *is bounded if and only if* $B \subseteq H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ *holds for some* $n \in \mathbb{N}$ *and* B *is bounded in* $H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$.

(iv) M_ζ *is bornological.*[9]

(v) M_ζ *is sequential.*[10]

(vi) *Let* E *be a locally convex topological vector space and* $A\colon M_\zeta \to E$ *a linear operator. Then, the following are equivalent.*

 (a) A *is continuous.*

 (b) A *is sequentially continuous.*

 (c) A *is bounded.*[11]

(vii) M_ζ *is complete.*

(viii) M_ζ *is barreled.*[12]

(ix) M_ζ *is ultrabornological*[13].

[9]A topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is called bornological if and only if it is locally convex and every absolutely convex bornivorous set is a neighborhood of zero. A set is called bornivorous if and only if it absorbs all bounded sets, i.e. let A be bounded and B a set then B is bornivorous if and only if there exists $\alpha \in \mathbb{R}_{>0}$ such that $\forall \lambda \in \mathbb{K}_{|\cdot|\geq\alpha}:\ A \subseteq \lambda B$.

[10]A subset U of a topological space is called sequentially open if and only if every sequence converging to a point in U is eventually in U. A topological space is called sequential if and only if every sequentially open set is open. Being sequential is the minimum requirement for a topological spaces such that sequences suffice to determine the topology.

[11]A bounded linear operator maps bounded sets into bounded sets.

[12]A topological vector is called barreled if and only if every barrel is a neighborhood of zero. A barrel is an absolutely convex, closed, and absorbing set. A set $A \subseteq E$ is called absorbing if and only if $\forall x \in E\ \exists \alpha \in \mathbb{R}_{>0}\ \forall \lambda \in \mathbb{K}_{|\cdot|\geq\alpha}:\ x \in \lambda A$.

[13]Let D be absolutely convex and bounded. D is called a Banach disk if and only if $\operatorname{lin} D$ equipped with the Minkowski functional $p_D(x) := \inf\{\lambda \in \mathbb{R}_{>0};\ \lambda x \in B\}$ is a Banach space. An absolutely convex set is called infrabornivorous if and only if it absorbs all Banach disks. A locally convex topological vector space is called ultrabornological if and only if every infrabornivorous set is a neighborhood of zero.

(x) *M_ζ is webbed.*[14]

(xi) *M_ζ is not metrizable.*

(xii) *M_ζ is not first-countable.*

(xiii) *M_ζ is not a Fréchet-Urysohn space.*[15]

PROOF. (i-iii) Theorem of Diedonné-Schwartz (cf. Theorem 9.7 in [**75**]).

(iv) Theorem 9.13 in [**75**] (the $H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ are bornological since metrizable).

(v) Follows from (iv) with Corollary 1.7 in [**23**].

(vi) "(a)⇒(b)" Let A be continuous. Then, we obtain $Ax_\alpha \to Ax$ in E whenever a net $(x_\alpha)_\alpha$ converges to x in M_ζ. In particular, this implies sequential continuity.

"(b)⇒(a)" Suppose A is not continuous. Then, we can find $U \subseteq_{\text{open}} E$ such that $[U]A$ is not open, i.e. not sequentially open. Let $(x_n)_{n\in\mathbb{N}} \in (M_\zeta \setminus [U]A)^{\mathbb{N}}$ satisfy $x_n \to: x \in [U]A$. Then, we obtain $\forall n \in \mathbb{N}: \ Ax_n \in E \setminus U$ and $Ax \in U$. In other words, A is not sequentially continuous.

"(a)⇔(c)" Proposition 6.13 in [**66**].

(vii) Köthe's theorem (cf. Theorem 9.17 in [**75**]).

(viii) Theorem 9.13 in [**75**] (the $H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$ are barreled since metrizable).

(ix) cf. below Corollary 4 in chapter 13.1 in [**41**]

(x) cf. §35.4(8) in [**46**]

[14]Let E be a topological vector space. A class $W = \{C_{n_1,\ldots,n_k} \subseteq E;\ k, n_j \in \mathbb{N}\}$ is called a web if and only if $\forall k \in \mathbb{N}\ \forall n_1,\ldots,n_k: \ C_{n_1,\ldots,n_k} = \bigcup_{n_{k+1}\in\mathbb{N}} C_{n_1,\ldots,n_{k+1}}$ and $E = \bigcup_{n_1\in\mathbb{N}} C_{n_1}$. W is called a $\mathcal{C}$-web if and only if for every fixed sequence $(n_k)_{k\in\mathbb{N}}$ there exists $(\varrho_k)_{k\in\mathbb{N}} \in (\mathbb{R}_{>0})^{\mathbb{N}}$ such that for all $\lambda_k \in [0,\varrho_k]$ and all $x_k \in C_{n_1,\ldots,n_k}$ the series $\sum_{k\in\mathbb{N}} \lambda_k x_k$ converges in E. E is called a webbed space if and only if there exists a $\mathcal{C}$-web on E.

[15]A space is called a Fréchet-Urysohn space if and only if the closure and the sequential closure of any subset coincide.

(xi) No metrizable strict LF-space can be complete by Corollary 5 in [**64**].

(xii) Follows directly from the fact that a Hausdorff topological vector space is first-countable if and only if it is metrizable (cf. §15.11(1) in [**45**]).

(xiii) If M_ζ were a Fréchet-Urysohn space, then it would be metrizable by Theorem 2.2 in [**10**].

□

These properties of M_ζ are sufficient for us to consider many Pettis integrals of ζ-functions. Even though the Pettis integral may not be a meromorphic function anymore, we still obtain the following proposition.

PROPOSITION 13.9. *Let $\Omega \in D$, $j \in \mathbb{Z}$, γ a cycle in Ω, and $\alpha \in \mathbb{C}$ with* $\mathrm{wind}_\alpha(\gamma) = 1$ *where* wind *denotes the winding number. Then, the Laurent coefficient map*

$$\mathrm{lc}_{j,\alpha,\gamma}:\ H_\zeta(\Omega) \to \mathbb{C};\ f \mapsto \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-\alpha)^{j+1}} dz$$

is continuous.

PROOF. $\mathrm{lc}_{j,\alpha,\gamma}$ is continuous if and only if it is sequentially continuous. Let $(f_n)_{n\in\mathbb{N}} \in H_\zeta(\Omega)^{\mathbb{N}}$ be convergent to $f \in H_\zeta(\Omega)$. Since the image $\mathrm{im}\,\gamma$ of γ is a compact subset of Ω, we obtain

$$\|f_n - f\|_{L_\infty(\mathrm{im}\,\gamma)} \to 0,$$

i.e.

$$\begin{aligned}|\mathrm{lc}_{j,\alpha,\gamma}(f_n) - \mathrm{lc}_{j,\alpha,\gamma}(f)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)-f(z)}{(z-\alpha)^{j+1}}\right| \\ &\leq \frac{1}{2\pi}\underbrace{\int_\gamma |z-\alpha|^{-j-1}\,dz}_{\in\mathbb{R}_{>0}}\|f_n - f\|_{L_\infty(\mathrm{im}\,\gamma)}\end{aligned}$$

shows the assertion.

□

EXAMPLE Returning to $H := \sqrt{|\Delta|}$ on $\mathbb{R}/_{2\pi\mathbb{Z}}$ where Δ is the Dirichlet-Laplacian and

$$f:\ [0,1] \to C^\omega(\mathbb{C},\Psi);\ x \mapsto (\mathbb{C} \ni z \mapsto H^{x+z} \in \Psi)$$

where Ψ denotes the set of pseudo-differential operators, we can interpret

$$\zeta(f(x))(z) = 2\zeta_R(-z-x)$$

as an element of $H_\zeta(\mathbb{C} \setminus [-2,-1])$.

Choosing a cycle γ in $\mathbb{C} \setminus [-2,-1]$ with $\mathrm{wind}_{-1}(\gamma) = 1$, we obtain

$$\forall x \in [0,1]:\ \mathrm{lc}_{-1,-1,\gamma}(\zeta(f(x))) = \mathrm{res}_{-1-x}(\zeta(f(x))).$$

Using Pettis integration in M_ζ and our extension of ζ to $\zeta_{R,\Omega}$, we find

$$\begin{aligned}\int_0^1 \mathrm{res}_{-1-x}(\zeta(f(x)))dx =& \int_0^1 \mathrm{lc}_{-1,-1,\gamma}(\zeta(f(x)))dx\\ =&\mathrm{lc}_{-1,-1,\gamma}\left(\int_0^1 \zeta(f(x))dx\right)\\ =&\mathrm{lc}_{-1,-1,\gamma}\left(\int_0^1 \zeta_{2,\Omega}(f(x))dx\right)\\ =&\mathrm{lc}_{-1,-1,\gamma}\circ\zeta_{2,\Omega}\left(\int_0^1 f(x)dx\right).\end{aligned}$$

Similarly, for $j \in \mathbb{N}_0$,

$$\int_0^1 \mathrm{lc}_{j,-1,\gamma}(\zeta(f(x)))dx = \mathrm{lc}_{j,-1,\gamma}\left(\int_0^1 \zeta(f(x))dx\right) = \mathrm{lc}_{j,-1,\gamma}\circ\zeta_{2,\Omega}\left(\int_0^1 f(x)dx\right).$$

■

EXAMPLE At this point, let us consider an orientable compact Riemannian C^∞-manifold (M,g) of dimension $N \in 2\mathbb{N}$. Let $|\Delta|$ be the non-negative Dirichlet-Laplacian on (M,g) and T the semigroup generated by $-|\Delta|$. For any multiplicative gauge $\mathfrak{g}$ we have seen that $\zeta(T(t)\mathfrak{g})(0)$ admits a Laurent expansion at zero with highest order negative Laurent coefficient $\frac{\mathrm{vol}_M(M)}{(4\pi t)^{\frac{N}{2}}}$ (in fact, if $\tilde{\mathfrak{g}}$ is another gauge, then $\zeta(T(t)\mathfrak{g})(0) = \zeta(T(t)\tilde{\mathfrak{g}})(0))$. The second highest order Laurent coefficient is given by $\frac{\text{total curvature}(M)}{3(4\pi)^{\frac{N}{2}}t^{\frac{N}{2}-1}}$.

Let us now assume the metric g is a measurable function on a Radon measure probability space K, that is, (M,g) is subject to random perturbations in the metric, such that $K \ni \omega \mapsto T(t)(\omega)$ is bounded and takes values in a separable subspace of $\mathcal{D}'_{\mathrm{id},1,\mathbb{C},\mathrm{plh}}$. Let $\mathbb{E}$ denote the expectation, i.e. integration in the probability space K. Then, we obtain

$$\mathbb{E}\zeta_{1,\mathbb{C}}(T(t)\mathfrak{g}) = \zeta_{1,\mathbb{C}}(\mathbb{E}(T(t)\mathfrak{g})) = \zeta_{1,\mathbb{C}}(\mathbb{E}(T(t))\mathfrak{g})$$

and, by continuity of δ_0 in $H(\mathbb{C})$,

$$\mathbb{E}\left(\zeta_{1,\mathbb{C}}(T(t)\mathfrak{g})(0)\right) = \zeta_{1,\mathbb{C}}(\mathbb{E}(T(t))\mathfrak{g})(0)$$

as well as

$$\forall j \in \mathbb{Z}: \ \mathbb{E}\left(\mathrm{lc}_{j,0}\left(\zeta_{1,\mathbb{C}}(T(t)\mathfrak{g})(0)\right)\right) = \mathrm{lc}_{j,0}\left(\zeta_{1,\mathbb{C}}(\mathbb{E}(T(t))\mathfrak{g})(0)\right)$$

where $\mathrm{lc}_{j,0}(f)$ denotes the j^{th} Laurent coefficient of f in zero, i.e. a meromorphic function f has the Laurent expansion $f(z) = \sum_{j\in\mathbb{Z}} \mathrm{lc}_{j,0}(f)z^j$ at zero. In particular, the expected volume and the expected mean curvature are determined by the operators $\mathbb{E}T(t)$.

Note that the Hörmander classes S^m and, hence, all Ψ^m are Fréchet spaces, i.e. $\mathbb{E}T(t) \in \Psi^m$ whenever all $T(t)$ are elements of Ψ^m. Since all $T(0)$ are the identity operator, we obtain $\mathbb{E}T(0) = 1 \in \Psi^0$. Since all $T(t)$ for $t \in \mathbb{R}_{>0}$ are in $\Psi^{-\infty}$, i.e. $\forall m \in \mathbb{R}: \ T(t) \in \Psi^m$, we conclude $\forall m \in \mathbb{R}: \ \mathbb{E}T(t) \in \Psi^m$, that is, $\mathbb{E}T(t) \in \Psi^{-\infty}$. In particular, the extension to $\zeta_{1,\mathbb{C}}$ is not even necessary to evaluate the ζ-functions. However, we need it in order to justify integration (note that the same works just as well for wave traces but, then, we will need the quasi-complete extensions of ζ).

On the other hand, $\mathbb{E}T(t)$ can be expressed using the holomorphic functional calculus

$$T(t) = e^{-t|\Delta|} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(\lambda - |\Delta|)^{-1}d\lambda,$$

that is, $T(t)$ has the kernel

$$k_{T(t)}(x,y) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle}e^{-t\sigma(|\Delta|)(x,y,\xi)}d\xi.$$

Hence, $\mathbb{E}(T(t))$ has the kernel

$$\mathbb{E}k_{T(t)}(x,y) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle}\mathbb{E}\left(e^{-t\sigma(|\Delta|)(x,y,\xi)}\right)d\xi.$$

Since the $T(t)$ are smoothing operators (save $T(0)$), we can largely reduce the assumptions on the measurable functions $\omega \mapsto T(t)(\omega)$. The important equality here is

$$\operatorname{tr}\mathbb{E}T(t) = \mathbb{E}\operatorname{tr}T(t)$$

which is also satisfied if tr is continuous and the $T(t)$ are Pettis integrable (above, we considered it in light of Hille's theorem). The assertion $\operatorname{tr} \in (\Psi^{-\infty})'$ is an application of the following version of the closed graph theorem (cf. Corollary 1 in chapter III.12 in [**73**]).

Theorem 13.10 (Closed Graph Theorem). *Let X be an LF-space, Y a Fréchet space, and $T:\ X \to Y$ a linear operator (everywhere defined). Then, the following are equivalent.*

(i) *T is continuous.*

(ii) *T is closed.*

(iii) *T is closable.*

Lemma 13.11. $\operatorname{tr}:\ \Psi^{-\infty} \to \mathbb{C}$ *is continuous.*

PROOF. It suffices to show that tr is closable. Let $(A_n)_{n\in\mathbb{N}} \in (\Psi^{-\infty})^{\mathbb{N}}$ such that $A_n \to 0$ and $\operatorname{tr} A_n \to t$. We need to show that $t = 0$. Let a_n be the symbol of A_n. Then, we have

$$\forall m \in \mathbb{R}: \; a_n \to 0 \text{ in } S^m(X \times X \times \mathbb{R}^N).$$

In particular, the set $\{a_n;\ n \in \mathbb{N}\} \subseteq S^m(X \times X \times \mathbb{R}^N)$ is bounded for each $m \in \mathbb{R}$. Let $m \in \mathbb{R}$ such that $m < -N - 1$. Then,

$$\tau: \; S^{m+1}(X \times X \times \mathbb{R}^N) \subseteq C^\infty(X \times X \times \mathbb{R}^N) \to \mathbb{C};\; f \mapsto \int_X \int_{\mathbb{R}^N} f(x,x,\xi) d\xi d\mathrm{vol}_X(x)$$

is continuous and $\forall n \in \mathbb{N}: \; \tau(a_n) = \operatorname{tr} A_n$. Since the topology of $C^\infty(X \times X \times \mathbb{R}^N)$ and $S^{m+1}(X \times X \times \mathbb{R}^N)$ coincide on bounded subsets of $S^m(X \times X \times \mathbb{R}^N)$ (cf. paragraph above Proposition 1.1.11 in [**39**]), we obtain $\tau(a_n) \to 0$ in $S^{m+1}(X \times X \times \mathbb{R}^N)$. In other words, the assertion follows from

$$t \leftarrow \operatorname{tr} A_n = \tau(a_n) \to 0.$$

□

Considering wave traces, we can follow the same idea as above but with the analytic semi-groups W generated by $-\sqrt{|\Delta|}$. Again, we will obtain that all expected Laurent coefficients are determined by the operators $\mathbb{E}W(t)$ and, in terms of the kernel, by

$$\mathbb{E}k_{W(t)}(x,y) = \frac{1}{(2\pi)^N} \mathbb{E} \int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle} e^{-t\sigma(\sqrt{|\Delta|})(x,y,\xi)} d\xi.$$

At this point it may be more convenient to not yet Fubini-ize this integral because we are interested in the extension to $t \in i\mathbb{R}$. For the sake of simplicity, let us assume $\sigma(\sqrt{|\Delta|})(x,y,\xi) = s(x,y) \left\|\xi\right\|_{\ell_2(N)}$. Then,

$$\mathbb{E}k_{W(t)}(x,y) = \frac{1}{(2\pi)^N} \mathbb{E} \int_{\mathbb{R}^N} e^{i\langle x-y,\xi\rangle} e^{-t\sigma(\sqrt{|\Delta|})(x,y,\xi)} d\xi$$

$$= \frac{1}{(2\pi)^N} \mathbb{E} \int_{\mathbb{R}^N} e^{i\left(\langle x-y,\xi\rangle - \Im(t)s(x,y)\|\xi\|_{\ell_2(N)}\right)} e^{-\Re(t)s(x,y)\|\xi\|_{\ell_2(N)}} d\xi$$

shows that we may also consider these integrals in an algebra of Fourier integral operators where our phase functions are

$$\vartheta_t(x,y,\xi) \coloneqq \langle x-y,\xi\rangle - \Im(t)s(x,y)\,\|\xi\|_{\ell_2(N)}\,.$$

■

Example Let Ω be a connected, compact, separable, metric space and (Ω,Σ,μ) a finite Radon measure space such that every open set has positive measure. Let Ell^m be the set of elliptic pseudo-differential operators of order m on a compact manifold without boundary and $E \subseteq \Psi^m$ a separable subspace of the pseudo-differential operators of order m with $E \cap \mathrm{Ell}^m \neq \varnothing$. Let $f \in L_1(\mu; E)$ take values in Ell^m, $\hat{\mu} \coloneqq \mu(\Omega)^{-1}\mu$, and $\mathbb{E}$ the expectation with respect to $\hat{\mu}$. Then,

$$\int_\Omega f d\mu = \mu(\Omega)\mathbb{E}f.$$

By Lusin's measurability theorem, there exists Ω_ε such that $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and $f|_{\Omega_\varepsilon}$ is continuous for every $\varepsilon \in \mathbb{R}_{>0}$. Let $\operatorname{ind} f(\omega)$ be the index of $f(\omega)$. Then, $\operatorname{ind} f$ is locally constant on each Ω_ε and $\bigcup_{\varepsilon\in\mathbb{R}_{>0}} \Omega_\varepsilon$ is dense in Ω because $\Omega \setminus \bigcup_{\varepsilon\in\mathbb{R}_{>0}} \Omega_\varepsilon$ cannot contain an open set. But $\operatorname{ind} f$ need not be a constant function.

Consider $\Omega \coloneqq \mathbb{R}/_{\mathbb{Z}}$ with the Borel Σ-algebra and the Lebesgue measure λ, and let

$$f \coloneqq A_0 1_{[0,\frac{1}{2}]} + A_1 1_{(\frac{1}{2},1)}.$$

Then, $\Omega_0 = (0,1) \setminus \left\{\frac{1}{2}\right\}$ and $f|_{\Omega_0}$ is locally constant. However,

$$\mathbb{E}f = \frac{A_0 + A_1}{2}$$

and

$$\begin{aligned}\mathbb{E}\operatorname{ind} f &= \int_\Omega \operatorname{ind}\circ\left(A_0 1_{[0,\frac{1}{2}]} + A_1 1_{(\frac{1}{2},1)}\right) d\lambda \\ &= \int_{[0,\frac{1}{2}]} \operatorname{ind}\circ\left(A_0 1_{[0,\frac{1}{2}]} + A_1 1_{(\frac{1}{2},1)}\right) d\lambda + \int_{[\frac{1}{2},1]} \operatorname{ind}\circ\left(A_0 1_{[0,\frac{1}{2}]} + A_1 1_{(\frac{1}{2},1)}\right) d\lambda \\ &= \int_{[0,\frac{1}{2}]} \operatorname{ind} A_0 d\lambda + \int_{[\frac{1}{2},1]} \operatorname{ind} A_1 d\lambda \\ &= \frac{\operatorname{ind} A_0 + \operatorname{ind} A_1}{2}.\end{aligned}$$

In particular, the expected index and the index of the expectation need not coincide.

Since Ψ^m is a Fréchet space, E is a separable metric space and $(\Omega,\Sigma,\mu;E)$ a Sombrero space. Thus, Fubini's theorem and Hille's theorem hold. Let $D\in\mathcal{M}(\mu;E)$ be a measurable family of Dirac operators (we may think of a manifold with random metric here) such that $e^{-tD^*D}, e^{-tDD^*}\in L_1(\mu;E)$ (e.g. $D\in L_\infty(\mu;E)$). Then, the pointwise index is given by

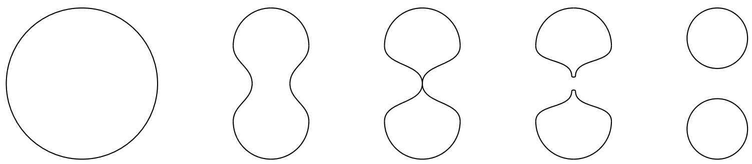
$$\operatorname{ind} D = \operatorname{tr}\left(e^{-tD^*D} - e^{-tDD^*}\right)$$

and we can use the fact that tr is a bounded linear operator on the smoothing operators $\Psi^{-\infty}$ (Lemma 13.11) to obtain

$$\mathbb{E}\operatorname{ind} D = \mathbb{E}\operatorname{tr}\left(e^{-tD^*D} - e^{-tDD^*}\right) = \operatorname{tr}\mathbb{E}\left(e^{-tD^*D} - e^{-tDD^*}\right)$$

where $\mathbb{E}\left(e^{-tD^*D} - e^{-tDD^*}\right)$ can also be taken in $\Psi^{-\infty}$, i.e. $\operatorname{tr}\mathbb{E}\left(e^{-tD^*D} - e^{-tDD^*}\right)$ is well-defined.

This becomes particularly interesting if we consider non-continuous deformations. Let (Ω,Σ,μ) be the space $([0,3],\mathcal{B}([0,3]),\mu)$ where $\mu=\frac{1}{3}\lambda$ and λ is the Lebesgue measure. Let M_0 be the 2-sphere and $(0,1)\ni\omega\mapsto M_\omega$ be a continuous deformation of M_0 such that the north and south pole converge to the origin and the pointwise limit M_1 exists. Furthermore, let $M_1\setminus B_{\mathbb{R}^3}(0,\varepsilon)$ be a manifold for every $\varepsilon \in \mathbb{R}_{>0}$. Let M_3 be a torus in $\mathbb{R}^3$ and $(1,3) \ni \omega \mapsto M_\omega$ a continuous deformation approximating M_1. In other words, $[0,3] \smallsetminus \{1\} \ni \omega \to M_\omega$ is a continuous deformation and M_1 exists as a limit but is not a manifold. For instance, we may think of rotations of the following.



For $\omega \in [0,3] \smallsetminus \{1\}$, let E_ω be the sum of even exterior powers of the cotangent bundle of M_ω, F_ω the sum of odd powers, and $D(\omega) \coloneqq d_\omega + d_\omega^*$ where d_ω is the exterior derivative on M_ω. Then, D is measurable (in fact, continuous on $[0,3] \smallsetminus \{1\}$) and $\operatorname{ind} D$ is locally constant. Since $\operatorname{ind} D(\omega)$ is the Euler characteristic of M_ω, it follows that $\operatorname{ind} D = 2 \cdot 1_{[0,1)}$ and we obtain the expected Euler characteristic

$$\mathbb{E} \operatorname{ind} D = \frac{2}{3}.$$

If, on the other hand, we wanted to consider this family on the geometric side of the index theorem, then we would look at integrals of the form

$$\mathbb{E}\left(\omega \mapsto \int_{M_\omega} \hat{A}_\omega \mathrm{ch}_\omega\right)$$

with no chance of applying Fubini here since the M_ω may not even be written as the same set and changing metric. The operator treatment is not faced by such problems giving us a tool to consider random manifolds and their expected characteristic values under discontinuous perturbations.

■

Remark Note that we do not expect ζ to be continuous/closable in general. Consider the Hörmander classes S^m. According to Proposition 1.1.11 in [**39**], $S^{-\infty}$ is dense in S^m with respect to the topology of $S^{m'}$ whenever $m' > m$. Let A

be a polyhomogeneous pseudo-differential operator with symbol in S^m and non-vanishing residue trace. Furthermore, let $\mathfrak{g}: \mathbb{C}_{\Re(\cdot)<1} \to S^1$ be gauged and $(A_n)_{n\in\mathbb{N}}$ a sequence of operators with symbols in $S^{-\infty}$ such that $A_n \to A$ with respect to their symbols in S^{m+1}. Then, we also have $\forall z \in \mathbb{C}_{\Re(\cdot)<1}: \ A_n\mathfrak{g}(z) \to A\mathfrak{g}(z)$ with respect to S^{m+1}. Hence, if ζ were continuous, we would obtain $\zeta(A_n\mathfrak{g}) \to \zeta(A\mathfrak{g})$ and, thus, $0 = \operatorname{res}_0 \zeta(A_n\mathfrak{g}) \to \operatorname{res}_0 \zeta(A\mathfrak{g}) \neq 0$.

In other words, obtaining quasi-complete extensions is the best we can do (in this generality).

■

Remark Note that the dependence on R in $\zeta_{R,\Omega}$ is (essentially) irrelevant if $\Omega \in D$, that is, Ω contains a subspace $\mathbb{C}_{\Re(\cdot)<r}$. If we consider an operator A with poly-log-homogeneous expansion $A = A_0 + \sum_{\iota\in I} A_\iota$, then each of the A_ι contributes a term $\frac{c_\iota}{(N+d_\iota+z)^{l_\iota}}$, i.e. we have poles at $-N - d_\iota$. Now, for $A_\iota(z)$ to not be of trace-class $\Re(d_\iota + z) \geq -N$ is necessary, i.e. $\Re(d_\iota) \geq -N - \Re(z)$. Hence, having no poles p with $\Re(p) < r$ implies

$$\Re(d_\iota) \geq -N - r \ \Rightarrow \ c_\iota = 0.$$

In other words, defining $\hat{I} := \{\iota \in I;\ \Re(d_\iota) \geq -N - r\}$ and $\tilde{I} := \{0\} \cup I \setminus \tilde{I}$ we can write $A = \tilde{A} + \hat{A}$ where $\tilde{A} := \sum_{\iota\in\tilde{I}} A_\iota \in \mathcal{D}'_{\Gamma,-N-r,\Omega,\mathrm{plh}}$ and $\hat{A} := \sum_{\iota\in\hat{I}} A_\iota$ is an operator whose ζ-function vanishes.

■

Finally, we will remark that the proof that ζ has a quasi-completion on $H(\Omega)$, does not extend to $H_\zeta(\Omega)$ (i.e. not to M_ζ) without further arguments since we cannot use Vitali's theorem because there my not be a dense and open subset of $\mathbb{C}$ which does not contain any poles of a sequence of ζ-function. As a counterexample, consider a bijection $q \in (i\mathbb{Q})^{\mathbb{N}}$. Then, $\zeta\left(z \mapsto \sqrt{|\Delta|}^{q_n+z}\right)(s) = 2\zeta_R(-s-q_n)$ has a pole at $-1-q_n$, i.e. there exists no open and connected $\Omega_0 \subseteq \mathbb{C}$ such that $-2 \in \Omega_0$ and $0 \in \Omega_0$. However, this counterexample violates that sequences used in the proof converge to zero. If, on the other hand, we define

$$\mathcal{D}'_{\Gamma,\mathrm{plh}} := \bigcup_{(m,n)\in\mathbb{Z}^2} \mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<n},\mathrm{plh}} \subseteq C^\omega(\mathbb{C},\mathcal{D}'_\Gamma),$$

then $\zeta \subseteq \mathcal{D}'_{\Gamma,\mathrm{plh}} \oplus M_\zeta$ is everywhere defined and we do obtain the following theorem.

THEOREM 13.12. *Let $(v_\alpha,\zeta(v_\alpha))_{\alpha\in A} \in \zeta^A$ be a bounded net, $(v_\alpha,\zeta(v_\alpha)) \to (0,s)$ in $\zeta \subseteq \mathcal{D}'_{\Gamma,\mathrm{plh}} \oplus M_\zeta$, and $\Omega_0 \subseteq \mathbb{C}$ open, connected, and dense such that $\forall \alpha \in A$: $\zeta(v_\alpha) \in H(\Omega_0)$. Then, $s = 0$.*

PROOF. Since each $\mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<n},\mathrm{plh}}$ has the subspace topology of $C^\omega(\mathbb{C},\mathcal{D}'_\Gamma)$ and $\left(\mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}\right)_{(m,n)\in(\mathbb{N}^2,\trianglelefteq)}$ with

$$\forall (m,n),(m',n') \in \mathbb{N}^2 : \ (m,n) \trianglelefteq (m',n') \ :\Leftrightarrow \ m \le m' \ \wedge \ n \le n'$$

is directed, we obtain that

$$\zeta = \bigcup_{(m,n)\in\mathbb{N}^2} \underbrace{\zeta|_{\mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}}}_{\subseteq \mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}} \oplus H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)} \subseteq \mathcal{D}'_{\Gamma,\mathrm{plh}} \oplus M_\zeta$$

is a strict inductive limit. Let B be a bounded subset of ζ. Then, the Theorem of Diedonné-Schwartz (cf. Theorem 9.7 in [**75**]) implies the existence of $(m,n) \in \mathbb{N}^2$ such that B is a bounded subset of $\zeta|_{\mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}}^{H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)}$ which is metrizable.

Let $(v_\alpha,\zeta(v_\alpha))_{\alpha\in A} \in \zeta^A$ be a bounded net such that $v_\alpha \to 0$ in $\mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}}$ and $\zeta(v_\alpha) \to: s$ in $H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$. Let $(m,n) \in \mathbb{N}^2$ be such that $(v_\alpha,\zeta(v_\alpha))_{\alpha\in A} \in \left(\mathcal{D}'_{\Gamma,m,\mathbb{C}_{\Re(\cdot)<-n},\mathrm{plh}} \oplus H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)\right)^A$, $V := \{v_\alpha;\ \alpha \in A\} \cup \{0\}$, $Z := \{\zeta(v_\alpha),\ \alpha \in A\} \cup (s)$, and d a metric on $V \times Z$. Then, $(v_\alpha,\zeta(v_\alpha))_{\alpha\in A} \to (0,s)$ is equivalent to

$(u_{\beta(k)}, s_{\beta(k)})_{k\in\mathbb{N}} \to (0,s)$ where $\beta:\ \mathbb{N}\to A$ is chosen such that

$$\forall k\in\mathbb{N}\ \forall \alpha\in A:\ \Big(\ \alpha\geq\beta(k)\ \Rightarrow\ d\big((u_\alpha,s_\alpha),(u_{\beta(k)},s_{\beta(k)})\big)\leq\frac{1}{k}\ \Big)$$

holds. Then, by definition of $H_\zeta\left(\mathbb{C}_{\Re(\cdot)<-n}\right)$, $(s_{\beta(k)})_{k\in\mathbb{N}}$ is a locally bounded sequence. Hence, Vitali's theorem yields that s is holomorphic in Ω_0 and by Theorem 13.4, we obtain $s_{\beta(k)}|_{\mathbb{C}_{\Re(\cdot)<-n}} \to 0$. In other words, $s=0$ almost everywhere.

$\square$

Concluding remarks

Based on Guillemin's work [**34**, **35**] on the residue trace for Fourier Integral Operators, we have developed an extension of the theory of ζ-functions for pseudo-differential operators to a large class of Fourier Integral Operators. By introducing the notion of gauged poly-log-homogeneous distributions explicitly and, thus, working in a generalized setting that shares the fundamental analytical structures that are preserved when replacing pseudo-differential operators with Fourier Integral Operators, we were able to study the Laurent expansion of Fourier Integral Operator ζ-functions and prove existence of a generalized Kontsevich-Vishik trace.

In conjunction with stationary phase expansion results for the Laurent coefficients and the kernel singularity structure, we have extended many known formulae from the pseudo-differential operator case to varying classes of Fourier Integral Operators. Furthermore, these considerations allowed us to identify non-trivial algebras of Fourier Integral Operators consisting purely of Hilbert-Schmidt operators with regular trace integrals, as well as utilize our unified approach to independently verify known results for special cases of Fourier Integral Operators. A particular special case that deserves highlighting are Boutet de Monvel's results [**7**] on generalized Szegő projectors since they gave rise to a class of Fourier Integral Operators whose generalized Kontsevich-Vishik trace is form-equivalent to the pseudo-differential operator case.

At this point, the lack of a holomorphic functional calculus in most algebras of Fourier Integral Operators became the limiting factor since many a consideration in the pseudo-differential case makes heavy use of the functional calculus. It was not even clear if we could replace phase functions in calculations that use holomorphic functional calculus and end up with an expression that is defined within a given algebra of Fourier Integral Operators. Hence, in part II, we had a look at Bochner-, Lebesgue-, and Pettis-integration in algebras of Fourier Integral Operators. We were, then, able to prove that replacement of phase functions is indeed possible and the integrals remain well-defined.

Furthermore, these integrals permit considerations of measurable functions of Fourier Integral Operators which extend the notion of continuous families of Fourier Integral Operators and whose "measurable index bundles" reduce to the Atiyah-Jänich bundle. In particular, these measurable Fourier Integral Operators raise the question whether or not it is possible to consider stochastic applications, e.g. randomly perturbed manifolds, directly (that is, without the need of the Colombeau algebra). We were able to give a positive answer to that question by calculating the expected volume of a randomly perturbed manifold as part of the expected heat- and wave-trace. Additionally, in appendix A, we have developed the basic theorems of probability in algebras of Fourier Integral Operators including versions of the law of large numbers and a Lindeberg type central limit theorem.

With a well-functioning integration theory in our hands, we returned to the ζ-functions. By introducing a topology on the set of ζ-functions, we proved the existence of quasi-complete extensions of certain restrictions on the ζ function. Hence, we obtained that the ζ-function and the integral commute in certain circumstances. Similarly, the extracting Laurent coefficients and taking the classical trace commutes with the integration (modulo some technical caveats); thus, validating that the expected heat- and wave-trace coefficients are, in fact, the coefficients of the trace of the expected semi-groups (a property that is very useful but far from obvious).

Of course, there are a number of open problems. For instance, we have obtained a notion of generalized ζ-determinants. However, it remains unclear for which classes of gauged Fourier Integral Operators these are actually determinants in the sense $\det_\zeta(AB) = \det_\zeta(A)\det_\zeta(B)$. Finding such classes of Fourier Integral Operators, as well as extending more known formulae from the pseudo-differential case, will probably need to make heavy use of integration techniques; at least if we want to stay fairly close to the known cases.

Regarding stochastic Fourier Integral Operators, essentially everything needs to be done. However, since we know that the index bundle is measurable, existence of measurable versions of the spectral flow, for instance, would follow directly from a proof of continuity/measurability of the first Chern character (in case of the spectral flow) with respect to the index bundle topology.

The most important open problem, however, is probably the case of Fourier Integral Operators on manifolds with boundary. While non-compact manifolds can easily be incorporated by assuming that the kernel representation as a series of oscillatory integrals is locally finite and by adding a condition that makes the series of local contributions to the ζ-function summable, considering manifolds with boundary is a far more complicated problem. Nevertheless, such further development will be left to future work.

Appendices

APPENDIX A

Probability in certain algebras of Fourier Integral Operators

Since we have seen that our integration theory allows us to consider random manifolds, it would be an interesting question whether or not it also permits theorems of classical stochastics, e.g. the central limit theorem. Hence, in this appendix, we will consider theorems from classical probability. From now on let $\mathcal{A}$ be an algebra of Fourier Integral Operators such that the integral on $L_1(\mu;\mathcal{A})$ takes values in $\mathcal{A}$ (e.g. an algebra associated with a Hörmander space $\mathcal{D}'_\Gamma$) and μ a probability measure (though some of the theorems work for finite measures or more general measures as well; mutatis mutandis). We will also continue to use the letter E if we do not use the algebra structure of $\mathcal{A}$ (so that, later on, we can easily consider subspaces of algebras which are not an algebra themselves). Furthermore, we will make no distinction between $\mathcal{A}$ and the corresponding space of kernels $\mathcal{D}'_{\mathcal{A}}$.

Recall that we assume that composition in the algebra is continuous, i.e.

$$\forall \iota \in I\ \exists \kappa, \lambda \in I\ \exists c \in \mathbb{R}_{\geq 0}\ \forall A, B \in \mathcal{A}:\ p_\iota(A \circ B) \leq c p_\kappa(A) p_\lambda(B).$$

The minimal constant c is also denoted by $\|\circ\|_{\iota,\kappa,\lambda}$. Similarly, we assume that the involution in a $*$-algebra is continuous. Furthermore, recall Hölder's inequality.

Theorem 9.11 (Hölder's inequality). *Let $A_i \in L_{p_i}(\mu;\mathcal{A})$ for $i \in \mathbb{N}_{\leq n}$ and $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{r}$. Then, $A_1 \circ A_2 \circ \ldots \circ A_n \in L_r(\mu;\mathcal{A})$ and*

$$\forall \iota \in I\ \exists \kappa \in I^n\ \exists c \in \mathbb{R}_{\geq 0}:\ p_\iota^{L_r(\mu;\mathcal{A})}(A_1 \circ A_2 \circ \ldots \circ A_n) \leq c \prod_{j=1}^n p_{\kappa_j}^{L_{p_j}(\mu;\mathcal{A})}(A_j).$$

OBSERVATION A.1. *Let E be a subspace of a $*$-algebra which is invariant with respect to the involution and $A \in L_p(\mu;E)$. Then, $A^* \in L_p(\mu;E)$ and*

$$\int_\Omega A^* d\mu = \left(\int_\Omega A d\mu\right)^*$$

for $p=1$. Furthermore, if $A \in \mathcal{S}L_p(\mu;E)$, then $A^* \in \mathcal{S}L_p(\mu;E)$.

PROOF. Since $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ is continuous, we obtain $A^* \in L_p(\mu;E)$ directly from

$$\forall \iota \in I\ \exists \kappa \in I:\ p_\iota \circ A^* \leq \|\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}\|_{\iota,\kappa} p_\kappa \circ A \in L_p(\mu).$$

$A^* \in \mathcal{S}L_p(\mu;E)$ follows from taking the adjoints of each of the simple functions approximating $A \in \mathcal{S}L_p(\mu;E)$. Finally,

$$\begin{aligned}\left\langle \left(\int_\Omega A d\mu\right)^* \varphi, \psi \right\rangle &= \left\langle \varphi, \int_\Omega A d\mu \psi \right\rangle \\ &= \int_\Omega \langle \varphi, A\psi \rangle d\mu \\ &= \int_\Omega \langle A^*\varphi, \psi \rangle d\mu \\ &= \left\langle \int_\Omega A^* d\mu \varphi, \psi \right\rangle\end{aligned}$$

implies $\int_\Omega A^* d\mu = \left(\int_\Omega A d\mu\right)^*$ for the Pettis integral. For the Bochner/Lebesgue integral it follows directly from the Pettis case (or applying Hille's theorem directly to the linear operator $A \mapsto A^*$).

□

Let us now define and study the most important property of classical probability; the notion of independence.

DEFINITION A.2. *Let $A \in \mathcal{M}(\mu;E)$ and $\mathcal{B}(E)$ the Borel σ-algebra on E. Then, we define the distribution μ_A of A with respect to μ to be the measure*

$$\forall S \in \mathcal{B}(E):\ \mu_A(S) := \mu([S]A).$$

We say that a family $(A_\kappa)_{\kappa\in K}\in E^K$ *is independent if and only if for every finite set* $k\subseteq_{\text{finite}} K$ *and every* $B\in\mathcal{B}(E)^k$

$$\mu\left(\bigcap_{\kappa\in k}[B_\kappa]A_\kappa\right)=\prod_{\kappa\in k}\mu\left([B_\kappa]A_\kappa\right)$$

holds, i.e. the joint distribution $\mu_{\bigoplus_{\kappa\in k}A_\kappa}$ *satisfies*

$$\mu_{\bigoplus_{\kappa\in k}A_\kappa}=\bigtimes_{\kappa\in k}\mu_{A_\kappa}.$$

Lemma A.3. *Let* $A_1,\dots,A_n\in\mathcal{M}(\mu;E)$ *be independent,* $k,m\in\mathbb{N}$, $k\le n$, *and* $f:\ E^k\to E^m$ *Borel-measurable. Then,* $g:=f\circ(A_1,\dots,A_k),A_{k+1},\dots,A_n$ *are independent.*

Proof. Let $S\subseteq E^m$ and $S_{k+1},\dots,S_n\subseteq E$ be Borel measurable sets. Then,

$$\begin{aligned}\mu\left([S]g\cap\bigcap_{j=k+1}^n[S_j]A_j\right)&=\mu\left([[S]f](A_1,\dots,A_k)\cap\bigcap_{j=k+1}^n[S_j]A_j\right)\\&=\mu\left(\left[[S]f\times\bigtimes_{j=k+1}^n S_j\right](A_1,\dots,A_n)\right)\\&=\mu\left(\left[[S]f\right](A_1,\dots,A_k)\right)\prod_{j=k+1}^n\mu\left([S_j]A_j\right)\\&=\mu\left([S]g\right)\prod_{j=k+1}^n\mu\left([S_j]A_j\right)\end{aligned}$$

□

An important application of Lemma A.3 is that the operations in our algebra/topological vector space preserve independence.

Corollary A.4.
(i) *Let* $A,B,C\in\mathcal{M}(\mu;E)$ *be independent. Then,* A *and* $B+C$ *are independent.*
(ii) *Let* $A,B,C\in\mathcal{M}(\mu;\mathcal{A})$ *be independent. Then,* A *and* BC *are independent.*
(iii) *Let* $\mathcal{A}$ *be a* $*$*-algebra and* $A,B\in\mathcal{M}(\mu;\mathcal{A})$ *be independent. Then,* A^* *and* B *are independent.*

PROOF. Follows directly from Lemma A.3 and the fact that addition, composition, and involution are continuous.

□

At this point the notion of independence turns out to be completely classical. Let us now consider the convolution since it is the main tool to study random variables and their distributions.

DEFINITION A.5. *Let μ, ν be Borel measures on E. Then, we define their convolution*

$$\forall S \in \mathcal{B}(E):\ (\mu * \nu)(S) := \int_E \mu(S - x) d\nu(x)$$

where $S - x := \{s - x \in E;\ s \in S\}$.

LEMMA A.6. *Let λ, μ, ν be σ-finite Borel measures on E. Then, the following are true.*

(i) $\mu * \nu = \nu * \mu$.

(ii) $\lambda * (\mu + \nu) = \lambda * \mu + \lambda * \nu$.

(iii) *Let $\alpha:\ E^2 \to E;\ (x, y) \mapsto x + y$. Then, $\mu * \nu$ is the push-forward measure of $\mu \times \nu$ under α.*

(iv) *If $f \in \overline{\mathcal{S}} L_1(\mu * \nu)$, then $\int_E f d\mu * \nu = \int_{E^2} f(x + y) d(\mu \times \nu)(x, y)$.*

(v) *Let λ be translation invariant and μ have a density $p \in \overline{\mathcal{S}} L_1(\lambda, E)$. Then, $\mu * \nu$ has the density $h(x) = \int_E p(x - y) d\nu(y)$.*

(vi) *Let λ be translation invariant, μ have a density $p \in \overline{\mathcal{S}} L_1(\lambda, E)$, and ν have a density $q \in \overline{\mathcal{S}} L_1(\lambda, E)$. Then, $\mu * \nu$ has the density*

$$h(x) = \int_E p(x - y) q(y) d\lambda(y) = \int_E q(x - y) p(y) d\lambda(y).$$

PROOF. (i) For $S \in \mathcal{B}(E)$, we obtain

$$\mu * \nu(S) = \int_E \mu(S - x) d\nu(x)$$

$$
\begin{aligned}
&= \int_E \int_E 1_{S-x}(y) d\mu(y) d\nu(x) \\
&= \int_E \int_E 1_S(x+y) d\mu(y) d\nu(x) \\
&= \int_{E^2} 1_S(x+y) d(\mu \times \nu)(x,y)
\end{aligned}
$$

by Fubini's theorem.

(ii) For $S \in \mathcal{B}(E)$, we obtain

$$
\begin{aligned}
\lambda * (\mu + \nu)(S) &= \int_E (\mu + \nu)(S - x) d\lambda(x) \\
&= \int_E \mu(S - x) d\lambda(x) + \int_E \nu(S - x) d\lambda(x) \\
&= \lambda * \mu(S) + \lambda * \nu(S).
\end{aligned}
$$

(iii) For $S \in \mathcal{B}(E)$, we obtain

$$
\begin{aligned}
\mu * \nu(S) &= \int_{E^2} 1_S(x+y) d(\mu \times \nu)(x,y) \\
&= \int_{E^2} 1_S \circ \alpha d\mu \times \nu \\
&= \int_{E^2} 1_{[S]\alpha} d\mu \times \nu \\
&= \mu \times \nu([S]\alpha).
\end{aligned}
$$

(iv) Let $f \in \overline{\mathcal{S}} L_1(\mu * \nu)$. Then,

$$
\int_E f d\mu * \nu = \int_{E^2} f \circ \alpha d\mu \times \nu = \int_{E^2} f(x+y) d(\mu \times \nu)(x,y).
$$

(v) For $S \in \mathcal{B}(E)$, we obtain

$$
\begin{aligned}
\mu * \nu(S) &= \int_E \int_E 1_S(x+y) d\mu(x) d\nu(y) \\
&= \int_E \int_E 1_S(x+y) p(x) d\lambda(x) d\nu(y) \\
&= \int_E \int_E 1_S(x) p(x-y) d\lambda(x) d\nu(y) \\
&= \int_E \int_E p(x-y) d\nu(y) 1_S(x) d\lambda(x)
\end{aligned}
$$

$$= \int_E 1_S(x)h(x)d\lambda(x)$$

(vi) Follows directly from (v).

□

LEMMA A.7. *Let $A, B \in \mathcal{M}(\mu; E)$ be independent. Then,*

$$\mu_{A+B} = \mu_A * \mu_B.$$

PROOF. Let $\alpha: E^2 \to E;\ (x,y) \mapsto x + y$. Then, we obtain for every $S \in \mathcal{B}(E)$

$$\begin{aligned}
\mu_{A+B}(S) &= \mu\left([S](A+B)\right) \\
&= \mu\left([S]\alpha \circ (A \oplus B)\right) \\
&= \mu\left([[S]\alpha]\, A \oplus B\right) \\
&= \mu_{A\oplus B}\left([S]\alpha\right) \\
&\overset{(*)}{=} \mu_A \times \mu_B\left([S]\alpha\right) \\
&\overset{(\dagger)}{=} \mu_A * \mu_B\left(S\right)
\end{aligned}$$

where $(*)$ uses the definition of independence and $(\dagger)$ is (iii) in Lemma A.6.

□

DEFINITION A.8. *Let $A \in L_1(\mu; E)$. Then, we define the expected value $\mathbb{E}(A)$ of A to be*

$$\mathbb{E}(A) := \int_\Omega A d\mu.$$

Furthermore, we define the variance of $A \in L_2(\mu; \mathcal{A})$ to be

$$\mathbb{V}(A) := \mathbb{E}\left((A - \mathbb{E}(A))^2\right).$$

Note, by Proposition 9.25, we obtain

$$\forall A \in L_1(\mu; E) : \ \mathbb{E}(A) = \int_\Omega A d\mu = \int_E x d\mu_A(x)$$

provided that the identity id $\in L(E)$ can be approximated by a net of simple functions, i.e. id $\in \overline{\mathcal{S}}\mathcal{M}(\mu_A; E)$.

LEMMA A.9. *Let $A, B \in L_1(\mu; \mathcal{A})$ independent such that* id $\in \overline{\mathcal{S}}L_1(\mu_A; \mathcal{A}) \cap \overline{\mathcal{S}}L_1(\mu_B; \mathcal{A})$ *and $\mathcal{A}^2 \ni (x,y) \mapsto xy \in \mathcal{A}$ is an element of $\overline{\mathcal{S}}L_1(\mu_A \times \mu_B; \mathcal{A})$. Then, $AB \in L_1(\mu; \mathcal{A})$ and*

$$\mathbb{E}(AB) = \mathbb{E}(A)\mathbb{E}(B).$$

PROOF.

$$\begin{aligned}
\mathbb{E}(A)\mathbb{E}(B) &= \int_\Omega A d\mu \int_\Omega B d\mu \\
&= \int_{\mathcal{A}} x d\mu_A \int_{\mathcal{A}} y d\mu_B \\
&= \int_{\mathcal{A}^2} xyd(\mu_A \times \mu_B)(x,y) \\
&= \int_{\mathcal{A}^2} xyd(\mu_{A \oplus B})(x,y) \\
&= \int_\Omega AB d\mu \\
&= \mathbb{E}(AB)
\end{aligned}$$

□

EXAMPLE Let f be a measurable family of m-forms, g a measurable family of n-form, and f, g independent. Then, we obtain

$$\begin{aligned}
&\mathbb{E}\left(f \wedge g\right)\left(v_1, \ldots, v_{m+n}\right) \\
=&\mathbb{E}\left(\left(f \wedge g\right)\left(v_1, \ldots, v_{m+n}\right)\right) \\
=&\mathbb{E}\left(\frac{1}{m!n!} \sum_{\sigma \in \mathrm{Sym}(m+n)} \mathrm{sgn}(\sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(m)}\right) g\left(v_{\sigma(m+1)}, \ldots, v_{\sigma(m+n)}\right)\right)
\end{aligned}$$

$$=\frac{1}{m!n!}\sum_{\sigma\in\mathrm{Sym}(m+n)}\mathrm{sgn}(\sigma)\mathbb{E}\left(f\right)\left(v_{\sigma(1)},\dots,v_{\sigma(m)}\right)\mathbb{E}\left(g\right)\left(v_{\sigma(m+1)},\dots,v_{\sigma(m+n)}\right)$$

$$=\left(\mathbb{E}\left(f\right)\wedge\mathbb{E}\left(g\right)\right)\left(v_1,\dots,v_{m+n}\right)$$

where Sym denotes the symmetric group and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ. Here we used that the functions $f\left(v_{\sigma(1)},\dots,v_{\sigma(m)}\right)$ and $g\left(v_{\sigma(m+1)},\dots,v_{\sigma(m+n)}\right)$ are $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ valued and that $(\Omega,\Sigma,\mu;\mathbb{K})$ is a Sombrero space, as well as continuity of point-evaluation.

∎

OBSERVATION A.10. *Let $A,B\in L_2(\mu;\mathcal{A})$ and $a,b\in\mathcal{A}$. Then, the following are true.*

(i) $\mathbb{V}(A)=\mathbb{E}(A^2)-\mathbb{E}(A)^2$

(ii) $\mathbb{V}(aA+b)=\mathbb{V}(aA)$

(iii) *Let $\tilde{A}:=A-\mathbb{E}(A)$ and $\tilde{B}:=B-\mathbb{E}(B)$. Then,*

$$\mathbb{V}(A+B)=\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}\left(\tilde{A}\tilde{B}+\tilde{B}\tilde{A}\right)$$

(iv) *If A,B are uncorrelated, that is,*

$$\mathbb{E}(AB)+\mathbb{E}(BA)=\mathbb{E}(A)\mathbb{E}(B)+\mathbb{E}(B)\mathbb{E}(A),$$

then $\mathbb{V}(A+B)=\mathbb{V}(A)+\mathbb{V}(B)$.

PROOF. (i)

$$\begin{aligned}\mathbb{V}(A)=&\mathbb{E}\left((A-\mathbb{E}(A))^2\right)\\=&\mathbb{E}\left(A^2-A\mathbb{E}(A)-\mathbb{E}(A)A+\mathbb{E}(A)^2\right)\\=&\mathbb{E}(A^2)-\mathbb{E}(A\mathbb{E}(A))-\mathbb{E}(\mathbb{E}(A)A)+\mathbb{E}(\mathbb{E}(A)^2)\\=&\mathbb{E}(A^2)-\mathbb{E}(A)^2.\end{aligned}$$

(ii)

$$
\begin{aligned}
&\mathbb{V}(aA+b)\\
=&\mathbb{E}\left((aA+b)^2\right)-\mathbb{E}(aA+b)^2\\
=&\mathbb{E}\left(aAaA\right)+\mathbb{E}(aAb)+\mathbb{E}(baA)+b^2-(a\mathbb{E}(A)+b)^2\\
=&\mathbb{E}\left((aA)^2\right)+a\mathbb{E}(A)b+ba\mathbb{E}(A)+b^2-\left(\mathbb{E}(aA)^2+a\mathbb{E}(A)b+ba\mathbb{E}(A)+b^2\right)\\
=&\mathbb{E}\left((aA)^2\right)-\mathbb{E}(aA)^2\\
=&\mathbb{V}(aA)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&\mathbb{V}(A+B)\\
=&\mathbb{E}\left(A^2+AB+BA+B^2\right)-\left(\mathbb{E}(A)^2+\mathbb{E}(A)\mathbb{E}(B)+\mathbb{E}(B)\mathbb{E}(A)+\mathbb{E}(B)^2\right)\\
=&\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}\left(AB+BA-\mathbb{E}(A)\mathbb{E}(B)-\mathbb{E}(B)\mathbb{E}(A)\right)\\
=&\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}\left(AB-\mathbb{E}(A)\mathbb{E}(B)+BA-\mathbb{E}(B)\mathbb{E}(A)\right)\\
=&\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}\left(AB-\mathbb{E}(A)\mathbb{E}(B)-\mathbb{E}(A)\mathbb{E}(B)+\mathbb{E}(A)\mathbb{E}(B)\right)\\
&+\mathbb{E}\left(BA-\mathbb{E}(B)\mathbb{E}(A)-\mathbb{E}(B)\mathbb{E}(A)+\mathbb{E}(B)\mathbb{E}(A)\right)\\
=&\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}\left(AB-\mathbb{E}(A)B-A\mathbb{E}(B)+\mathbb{E}(A)\mathbb{E}(B)\right)\\
&+\mathbb{E}\left(BA-\mathbb{E}(B)A-B\mathbb{E}(A)+\mathbb{E}(B)\mathbb{E}(A)\right)\\
=&\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}\left((A-\mathbb{E}(A))(B-\mathbb{E}(B))+(B-\mathbb{E}(B))(A-\mathbb{E}(A))\right)
\end{aligned}
$$

(iv)

$$
\begin{aligned}
&\mathbb{V}(A+B)\\
=&\mathbb{E}\left(A^2+AB+BA+B^2\right)-\left(\mathbb{E}(A)^2+\mathbb{E}(A)\mathbb{E}(B)+\mathbb{E}(B)\mathbb{E}(A)+\mathbb{E}(B)^2\right)\\
=&\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}\left(AB+BA-\mathbb{E}(A)\mathbb{E}(B)-\mathbb{E}(B)\mathbb{E}(A)\right)
\end{aligned}
$$

$$=\mathbb{V}(A)+\mathbb{V}(B)+\mathbb{E}(A)\mathbb{E}(B)+\mathbb{E}(B)\mathbb{E}(A)-\mathbb{E}(A)\mathbb{E}(B)-\mathbb{E}(B)\mathbb{E}(A)$$

$$=\mathbb{V}(A)+\mathbb{V}(B)$$

□

DEFINITION A.11. *Let $\mathcal{A}$ be a $*$-algebra and $A \in L_2(\mu;\mathcal{A})$. Then, we define the symmetric variance of A to be*

$$\mathbb{V}_{\mathrm{sym}}(A) := \mathbb{E}\left((A-\mathbb{E}(A))\left(A^*-\mathbb{E}(A^*)\right)\right).$$

OBSERVATION A.12. *Let $A, B \in L_2(\mu;\mathcal{A})$ and $a, b \in \mathcal{A}$. Then, the following are true.*

(i) $\mathbb{V}_{\mathrm{sym}}(A) = \mathbb{E}(AA^*) - \mathbb{E}(A)\mathbb{E}(A^*)$

(ii) $\mathbb{V}_{\mathrm{sym}}(aA+b) = \mathbb{V}_{\mathrm{sym}}(aA) = a\mathbb{V}_{\mathrm{sym}}(A)a^*$

(iii) *Let $\tilde{A} := A - \mathbb{E}(A)$ and $\tilde{B} := B - \mathbb{E}(B)$. Then,*

$$\mathbb{V}_{\mathrm{sym}}(A+B) = \mathbb{V}_{\mathrm{sym}}(A) + \mathbb{V}_{\mathrm{sym}}(B) + \mathbb{E}\left(\tilde{A}\tilde{B}^* + \tilde{B}\tilde{A}^*\right)$$

(iv) *If A, B are skew-uncorrelated, that is,*

$$\mathbb{E}(AB^*) + \mathbb{E}(BA^*) = \mathbb{E}(A)\mathbb{E}(B^*) + \mathbb{E}(B)\mathbb{E}(A^*),$$

then $\mathbb{V}_{\mathrm{sym}}(A+B) = \mathbb{V}_{\mathrm{sym}}(A) + \mathbb{V}_{\mathrm{sym}}(B)$.

PROOF. (i)

$$\begin{aligned}\mathbb{V}_{\mathrm{sym}}(A) =& \mathbb{E}\left((A-\mathbb{E}(A))\left(A^*-\mathbb{E}(A^*)\right)\right)\\ =& \mathbb{E}\left(AA^* - A\mathbb{E}(A^*) - \mathbb{E}(A)A^* + \mathbb{E}(A)\mathbb{E}(A^*)\right)\\ =& \mathbb{E}(AA^*) - \mathbb{E}(A\mathbb{E}(A^*)) - \mathbb{E}(\mathbb{E}(A)A^*) + \mathbb{E}(\mathbb{E}(A)\mathbb{E}(A^*))\\ =& \mathbb{E}(AA^*) - \mathbb{E}(A)\mathbb{E}(A^*).\end{aligned}$$

(ii)

$$
\begin{aligned}
&\mathbb{V}_{\mathrm{sym}}(aA+b)\\
=&\mathbb{E}\left((aA+b)(aA+b)^*\right)-\mathbb{E}(aA+b)\mathbb{E}((aA+b)^*)\\
=&\mathbb{E}\left(aAA^*a^*\right)+\mathbb{E}(aAb^*)+\mathbb{E}(bA^*a^*)+bb^*-(a\mathbb{E}(A)+b)(\mathbb{E}(A^*)a^*+b^*)\\
=&\mathbb{E}\left((aA)(aA)^*\right)+a\mathbb{E}(A)b^*+b\mathbb{E}(A^*)a^*+bb^*\\
&-(\mathbb{E}(aA)\mathbb{E}((aA)^*)+a\mathbb{E}(A)b^*+b\mathbb{E}(A^*)a^*+bb^*)\\
=&\mathbb{E}\left((aA)(aA)^*\right)-\mathbb{E}(aA)\mathbb{E}((aA)^*)\\
=&\mathbb{V}_{\mathrm{sym}}(aA)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&\mathbb{V}_{\mathrm{sym}}(A+B)\\
=&\mathbb{E}\left(AA^*+AB^*+BA^*+BB^*\right)\\
&-(\mathbb{E}(A)\mathbb{E}(A^*)+\mathbb{E}(A)\mathbb{E}(B^*)+\mathbb{E}(B)\mathbb{E}(A^*)+\mathbb{E}(B)\mathbb{E}(B^*))\\
=&\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)+\mathbb{E}\left(AB^*+BA^*-\mathbb{E}(A)\mathbb{E}(B^*)-\mathbb{E}(B)\mathbb{E}(A^*)\right)\\
=&\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)+\mathbb{E}\left(AB^*-\mathbb{E}(A)\mathbb{E}(B^*)+BA^*-\mathbb{E}(B)\mathbb{E}(A^*)\right)\\
=&\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)+\mathbb{E}\left(AB^*-\mathbb{E}(A)\mathbb{E}(B^*)-\mathbb{E}(A)\mathbb{E}(B^*)+\mathbb{E}(A)\mathbb{E}(B^*)\right)\\
&+\mathbb{E}\left(BA^*-\mathbb{E}(B)\mathbb{E}(A^*)-\mathbb{E}(B)\mathbb{E}(A^*)+\mathbb{E}(B)\mathbb{E}(A^*)\right)\\
=&\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)+\mathbb{E}\left(AB^*-\mathbb{E}(A)B^*-A\mathbb{E}(B^*)+\mathbb{E}(A)\mathbb{E}(B^*)\right)\\
&+\mathbb{E}\left(BA^*-\mathbb{E}(B)A^*-B\mathbb{E}(A^*)+\mathbb{E}(B)\mathbb{E}(A^*)\right)\\
=&\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)+\mathbb{E}\left((A-\mathbb{E}(A))(B^*-\mathbb{E}(B^*))+(B-\mathbb{E}(B))(A^*-\mathbb{E}(A^*))\right)
\end{aligned}
$$

(iv)

$$\mathbb{V}_{\mathrm{sym}}(A+B)$$

$$=\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)+\mathbb{E}\left(AB^*+BA^*-\mathbb{E}(A)\mathbb{E}(B^*)-\mathbb{E}(B)\mathbb{E}(A^*)\right)$$

$$=\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)+\mathbb{E}\left(AB^*\right)+\mathbb{E}\left(BA^*\right)-\mathbb{E}(A)\mathbb{E}(B^*)-\mathbb{E}(B)\mathbb{E}(A^*)$$

$$=\mathbb{V}_{\mathrm{sym}}(A)+\mathbb{V}_{\mathrm{sym}}(B)$$

□

Definition A.13. *Let $A,B\in L_2(\mu;\mathcal{A})$. Then, we define the covariance*

$$\begin{aligned}\operatorname{cov}(A,B):=&\frac{\mathbb{E}\left((A-\mathbb{E}(A))(B-\mathbb{E}(B))\right)+\mathbb{E}\left((B-\mathbb{E}(B))(A-\mathbb{E}(A))\right)}{2}\\=&\frac{\mathbb{E}(AB)+\mathbb{E}(BA)-\mathbb{E}(A)\mathbb{E}(B)-\mathbb{E}(B)\mathbb{E}(A)}{2}.\end{aligned}$$

A and B are called uncorrelated if and only if $\operatorname{cov}(A,B)=0$.

If $\mathcal{A}$ is a $*$-algebra, then we also define the symmetric covariance

$$\begin{aligned}\operatorname{cov}_{\mathrm{sym}}(A,B):=&\frac{\mathbb{E}\left((A-\mathbb{E}(A))(B^*-\mathbb{E}(B^*))\right)+\mathbb{E}\left((B-\mathbb{E}(B))(A^*-\mathbb{E}(A^*))\right)}{2}\\=&\frac{\mathbb{E}(AB^*)+\mathbb{E}(BA^*)-\mathbb{E}(A)\mathbb{E}(B^*)-\mathbb{E}(B)\mathbb{E}(A^*)}{2}\end{aligned}$$

and A and B are called skew-uncorrelated if and only if $\operatorname{cov}_{\mathrm{sym}}(A,B)=0$.

Remark Note that there are other approaches to the covariance on a topological vector space E (cf. e.g. Definition 2.2.7 in [**5**]). Let μ be a probability Borel measure on $(E,\sigma(E,E'))$ ($\sigma(E,E')$ is the weak topology in E, i.e. the coarsest topology such that all linear functionals in the topological dual E' are continuous) such that $E'\subseteq L_2(\mu)$. Then, the mean of μ is defined as an element a_μ of $(E')^*$ (the algebraic dual of E') via

$$\forall f\in E':\ a_\mu(f):=\int_E f(x)d\mu(x).$$

Furthermore, we define the covariance operator R_μ by

$$R_\mu:\ E'\to(E')^*;\ f\mapsto\left(\ g\mapsto\int_E(f(x)-a_\mu(f))(g(x)-a_\mu(g))^*d\mu\ \right)$$

and the covariance of μ is the corresponding quadratic form on E', i.e.

$$C_\mu : \; (E')^2 \to \mathbb{K}; \; (f,g) \mapsto \int_E (f(x) - a_\mu(f))(g(x) - a_\mu(g))^* d\mu.$$

However, this means that we will have to work with the distribution of a random operator rather than the operators themselves. In particular, the assumptions needed to define these operators are much more technically involved (for instance, how will one check that $E' \subseteq L_2(\mu_A)$ holds for some $A \in L_1(\mu; E)$?). Hence, we are using the notion of covariances which can be defined in algebras rather than the one coming from topological vector spaces.

■

With those definitions, we can also write

$$\mathbb{V}(A_1 + A_2) = \mathbb{V}(A_1) + \mathbb{V}(A_2) + 2\mathrm{cov}(A_1, A_2),$$

$$\mathbb{V}_{\mathrm{sym}}(A_1 + A_2) = \mathbb{V}_{\mathrm{sym}}(A_1) + \mathbb{V}_{\mathrm{sym}}(A_2) + 2\mathrm{cov}_{\mathrm{sym}}(A_1, A_2).$$

Since the covariances are bi-linear[1], we obtain by induction

$$\begin{aligned}
\mathbb{V}_{(\mathrm{sym})}\left(\sum_{i=1}^{n} A_i\right) &= \mathbb{V}_{(\mathrm{sym})}\left(\sum_{i=1}^{n-1} A_i\right) + \mathbb{V}_{(\mathrm{sym})}(A_n) + 2\mathrm{cov}_{(\mathrm{sym})}\left(\sum_{i=1}^{n-1} A_i, A_n\right) \\
&= \sum_{i=1}^{n-1} \mathbb{V}_{(\mathrm{sym})}(A_i) + 2\sum_{i<j<n} \mathrm{cov}_{(\mathrm{sym})}(A_i, A_j) + \mathbb{V}_{(\mathrm{sym})}(A_n) \\
&\quad + 2\sum_{i=1}^{n-1} \mathrm{cov}_{(\mathrm{sym})}(A_i, A_n) \\
&= \sum_{i=1}^{n} \mathbb{V}_{(\mathrm{sym})}(A_i) + 2\sum_{i<j} \mathrm{cov}_{(\mathrm{sym})}(A_i, A_j) \\
&= \sum_{i,j=1}^{n} \mathrm{cov}_{(\mathrm{sym})}(A_i, A_j)
\end{aligned}$$

where we used $\mathbb{V}_{(\mathrm{sym})}(A) = \mathrm{cov}_{(\mathrm{sym})}(A, A)$. We also observe that independent variables are uncorrelated and skew-uncorrelated (whenever that makes sense).

[1] For $\mathrm{cov}_{\mathrm{sym}}$ we need to assume $(A+B)^* = A^* + B^*$ to show linearity; in general, for densely defined $A+B$ (A and B are operators between Hilbert spaces) we only have $A^* + B^* \subseteq (A+B)^*$ - similarly, $A^*B^* \subseteq (BA)^*$ - with equality if at least one of A and B are bounded.

So far, we have seen that many of the numerical characteristics of real probability theory still exist, though the assumptions on many theorems might be more restrictive. However, it is still sufficiently nice for us to have a look at some more interesting theorems.

PROPOSITION A.14. *Let E be metrizable and $(A_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mu;E)^{\mathbb{N}}$ independent. Then,*

$$A_n \to 0 \ \mu\text{-almost everywhere} \Leftrightarrow \ \forall\iota\in I \ \forall k\in\mathbb{N}: \ \sum_{n\in\mathbb{N}} \mu\left(\left[\mathbb{R}_{\geq\frac{1}{k}}\right](p_\iota\circ A_n)\right) < \infty.$$

PROOF. The set $\Omega_0 := \{\omega\in\Omega;\ A_n(\omega)\to 0\}$ is measurable (for every choice of representatives) because

$$\begin{aligned}\Omega_0 &= \left\{\omega\in\Omega;\ \forall\iota\in I\ \forall k\in\mathbb{N}\ \exists n\in\mathbb{N}\ \forall m\in\mathbb{N}_{\geq n}:\ p_\iota(A_m(\omega))\leq\frac{1}{k}\right\}\\ &= \bigcap_{\iota\in I}\bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{m\in\mathbb{N}_{\geq n}}\left\{\omega\in\Omega;\ p_\iota(A_m(\omega))\leq\frac{1}{k}\right\}\end{aligned}$$

which is measurable because I is countable. Hence, Borel-Cantelli[2] yields

$$\begin{aligned}&\forall\iota\in I\ \forall k\in\mathbb{N}:\ \sum_{n\in\mathbb{N}}\mu\left(\left[\mathbb{R}_{\geq\frac{1}{k}}\right](p_\iota\circ A_n)\right)<\infty\\ \Leftrightarrow\ &\forall\iota\in I\ \forall k\in\mathbb{N}:\ \mu\left(\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}_{\geq n}}\left\{\omega\in\Omega;\ p_\iota(A_m(\omega))\geq\frac{1}{k}\right\}\right)=0\\ \Leftrightarrow\ &\mu\left(\bigcup_{\iota\in I}\bigcup_{k\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}_{\geq n}}\left\{\omega\in\Omega;\ p_\iota(A_m(\omega))\geq\frac{1}{k}\right\}\right)=0\\ \Leftrightarrow\ &\mu\left(\bigcap_{\iota\in I}\bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{m\in\mathbb{N}_{\geq n}}\left\{\omega\in\Omega;\ p_\iota(A_m(\omega))<\frac{1}{k}\right\}\right)=1\end{aligned}$$

---

[2]cf. Theorem 18.9 in **[65]**

THEOREM (BOREL-CANTELLI). *Let (Ω,Σ,μ) be a probability space and $(S_j)_{j\in\mathbb{N}}\in\Sigma^{\mathbb{N}}$. Then,*

$$\sum_{j\in\mathbb{N}}\mu(S_j)<\infty\ \Rightarrow\ \mu\left(\bigcap_{k\in\mathbb{N}}\bigcup_{j\in\mathbb{N}_{\geq k}}S_j\right)=0.$$

If the sets S_j are pairwise independent, i.e. $\forall j,k\in\mathbb{N}:\ \mu(S_j\cap S_k)=\mu(S_j)\mu(S_k)$, then

$$\sum_{j\in\mathbb{N}}\mu(S_j)=\infty\ \Rightarrow\ \mu\left(\bigcap_{k\in\mathbb{N}}\bigcup_{j\in\mathbb{N}_{\geq k}}S_j\right)=1.$$

$$\Leftrightarrow \mu\left(\Omega_0\right) = 1$$

□

REMARK The proof shows that the set Ω_0 might not be measurable if E is not metrizable. Even if it were, as it would be the case choosing the $A_i \in \mathcal{SM}(\mu; E)$, then the union of uncountably many null sets need not be a null set anymore.

■

THEOREM A.15 (Hájek-Rènyi). *Let $\mathcal{A}$ be a $*$-algebra of densely defined linear operators on a Hilbert space H, and $A_1, \ldots, A_n \in \overline{\mathcal{S}}L_2(\mu; \mathcal{A})$ independent. Furthermore, let $r_1 \geq r_2 \geq \ldots \geq r_n \in \mathbb{R}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$, $D := \bigcap_{i=1}^{n} D(A_n^*)$, and*

$$\forall i \in \mathbb{N}_{\leq n}: \ S_i := \sum_{k=1}^{i} (A_k - \mathbb{E}(A_k))$$

such that the S_i and A_i are uncorrelated and skew-uncorrelated. For $\varphi \in D$ and $m \in \mathbb{N}_{\leq n}$, let

$$\hat{\Omega} := \left\{\omega \in \Omega; \ \max_{m \leq i \leq n} r_i \left\|S_i(\omega)^* \varphi\right\|_H \geq \varepsilon\right\}.$$

Then,

$$\mu\left(\hat{\Omega}\right) \leq \frac{1}{\varepsilon^2}\left(r_m^2 \sum_{j=1}^{m} \mathbb{E}\left(\left\|(A_j^* - \mathbb{E}(A_j^*))\varphi\right\|_H^2\right) + \sum_{j=m+1}^{n} r_j^2 \mathbb{E}\left(\left\|(A_j^* - \mathbb{E}(A_j^*))\varphi\right\|_H^2\right)\right).$$

PROOF. Let

$$\Omega_i := \{\omega \in \Omega; \ r_i \left\|S_i(\omega)^* \varphi\right\|_H \geq \varepsilon\}$$

and

$$\omega_i := \Omega_i \setminus \bigcup_{j=m}^{i-1} \Omega_j.$$

Then, we have $\hat{\Omega} = \dot{\bigcup}_{j=m}^{n} \omega_j$ ($\dot{\bigcup}$ denotes the disjoint union) and $\omega_j \subseteq \Omega_j$. Let $Z := \sum_{j=m}^{n}(r_j^2 - r_{j+1}^2)S_jS_j^*$ with $r_{n+1} = 0$. Then,

$$\begin{aligned}
\mathbb{E}(Z) =& \sum_{i=m}^{n}(r_i^2 - r_{i+1}^2)\mathbb{V}_{\mathrm{sym}}(S_i)\\
=& \sum_{i=m}^{n}\sum_{j=1}^{i}(r_i^2 - r_{i+1}^2)\mathbb{V}_{\mathrm{sym}}(A_j)\\
=& \sum_{i=m}^{n}\sum_{j=1}^{m}(r_i^2 - r_{i+1}^2)\mathbb{V}_{\mathrm{sym}}(A_j) + \sum_{i=m}^{n}\sum_{j=m+1}^{i}(r_i^2 - r_{i+1}^2)\mathbb{V}_{\mathrm{sym}}(A_j)\\
=& r_m^2\sum_{j=1}^{m}\mathbb{V}_{\mathrm{sym}}(A_j) + \sum_{j=m+1}^{n}\sum_{i=j}^{n}(r_i^2 - r_{i+1}^2)\mathbb{V}_{\mathrm{sym}}(A_j)\\
=& r_m^2\sum_{j=1}^{m}\mathbb{V}_{\mathrm{sym}}(A_j) + \sum_{j=m+1}^{n} r_j^2\mathbb{V}_{\mathrm{sym}}(A_j)
\end{aligned}$$

implies

$$\begin{aligned}
\mathbb{E}\langle Z\varphi,\varphi\rangle_H =& \langle \mathbb{E}(Z)\varphi,\varphi\rangle_H\\
=& r_m^2\sum_{j=1}^{m}\left\langle \mathbb{V}_{\mathrm{sym}}(A_j)\varphi,\varphi\right\rangle_H + \sum_{j=m+1}^{n} r_j^2\left\langle \mathbb{V}_{\mathrm{sym}}(A_j)\varphi,\varphi\right\rangle_H\\
=& r_m^2\sum_{j=1}^{m}\left\langle \mathbb{E}\left((A_j - \mathbb{E}(A_j))(A_j^* - \mathbb{E}(A_j^*))\right)\varphi,\varphi\right\rangle_H\\
&+ \sum_{j=m+1}^{n} r_j^2\left\langle \mathbb{E}\left((A_j - \mathbb{E}(A_j))(A_j^* - \mathbb{E}(A_j^*))\right)\varphi,\varphi\right\rangle_H\\
=& r_m^2\sum_{j=1}^{m}\mathbb{E}\left\langle (A_j - \mathbb{E}(A_j))(A_j^* - \mathbb{E}(A_j^*))\varphi,\varphi\right\rangle_H\\
&+ \sum_{j=m+1}^{n} r_j^2\mathbb{E}\left\langle (A_j - \mathbb{E}(A_j))(A_j^* - \mathbb{E}(A_j^*))\varphi,\varphi\right\rangle_H\\
=& r_m^2\sum_{j=1}^{m}\mathbb{E}\left(\left\|(A_j^* - \mathbb{E}(A_j^*))\varphi\right\|_H^2\right) + \sum_{j=m+1}^{n} r_j^2\mathbb{E}\left(\left\|(A_j^* - \mathbb{E}(A_j^*))\varphi\right\|_H^2\right).
\end{aligned}$$

On the other hand, we obtain, since $\langle Z\varphi,\varphi\rangle_H$ is non-negative,

$$\begin{aligned}
\mathbb{E}\langle Z\varphi,\varphi\rangle_H \geq& \mathbb{E}\left(\sum_{i=m}^{n} 1_{\omega_i}\langle Z\varphi,\varphi\rangle_H\right)\\
=& \sum_{i=m}^{n}\sum_{j=m}^{n}(r_j^2 - r_{j+1}^2)\mathbb{E}\left(1_{\omega_i}\left\|S_j^*\varphi\right\|_H^2\right)\\
\geq& \sum_{i=m}^{n}\sum_{j=i}^{n}(r_j^2 - r_{j+1}^2)\mathbb{E}\left(1_{\omega_i}\left\|S_j^*\varphi\right\|_H^2\right)
\end{aligned}$$

$$\geq \sum_{i=m}^{n}\sum_{j=i}^{n}(r_j^2 - r_{j+1}^2)\frac{\varepsilon^2}{r_i^2}\mu(\omega_i)$$

$$= \sum_{i=m}^{n}\varepsilon^2\mu(\omega_i)$$

$$= \varepsilon^2\mu\left(\hat{\Omega}\right),$$

i.e.

$$\mu\left(\hat{\Omega}\right) \leq \frac{1}{\varepsilon^2}\left(r_m^2\sum_{j=1}^{m}\mathbb{E}\left(\left\|(A_j^* - \mathbb{E}(A_j^*))\varphi\right\|_H^2\right) + \sum_{j=m+1}^{n} r_j^2\mathbb{E}\left(\left\|(A_j^* - \mathbb{E}(A_j^*))\varphi\right\|_H^2\right)\right).$$

□

Corollary A.16 (Kolmogorov). *Let $\mathcal{A}$ be a $*$-algebra of densely defined linear operators on a Hilbert space H, $A_1, \ldots, A_n \in \overline{\mathcal{S}}L_2(\mu;\mathcal{A})$ independent, $\varepsilon \in \mathbb{R}_{>0}$, $D := \bigcap_{i=1}^{n} D(A_n^*)$,*

$$\forall i \in \mathbb{N}_{\leq n}: \; S_i := \sum_{k=1}^{i}(A_k - \mathbb{E}(A_k))$$

such that the S_i and A_i are uncorrelated and skew-uncorrelated, and $\varphi \in D$. Then,

$$\mu\left(\left\{\omega \in \Omega;\; \max_{i\leq n}\|S_i(\omega)^*\varphi\|_H \geq \varepsilon\right\}\right) \leq \frac{1}{\varepsilon^2}\sum_{j=1}^{n}\mathbb{E}\left(\left\|(A_j^* - \mathbb{E}(A_j^*))\varphi\right\|_H^2\right).$$

Corollary A.17 (Chebyshev). *Let $\mathcal{A}$ be a $*$-algebra of densely defined linear operators on a Hilbert space H, $A \in \overline{\mathcal{S}}L_2(\mu;\mathcal{A})$, $\varepsilon \in \mathbb{R}_{>0}$, and $\varphi \in D(A^*)$. Then,*

$$\mu\left(\{\omega \in \Omega;\; \|(A(\omega)^* - \mathbb{E}(A^*))\varphi\|_H \geq \varepsilon\}\right) \leq \frac{1}{\varepsilon^2}\mathbb{E}\left(\|(A^* - \mathbb{E}(A^*))\varphi\|_H^2\right).$$

We may also state the Hájek-Rènyi inequality for closed linear relations.

Theorem A.18 (Hájek-Rènyi for relations). *Let $\mathcal{A}$ be a $*$-algebra of closed linear relations in a Hilbert space H, and $A_1, \ldots, A_n \in \overline{\mathcal{S}}L_2(\mu;\mathcal{A})$ independent. Furthermore, let $r_1 \geq r_2 \geq \ldots \geq r_n \in \mathbb{R}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$, $D := \bigcap_{i=1}^{n} D(A_n \circ A_n^*)$, $\varphi \in D$, and $\chi_j, \psi_{ij}, \Psi_{ij} \in H$ such that $(\varphi, \chi_j) \in A_j^*$, $(\chi_j, \psi_{ij}) \in A_i$, $(\mathbb{E}\chi_j, \Psi_{ij}) \in A_i$,*

$(\chi_j, \hat{\psi}_{ij}) \in \mathbb{E}A_i$, and $(\mathbb{E}\chi_j, \hat{\Psi}_{ij}) \in \mathbb{E}A_i$ for every $j \in \mathbb{N}_{\leq n}$. Let $\tau_j := \sum_{i=1}^{j} \chi_i - \mathbb{E}\chi_i$, $\sigma_j := \sum_{i=1}^{j} \sum_{k=1}^{j} \psi_{ik} - \Psi_{ik} - \hat{\psi}_{ik} + \hat{\Psi}_{ik}$, and

$$\forall i \in \mathbb{N}_{\leq n}: \ S_i := \sum_{k=1}^{i} (A_k - \mathbb{E}(A_k))$$

such that the S_i and A_i are uncorrelated and skew-uncorrelated, i.e.

$$\mathbb{E}\sigma_j = \sum_{i=1}^{j} \mathbb{E}\left(\psi_{ii} - \Psi_{ii} - \hat{\psi}_{ii} + \hat{\Psi}_{ii}\right).$$

For $m \in \mathbb{N}_{\leq n}$, let

$$\hat{\Omega} := \left\{\omega \in \Omega;\ \max_{m \leq i \leq n} r_i \left\| \sum_{j=1}^{i} \chi_j(\omega) - \mathbb{E}\chi_j \right\|_H \geq \varepsilon \right\}.$$

Then,

$$\mu\left(\hat{\Omega}\right) \leq \frac{1}{\varepsilon^2} \left(r_m^2 \sum_{j=1}^{m} \mathbb{E}\left(\|\chi_j - \mathbb{E}\chi_j\|_H^2 \right) + \sum_{j=m+1}^{n} r_j^2 \mathbb{E}\left(\|\chi_j - \mathbb{E}\chi_j\|_H^2 \right) \right).$$

Proof. Let

$$\Omega_i := \left\{\omega \in \Omega;\ r_i \left\| \sum_{j=1}^{i} \chi_j(\omega) - \mathbb{E}\chi_j \right\|_H \geq \varepsilon \right\}$$

and

$$\omega_i := \Omega_i \setminus \bigcup_{j=m}^{i-1} \Omega_j.$$

Then, we have $\hat{\Omega} = \dot{\bigcup}_{j=m}^{n} \omega_j$ ($\dot{\bigcup}$ denotes the disjoint union) and $\omega_j \subseteq \Omega_j$. Furthermore, let $r_{n+1} := 0$. Then,

$$\begin{aligned}
& r_m^2 \sum_{j=1}^{m} \mathbb{E}\left(\|\chi_j - \mathbb{E}\chi_j\|_H^2 \right) + \sum_{j=m+1}^{n} r_j^2 \mathbb{E}\left(\|\chi_j^* - \mathbb{E}\chi_j\|_H^2 \right) \\
=& r_m^2 \sum_{j=1}^{m} \mathbb{E}\left\langle \psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}, \varphi \right\rangle_H \\
& + \sum_{j=m+1}^{n} r_j^2 \mathbb{E}\left\langle \psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}, \varphi \right\rangle_H \\
=& \left\langle r_m^2 \sum_{j=1}^{m} \mathbb{E}\left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj} \right), \varphi \right\rangle_H \\
& + \left\langle \sum_{j=m+1}^{n} r_j^2 \mathbb{E}\left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj} \right), \varphi \right\rangle_H
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle r_m^2 \sum_{j=1}^{m} \mathbb{E}\left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}\right), \varphi \right\rangle_H \\
&\quad + \left\langle \sum_{j=m+1}^{n} \sum_{i=j}^{n} (r_i^2 - r_{i+1}^2) \mathbb{E}\left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}\right), \varphi \right\rangle_H \\
&= \left\langle \sum_{i=m}^{n} \sum_{j=1}^{m} (r_i^2 - r_{i+1}^2) \mathbb{E}\left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}\right), \varphi \right\rangle_H \\
&\quad + \left\langle \sum_{i=m}^{n} \sum_{j=m+1}^{i} (r_i^2 - r_{i+1}^2) \mathbb{E}\left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}\right), \varphi \right\rangle_H \\
&= \left\langle \sum_{i=m}^{n} (r_i^2 - r_{i+1}^2) \sum_{j=1}^{i} \mathbb{E}\left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}\right), \varphi \right\rangle_H \\
&= \mathbb{E} \left\langle \sum_{i=m}^{n} (r_i^2 - r_{i+1}^2) \sum_{j=1}^{i} \left(\psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}\right), \varphi \right\rangle_H \\
&= \sum_{i=m}^{n} (r_i^2 - r_{i+1}^2) \sum_{j=1}^{i} \mathbb{E} \left\langle \psi_{jj} - \Psi_{jj} - \hat{\psi}_{jj} + \hat{\Psi}_{jj}, \varphi \right\rangle_H \\
&= \sum_{i=m}^{n} \sum_{j=m}^{n} (r_j^2 - r_{j+1}^2) \mathbb{E} \underbrace{\langle \sigma_i, \varphi \rangle_H}_{\geq 0} \\
&\geq \sum_{i=m}^{n} \sum_{j=m}^{n} (r_j^2 - r_{j+1}^2) \mathbb{E} 1_{\omega_i} \langle \sigma_i, \varphi \rangle_H \\
&= \left\langle \sum_{i=m}^{n} \sum_{j=m}^{n} (r_j^2 - r_{j+1}^2) \mathbb{E} \left(1_{\omega_i} \sum_{k=1}^{i} \sum_{l=1}^{i} \psi_{lk} - \Psi_{lk} - \hat{\psi}_{lk} + \hat{\Psi}_{lk} \right), \varphi \right\rangle_H \\
&= \sum_{i=m}^{n} \sum_{j=m}^{n} (r_j^2 - r_{j+1}^2) \sum_{k=1}^{i} \sum_{l=1}^{i} \mathbb{E}\left(1_{\omega_i} \left\langle \psi_{lk} - \Psi_{lk} - \hat{\psi}_{lk} + \hat{\Psi}_{lk}, \varphi \right\rangle_H \right) \\
&= \sum_{i=m}^{n} \sum_{j=m}^{n} (r_j^2 - r_{j+1}^2) \sum_{k=1}^{i} \sum_{l=1}^{i} \mathbb{E}\left(1_{\omega_i} \langle \chi_k - \mathbb{E}\chi_k, \chi_l - \mathbb{E}\chi_l \rangle_H \right) \\
&= \sum_{i=m}^{n} \sum_{j=m}^{n} (r_j^2 - r_{j+1}^2) \mathbb{E} \left(1_{\omega_i} \left\| \sum_{k=1}^{i} \chi_k - \mathbb{E}\chi_k \right\|_H^2 \right) \\
&\geq \sum_{i=m}^{n} \sum_{j=i}^{n} (r_j^2 - r_{j+1}^2) \mathbb{E} \left(1_{\omega_i} \left\| \sum_{k=1}^{i} \chi_k - \mathbb{E}\chi_k \right\|_H^2 \right) \\
&\geq \sum_{i=m}^{n} \sum_{j=i}^{n} (r_j^2 - r_{j+1}^2) \frac{\varepsilon^2}{r_i^2} \mu(\omega_i) \\
&= \sum_{i=m}^{n} \varepsilon^2 \mu(\omega_i) \\
&= \varepsilon^2 \mu\left(\hat{\Omega}\right).
\end{aligned}
$$

□

PROPOSITION A.19 (Weak Law of Large Numbers; Strong Operator Topology). *Let $\mathcal{A}$ be a $*$-algebra of densely defined operators in a Hilbert space H, $(A_n)_{n\in\mathbb{N}} \in \overline{\mathcal{S}}L_2(\mu;\mathcal{A})^{\mathbb{N}}$ skew-uncorrelated, $D := \bigcap_{n\in\mathbb{N}} D(A_n^*)$,*

$$\forall n \in \mathbb{N}: \; S_n := \frac{1}{n}\sum_{k=1}^{n}(A_k - \mathbb{E}(A_k)),$$

$\varphi \in D$, and $\frac{1}{n^2}\sum_{j=1}^{n}\left\langle \mathbb{V}_{\mathrm{sym}}(A_j)\varphi,\varphi\right\rangle_H \to 0$. Then,

$$\mu\left(\{\omega \in \Omega;\; \|S_n(\omega)^*\varphi\|_H \geq \varepsilon\}\right) \to 0 \qquad (n \to \infty).$$

PROOF. Chebyshev's inequality yields

$$\begin{aligned}
\mu\left(\{\omega \in \Omega;\; \|S_n(\omega)^*\varphi\|_H \geq \varepsilon\}\right) =& \mu\left(\{\omega \in \Omega;\; \|(S_n(\omega)^* - \mathbb{E}(S_n^*))\varphi\|_H \geq \varepsilon\}\right)\\
\leq& \frac{1}{\varepsilon^2}\mathbb{E}\left(\|(S_n^* - \mathbb{E}(S_n^*))\varphi\|_H^2\right)\\
=& \frac{1}{\varepsilon^2}\mathbb{E}\left(\|S_n^*\varphi\|_H^2\right)\\
=& \frac{1}{n^2\varepsilon^2}\mathbb{E}\left(\left\|\sum_{k=1}^{n}(A_k^* - \mathbb{E}(A_k^*))\varphi\right\|_H^2\right)\\
=& \frac{1}{n^2\varepsilon^2}\mathbb{E}\left(\left\langle\sum_{k=1}^{n}(A_k^* - \mathbb{E}(A_k^*))\varphi, \sum_{k=1}^{n}(A_k^* - \mathbb{E}(A_k^*))\varphi\right\rangle_H\right)\\
=& \frac{1}{n^2\varepsilon^2}\sum_{k=1}^{n}\left\langle \mathbb{V}_{\mathrm{sym}}(A_k)\varphi,\varphi\right\rangle_H\\
\to& 0.
\end{aligned}$$

□

Note that the strong operator topology refers to convergence of S_n^*, i.e. S_n^* converges to zero in measure with respect to the strong operator topology. With the following lemma we can also formulate the strong law of large numbers in the strong operator topology.

LEMMA A.20. *Let $(A_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $t_n \searrow 0$ and $\sum_{n\in\mathbb{N}} t_n A_n$ is Cauchy. Then, $t_n \sum_{k=1}^{n} A_k \to 0$.*

PROOF. For $n \in \mathbb{N}_0$, let $B_n := \sum_{k=1}^n t_k A_k$. Then, $A_n = \frac{B_n - B_{n-1}}{t_n}$ and

$$\begin{aligned} t_n \sum_{k=1}^n A_k =& t_n \sum_{k=1}^n \frac{B_k - B_{k-1}}{t_k} \\ =& B_n - \sum_{k=2}^n B_{k-1} \frac{t_n}{t_k} + \sum_{k=1}^{n-1} B_k \frac{t_n}{t_k} \\ =& B_n - \sum_{k=1}^{n-1} B_k \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right). \end{aligned}$$

Let $\iota \in I$ and $\varepsilon \in \mathbb{R}_{>0}$. Then, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}_{\geq n_0}$

$$p_\iota(B_m - B_n) < \varepsilon$$

holds. Hence,

$$\begin{aligned} & p_\iota \left(B_n - \sum_{k=1}^{n-1} B_k \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) \right) \\ =& p_\iota \left(B_n - \sum_{k=1}^{n-1} (B_k - B_n) \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) - \sum_{k=1}^{n-1} B_n \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) \right) \\ =& p_\iota \left(B_n \left(1 - \sum_{k=1}^{n-1} \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) \right) - \sum_{k=1}^{n-1} (B_k - B_n) \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) \right) \\ =& p_\iota \left(B_n \frac{t_n}{t_1} - \sum_{k=1}^{n_0-1} (B_k - B_n) \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) - \sum_{k=n_0}^{n-1} (B_k - B_n) \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) \right) \\ \leq& p_\iota(B_n) \frac{t_n}{t_1} + \sum_{k=1}^{n_0-1} p_\iota(B_k - B_n) \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) + \sum_{k=n_0}^{n-1} p_\iota(B_k - B_n) \left(\frac{t_n}{t_{k+1}} - \frac{t_n}{t_k} \right) \\ <& p_\iota(B_n) \frac{t_n}{t_1} + t_n \sum_{k=1}^{n_0-1} p_\iota(B_k - B_n) \left(\frac{1}{t_{k+1}} - \frac{1}{t_k} \right) + \varepsilon t_n \sum_{k=n_0}^{n-1} \left(\frac{1}{t_{k+1}} - \frac{1}{t_k} \right) \\ \leq& p_\iota(B_n) \frac{t_n}{t_1} + \underbrace{t_n \sum_{k=1}^{n_0-1} \underbrace{(p_\iota(B_k - B_{n_0}) + p_\iota(B_{n_0} - B_n))}_{\leq 2 \max_{k \leq n_0} p_\iota(B_k - B_{n_0})} \left(\frac{1}{t_{k+1}} - \frac{1}{t_k} \right)}_{\to 0 \ (n \to \infty)} + \underbrace{\varepsilon \left(1 - \frac{t_n}{t_{n_0}} \right)}_{\to \varepsilon \ (n \to \infty)} \end{aligned}$$

holds. Furthermore,

$$\exists N \in \mathbb{N} \ \forall m, n \in \mathbb{N}_{\geq N} : \ p_\iota \left(\sum_{j=m}^n t_j A_j \right) < \varepsilon$$

implies

$$p_\iota(B_n) \frac{t_n}{t_1} = \frac{t_n}{t_1} p_\iota \left(\sum_{k=1}^n t_k A_k \right)$$

$$
\begin{aligned}
&\leq \frac{t_n}{t_1}\left(p_\iota\left(B_N\right)+p_\iota\left(\sum_{j=N+1}^{n} t_j A_j\right)\right) \\
&< \frac{t_n}{t_1}\left(p_\iota\left(B_N\right)+\varepsilon\right) \\
&\to 0 \qquad (n\to\infty),
\end{aligned}
$$

i.e.

$$
\lim_{n\to\infty} p_\iota\left(t_n \sum_{k=1}^{n} A_k\right) = \lim_{n\to\infty} p_\iota\left(B_n - \sum_{k=1}^{n-1} B_k\left(\frac{t_n}{t_{k+1}}-\frac{t_n}{t_k}\right)\right) < \varepsilon.
$$

□

THEOREM A.21 (Strong Law of Large Numbers; Strong Operator Topology). *Let $\mathcal{A}$ be a $*$-algebra of densely defined operators in a Hilbert space H, $(A_n)_{n\in\mathbb{N}} \in \overline{\mathcal{S}}L_2(\mu;\mathcal{A})^{\mathbb{N}}$ independent, $D := \bigcap_{n\in\mathbb{N}} D(A_n^*)$, $\sum_{n\in\mathbb{N}} \frac{\mathbb{V}_{\mathrm{sym}}(A_n)}{n^2}$ Cauchy, $\varphi \in D$, and*

$$
\forall n\in\mathbb{N}:\ S_n := \frac{1}{n}\sum_{k=1}^{n}(A_k - \mathbb{E}(A_k))
$$

such that the S_n and A_n are uncorrelated and skew-uncorrelated. Then,

$$
\|S_n^*\varphi\|_H \to 0 \ \mu\text{-almost everywhere} \qquad (n\to\infty).
$$

PROOF. Since $\sum_{n\in\mathbb{N}} \frac{\mathbb{V}_{\mathrm{sym}}(A_n)}{n^2}$ is Cauchy, we obtain

$$
\frac{1}{n^2}\sum_{k=1}^{n}\mathbb{V}_{\mathrm{sym}}(A_k) \to 0.
$$

Thus, for $\varepsilon \in \mathbb{R}_{>0}$,

$$
\begin{aligned}
&\mu\left(\left\{\omega\in\Omega;\ \sup_{i\in\mathbb{N}_{\geq m}} \|S_i(\omega)^*\varphi\|_H \geq \varepsilon\right\}\right) \\
=&\mu\left(\left\{\omega\in\Omega;\ \lim_{n\to\infty}\max_{m\leq i\leq n} \|S_i(\omega)^*\varphi\|_H \geq \varepsilon\right\}\right) \\
=&\lim_{n\to\infty}\mu\left(\left\{\omega\in\Omega;\ \max_{m\leq i\leq n} \|S_i(\omega)^*\varphi\|_H \geq \varepsilon\right\}\right) \\
\leq&\lim_{n\to\infty}\frac{1}{\varepsilon^2}\left(\frac{1}{m^2}\sum_{j=1}^{m}\mathbb{E}\left(\left\|(A_j^*-\mathbb{E}(A_j^*))\varphi\right\|_H^2\right) + \sum_{j=m+1}^{n}\frac{\mathbb{E}\left(\left\|(A_j^*-\mathbb{E}(A_j^*))\varphi\right\|_H^2\right)}{j^2}\right)
\end{aligned}
$$

$$= \frac{1}{\varepsilon^2}\left(\left\langle \frac{1}{m^2}\sum_{j=1}^{m} \mathbb{V}_{\mathrm{sym}}(A_j)\varphi,\varphi\right\rangle_H + \left\langle \sum_{j\in\mathbb{N}_{>m}} \frac{\mathbb{V}_{\mathrm{sym}}(A_j)}{j^2}\varphi,\varphi\right\rangle_H\right)$$

holds[3] and implies

$$\lim_{m\to\infty}\mu\left(\left\{\omega\in\Omega;\ \sup_{i\in\mathbb{N}_{\geq m}}\|S_i(\omega)^*\varphi\|_H\geq\varepsilon\right\}\right)=0.$$

Hence,

$$\begin{aligned}
0 &= \lim_{m\to\infty}\mu\left(\left\{\omega\in\Omega;\ \sup_{i\in\mathbb{N}_{\geq m}}\|S_i(\omega)^*\varphi\|_H\geq\varepsilon\right\}\right)\\
&= \lim_{m\to\infty}\mu\left(\bigcup_{i\in\mathbb{N}_{\geq m}}\{\omega\in\Omega;\ \|S_i(\omega)^*\varphi\|_H\geq\varepsilon\}\right)\\
&= \mu\left(\bigcap_{m\in\mathbb{N}}\bigcup_{i\in\mathbb{N}_{\geq m}}\{\omega\in\Omega;\ \|S_i(\omega)^*\varphi\|_H\geq\varepsilon\}\right)\\
&= \mu\left(\{\omega\in\Omega;\ \|S_i(\omega)^*\varphi\|_H\geq\varepsilon \text{ infinitely often}\}\right)
\end{aligned}$$

implies

$$\mu\left(\{\omega\in\Omega;\ \|S_i(\omega)^*\varphi\|_H<\varepsilon \text{ at most finitely often}\}\right)=1,$$

i.e. $(\|S_n^*\varphi\|_H)_{n\in\mathbb{N}}$ converges to zero μ-almost everywhere.

□

Example Since L_2 spaces over separable measure spaces are separable, the generalized Sombrero Lemma 9.8 yields that $(\Omega,\Sigma,\mu;L_2)$ is a Sombrero space for every Radon measure space (Ω,Σ,μ) and every algebra of Fourier Integral Operators associated with a canonically idempotent canonical relation has the strong law of large numbers with respect to the strong operator topology in a separable L_2.

■

[3]Recall that measures are continuous from below, i.e. $(S_j)_{j\in\mathbb{N}}\in\Sigma^{\mathbb{N}}$ with $S_j\nearrow: S\in\Sigma$ implies $\mu(S)=\lim_{n\to\infty}\mu(S_j)$; cf. Theorem 4.4 in **[65]**. Similarly, all measures are continuous from above.

If we want a to obtain a weaker formulation of the law of large numbers, then we will need the following weaker Hájek-Rènyi inequality which is simply the Hájek-Rènyi inequality for real random variables.

THEOREM A.22 (Hájek-Rènyi; classical). *Let* $A_1, \ldots, A_n \in L_2(\mu; \mathcal{A})$ *independent,* $r_1 \geq r_2 \geq \ldots \geq r_n \in \mathbb{R}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$, *and*

$$\forall i \in \mathbb{N}_{\leq n}: \; S_i := \sum_{k=1}^{i} (A_k - \mathbb{E}(A_k)).$$

For $\varphi \in L(\mathcal{A}, \mathbb{K})$ *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $m \in \mathbb{N}_{\leq n}$, *let*

$$\hat{\Omega} := \left\{ \omega \in \Omega; \; \max_{m \leq i \leq n} r_i \left| \varphi\left(S_i(\omega)\right)\right| \geq \varepsilon \right\}.$$

Then,

$$\mu\left(\hat{\Omega}\right) \leq \frac{1}{\varepsilon^2} \left(r_m^2 \sum_{j=1}^{m} \mathbb{V}_{\text{sym}}\left(\varphi \circ A_j\right) + \sum_{j=m+1}^{n} r_j^2 \mathbb{V}_{\text{sym}}\left(\varphi \circ A_j\right) \right).$$

In particular, if φ *is a character,*[4] *then we obtain*

$$\mu\left(\hat{\Omega}\right) \leq \frac{1}{\varepsilon^2} \left(r_m^2 \sum_{j=1}^{m} \varphi\left(\mathbb{V}_{\text{sym}}\left(A_j\right)\right) + \sum_{j=m+1}^{n} r_j^2 \varphi\left(\mathbb{V}_{\text{sym}}\left(A_j\right)\right) \right).$$

PROOF.

$$\mu\left(\hat{\Omega}\right) \leq \frac{1}{\varepsilon^2} \left(r_m^2 \sum_{j=1}^{m} \mathbb{V}_{\text{sym}}\left(\varphi \circ A_j\right) + \sum_{j=m+1}^{n} r_j^2 \mathbb{V}_{\text{sym}}\left(\varphi \circ A_j\right) \right).$$

is simply the Hájek-Rènyi inequality over $\mathbb{K}$ (the statement follows from the Hájek-Rènyi inequality above with $\mathcal{A} = \mathbb{K}$ and noting that the $\varphi \circ A_j \in L_2(\mu) = \mathcal{S}L_2(\mu)$ are independent, thus, (skew-)uncorrelated). If φ is a character, then

$$\begin{aligned}
\mathbb{V}_{\text{sym}}(\varphi \circ A) &= \mathbb{E}\left(\left(\varphi \circ A\right)\left(\varphi \circ A\right)^*\right) - \left(\mathbb{E}\left(\varphi \circ A\right)\right)\left(\mathbb{E}\left(\varphi \circ A\right)\right)^* \\
&= \mathbb{E}\left(\varphi \circ \left(AA^*\right)\right) - \varphi\left(\mathbb{E}A\right)\varphi\left(\mathbb{E}A\right)^*
\end{aligned}$$

[4] $\varphi \in L(\mathcal{A}, \mathbb{K})$ is called a character if and only if φ is a homomorphism, i.e. $\forall A, B \in \mathcal{A}$: $\varphi(AB) = \varphi(A)\varphi(B) \;\wedge\; \varphi(A^*) = \varphi(A)^*$.

$$=\varphi\left(\mathbb{E}\left(AA^*\right)\right)-\varphi\left(\mathbb{E}A\mathbb{E}A^*\right)$$

$$=\varphi(\mathbb{V}_{\mathrm{sym}}(A))$$

shows the assertion.

□

Corollary A.23 (Kolmogorov; classical). *Let $A_1,\dots,A_n\in L_2(\mu;\mathcal{A})$ independent, $\varepsilon\in\mathbb{R}_{>0}$,*

$$\forall i\in\mathbb{N}_{\le n}:\ S_i:=\sum_{k=1}^{i}(A_k-\mathbb{E}(A_k)),$$

and $\varphi\in L(\mathcal{A},\mathbb{K})$ with $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Then,

$$\mu\left(\left\{\omega\in\Omega;\ \max_{i\le n}\left|\varphi\left(S_i(\omega)\right)\right|\ge\varepsilon\right\}\right)\le\frac{1}{\varepsilon^2}\sum_{j=1}^{n}\mathbb{V}_{\mathrm{sym}}\left(\varphi\circ A_j\right).$$

In particular, if φ is a character, then we obtain

$$\mu\left(\left\{\omega\in\Omega;\ \max_{i\le n}\left|\varphi\left(S_i(\omega)\right)\right|\ge\varepsilon\right\}\right)\le\frac{1}{\varepsilon^2}\sum_{j=1}^{n}\varphi\left(\mathbb{V}_{\mathrm{sym}}\left(A_j\right)\right).$$

Corollary A.24 (Chebyshev; classical). *Let $A\in L_2(\mu;\mathcal{A})$, $\varepsilon\in\mathbb{R}_{>0}$, and $\varphi\in L(\mathcal{A},\mathbb{K})$ with $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Then,*

$$\mu\left(\left\{\omega\in\Omega;\ \left|\varphi(A(\omega)-\mathbb{E}(A))\right|\ge\varepsilon\right\}\right)\le\frac{1}{\varepsilon^2}\mathbb{V}_{\mathrm{sym}}\left(\varphi\circ A\right).$$

In particular, if φ is a character, then we obtain

$$\mu\left(\left\{\omega\in\Omega;\ \left|\varphi(A(\omega)-\mathbb{E}(A))\right|\ge\varepsilon\right\}\right)\le\frac{1}{\varepsilon^2}\varphi\left(\mathbb{V}_{\mathrm{sym}}\left(A\right)\right).$$

Corollary A.25 (Strong Law of Large Numbers; weak topology). *Let $\varphi\in L(\mathcal{A},\mathbb{K})$ with $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $(A_n)_{n\in\mathbb{N}}\in L_2(\mu;\mathcal{A})^{\mathbb{N}}$ independent,*

$$\forall n\in\mathbb{N}:\ S_n:=\frac{1}{n}\sum_{k=1}^{n}(A_k-\mathbb{E}(A_k)),$$

$\sum_{n\in\mathbb{N}} \frac{\mathbb{V}_{\mathrm{sym}}(\varphi\circ A_n)}{n^2}$ *Cauchy. Then,*

$$\varphi \circ S_n \to 0 \ \mu\text{-almost everywhere} \qquad (n \to \infty).$$

Considering convergent sums $\frac{1}{n}\sum_{k=1}^{n} A_k$, we obtain the notion of tail events.

Definition A.26. *Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mu;E)^{\mathbb{N}}$. $S \in \Sigma$ is called a tail event if and only if $S \in \sigma\left((A_n)_{n\in\mathbb{N}_{\geq m}}\right)$ for all $m \in \mathbb{N}$ where $\sigma\left((A_n)_{n\in\mathbb{N}_{\geq m}}\right)$ denotes the σ-algebra generated by $\bigcup_{n\in\mathbb{N}_{\geq m}}\{[S']A_n;\ S' \in \mathcal{B}(E)\}$.*

Remark Note that $S \in \sigma\left((A_n)_{n\in\mathbb{N}_{\geq m}}\right)$ and $(A_n)_{n\in\mathbb{N}}$ independent imply independence of S and $\sigma(A_1,\ldots,A_{m-1})$. S being independent of $\sigma(A_1,\ldots,A_n)$ for all $n \in \mathbb{N}$ implies, thus, independence of S and $\sigma\left((A_n)_{n\in\mathbb{N}}\right)$.

■

Proposition A.27 (Kolmogorov's 0-1-Law). *Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mu;E)^{\mathbb{N}}$ be independent and $S \in \Sigma$ a tail event. Then, $\mu(S) \in \{0,1\}$.*

Proof. Since $S \in \sigma\left((A_n)_{n\in\mathbb{N}_{\geq m}}\right)$ for all $m \in \mathbb{N}$, we obtain that S is independent of all $\sigma(A_1,\ldots,A_n)$, i.e. independent of $\sigma\left((A_n)_{n\in\mathbb{N}}\right)$. However, $S \in \sigma\left((A_n)_{n\in\mathbb{N}}\right)$. Hence, S is independent of itself, that is,

$$\mu(S) = \mu(S\cap S) = \mu(S)\mu(S)$$

which implies $\mu(S) \in \{0,1\}$.

□

Proposition A.28. *Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mu;E)^{\mathbb{N}}$ be independent and identically distributed, $B \in \mathcal{M}(\mu;E)$, and $\frac{1}{n}\sum_{k=1}^{n} A_k \to B$ μ-almost everywhere. Then, $\forall n \in \mathbb{N}:\ A_n \in L_1(\mu;E)$ and $B \in L_1(\mu;E)$.*

If $(A_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mu;E)^{\mathbb{N}}$ *furthermore satisfies the weak law of large numbers in* E*, i.e.* $\frac{1}{n}\sum_{k=1}^{n}(A_k - \mathbb{E}(A_k)) \to 0$ *in measure, and* $\mathrm{id} \in L(E) \cap \overline{\mathcal{S}}\mathcal{M}(\mu_{A_n};E)$ *for every* $n\in\mathbb{N}$*, then* $\forall n \in \mathbb{N}:\ B = \mathbb{E}(A_n)$ μ*-almost everywhere.*

Proof. Let

$$S_n := \frac{1}{n}\sum_{k=1}^{n} A_k$$

and, for $\iota \in I$ and $k, n \in \mathbb{N}$,

$$\Omega_{k,n,\iota} := \{\omega \in \Omega;\ p_\iota\left(A_k(\omega)\right) \geq n\}.$$

Then, $\forall k,m,n \in \mathbb{N}\ \forall \iota \in I:$

$$\mu\left(\Omega_{k,n,\iota}\right) = \mu\left(\left[\left[\mathbb{R}_{\geq n}\right]p_\iota\right]A_k\right) = \mu_{A_k}\left(\left[\mathbb{R}_{\geq n}\right]p_\iota\right) = \mu_{A_m}\left(\left[\mathbb{R}_{\geq n}\right]p_\iota\right) = \mu\left(\Omega_{m,n,\iota}\right).$$

Hence, $\frac{1}{n}A_n = S_n - \frac{n-1}{n}S_{n-1} \to 0$ μ-almost everywhere implies that the set

$$\left\{\omega \in \Omega;\ \exists j \in \mathbb{N}^{\mathbb{N}}\ \forall k \in \mathbb{N}:\ j_k < j_{k+1}\ \wedge\ \omega \in \Omega_{j_k,j_k,\iota}\right\}$$

must have probability zero, i.e.

$$\mu\left(\bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}_{\geq k}}\Omega_{n,n,\iota}\right) = 0.$$

Thus, by Borel-Cantelli,

$$\forall k \in \mathbb{N}\ \forall \iota \in I:\ \sum_{n\in\mathbb{N}}\mu\left(\Omega_{k,n,\iota}\right) = \sum_{n\in\mathbb{N}}\mu\left(\Omega_{n,n,\iota}\right) < \infty.$$

However, for real random variables X the inequality

$$\sum_{n\in\mathbb{N}}\mu\left(\left[\mathbb{R}_{\geq n}\right]|X|\right) \leq \mathbb{E}(|X|) \leq 1 + \sum_{n\in\mathbb{N}}\mu\left(\left[\mathbb{R}_{\geq n}\right]|X|\right)$$

holds,[5] which implies

$$\forall k \in \mathbb{N}\ \forall \iota \in I: \ \mathbb{E}(p_\iota \circ A_k) \leq 1 + \sum_{n \in \mathbb{N}} \mu\left(\Omega_{k,n,\iota}\right) < \infty,$$

i.e. $A_k \in L_1(\mu; E)$. Furthermore,

$$\mathbb{E}\left(p_\iota \circ B\right) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left(p_\iota \circ A_k\right) \leq \max_{k \in \mathbb{N}_{\leq n}} \mathbb{E}\left(p_\iota \circ A_k\right) \leq 1 + \sum_{n \in \mathbb{N}} \mu\left(\Omega_{n,n,\iota}\right) < \infty$$

shows $B \in L_1(\mu; E)$.

Note that $(A_n)_{n \in \mathbb{N}} \in \mathcal{M}(\mu; E)^{\mathbb{N}}$ being identically distributed means $\forall k, n \in \mathbb{N}:$ $\mu_{A_k} = \mu_{A_n}$ which implies

$$\forall k, n \in \mathbb{N}: \ \mathbb{E}(A_k) = \int_E x d\mu_{A_k} = \int_E x d\mu_{A_n} = \mathbb{E}(A_n)$$

provided that the A_n are integrable.

---

[5]PROOF. Without loss of generality, let $X \geq 0$. Then, we obtain $\Omega = \dot{\bigcup}_{n \in \mathbb{N}_0} A_n$ with $A_n :=$ $[[n, n+1)]\, X$, and $\mathbb{E}X = \sum_{n \in \mathbb{N}_0} \int_{A_n} X d\mu$ implies

$$\sum_{n \in \mathbb{N}_0} n\mu(A_n) \leq \mathbb{E}X \leq \sum_{n \in \mathbb{N}_0} (n+1)\mu(A_n) = 1 + \sum_{n \in \mathbb{N}_0} n\mu(A_n).$$

Let $B_n := [\mathbb{R}_{\geq n}]\, X$. Then, $A_n = B_n \setminus B_{n+1}$ and $\forall N \in \mathbb{N}:$

$$\begin{aligned}\sum_{n=1}^{N} n\mu(A_n) &= \sum_{n=1}^{N} n\mu(B_n) - \sum_{n=1}^{N} n\mu(B_{n+1}) = \sum_{n=1}^{N} n\mu(B_n) - \sum_{n=1}^{N} (n-1)\mu(B_n) - N\mu(B_{N+1}) \\ &= \sum_{n=1}^{N} \mu(B_n) - N\mu(B_{N+1})\end{aligned}$$

If $\mathbb{E}X < \infty$, then $\mu\left(\bigcap_{n \in \mathbb{N}_0} B_n\right) = 0$, i.e.

$$0 \leq N\mu(B_{N+1}) \leq \int_{B_{N+1}} X d\mu \to 0$$

implies $\sum_{n \in \mathbb{N}_0} n\mu(A_n) = \sum_{n \in \mathbb{N}} \mu(B_n)$, that is, the assertion. If $\mathbb{E}X = \infty$, then $\sum_{n \in \mathbb{N}_0} n\mu(A_n) = \infty$ implies $\sum_{n=1}^{N} \mu(B_n)$ and, hence, the assertion.

$\square$

If $\frac{1}{n}\sum_{k=1}^{n}\left(A_k - \mathbb{E}(A_k)\right) \to 0$ in measure, then

$$\begin{aligned}
B - \mathbb{E}(A_n) &= B - \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(A_n) \\
&= B - \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(A_k) \\
&= \underbrace{B - \frac{1}{n}\sum_{k=1}^{n} A_k}_{\to 0\ \mu\text{-a.e.}} + \underbrace{\frac{1}{n}\sum_{k=1}^{n}\left(A_k - \mathbb{E}(A_k)\right)}_{\to 0 \text{ in measure}} \\
&\to 0 \qquad \text{in measure.}
\end{aligned}$$

In other words, $B = \mathbb{E}(A_n)$.

□

Finally, we will define characteristic functions. These will lead directly to a central limit theorem.

Definition A.29. *Let $A \in \mathcal{M}(\mu; E)$. Then, we call*

$$\operatorname{char}_A : \ L(E, \mathbb{R}) \to \mathbb{C}; \ t \mapsto \mathbb{E}\left(\exp \circ (it) \circ A\right)$$

the characteristic function of A.

Corollary A.30. *Let $A \in \mathcal{M}(\mu; E)$. Then, the following are true.*

(i) $\operatorname{char}_A(0) = 1$

(ii) $\forall t \in L(E, \mathbb{R}) : \ \operatorname{char}_A(-t) = \operatorname{char}_A(t)^*$

(iii) $|\operatorname{char}_A(t)| \le 1$

Remark Note that for algebras $\mathcal{A}_\Gamma$ the functions $t_1 = \Re \operatorname{tr}(T\cdot)$, $t_2 = \Re \operatorname{tr}(T^*\cdot)$, $t_3 = \Im \operatorname{tr}(T\cdot)$, and $t_4 = \Im \operatorname{tr}(T^*\cdot)$ for smoothing T are interesting. For non-smoothing T we may also think of choosing a different trace function tr.

■

OBSERVATION A.31. *Let $A, B \in \mathcal{M}(\mu; E)$ be independent. Then,*

$$\forall t \in L(E, \mathbb{R}): \ \operatorname{char}_{A+B}(t) = \operatorname{char}_A(t) \operatorname{char}_B(t).$$

PROOF.

$$\begin{aligned}\operatorname{char}_{A+B}(t) &= \mathbb{E}\left(\exp\left(it \circ (A+B)\right)\right)\\ &= \mathbb{E}\left(\exp\left(it \circ A + it \circ B\right)\right)\\ &= \mathbb{E}\left(\exp\left(it \circ A\right) \exp\left(it \circ B\right)\right)\\ &= \mathbb{E}\left(\exp\left(it \circ A\right)\right) \mathbb{E}\left(\exp\left(it \circ B\right)\right)\\ &= \operatorname{char}_A(t) \operatorname{char}_B(t)\end{aligned}$$

□

DEFINITION A.32. *$A \in \mathcal{M}(\mu; E)$ is called Gaussian if and only if*

$$\forall t \in L(E, \mathbb{R}) \setminus \{0\}: \ t \circ A \text{ is normally distributed.}$$

A is called degenerate Gaussian if and only if there exists a subspace $F \subsetneq E$ such that $F \neq \{0\}$, A takes μ-almost every value in F, and $A \in \mathcal{M}(\mu; F)$ is Gaussian.

Thus, for $A \in L_1(\mu; E)$ Gaussian, we obtain $\mathbb{E}(t \circ A) = t(\mathbb{E}(A))$ in the Pettis sense and, using $\alpha \in \mathbb{R}$ and

$$\begin{aligned}\mathbb{E}\left(\exp\left(i\alpha t \circ A\right)\right) &= \exp\left(i\alpha \mathbb{E}(t \circ A) - \frac{\alpha^2 \mathbb{V}(t \circ A)}{2}\right)\\ &= \exp\left(i\alpha t \mathbb{E} A - \frac{\mathbb{V}(\alpha t \circ A)}{2}\right),\end{aligned}$$

the following statement (note that $\mathbb{V}(t \circ A) = R_{\mu_A}(t)(t)$ in the general covariance of topological vector spaces sense).

LEMMA A.33. *Let $A \in L_1(\mu; E)$. Then, the following are equivalent.*

(i) *A is Gaussian.*

(ii) $\forall t_1, t_2 \in L(E,\mathbb{R})\ \forall \alpha \in \mathbb{R}:$

$$\mathbb{E}\left(\exp\left(i(\alpha t_1 + t_2)\circ A\right)\right) = \exp\left(i(\alpha t_1 + t_2)\mathbb{E}A - \frac{\mathbb{V}((\alpha t_1 + t_2)\circ A)}{2}\right).$$

(iii) $\forall t \in L(E,\mathbb{R}):\ \mathbb{E}\left(\exp\left(it\circ A\right)\right) = \exp\left(it\mathbb{E}A - \frac{\mathbb{V}(t\circ A)}{2}\right).$

Remark More generally, we can define a measure μ on E to be Gaussian if and only if for every $f \in E'$ the push-forward μ_f is Gaussian. We can furthermore define the characteristic function of a measure to be

$$\operatorname{char}(\mu) := \left(\ L(E,\mathbb{R}) \ni f \mapsto \int_E \exp\left(if(x)\right) d\mu(x) \in \mathbb{C}\right).$$

In that setting, it can be shown that a measure μ on a locally convex space E is Gaussian if and only if there exists $L \in L(L(E,\mathbb{R}),\mathbb{R})$ and a symmetric bilinear form B on $L(E,\mathbb{R})$ such that $f \mapsto B(f,f)$ is non-negative and

$$\operatorname{char}(\mu)(f) = \exp\left(iL(f) - \frac{1}{2}B(f,f)\right);$$

cf. e.g. Theorem 2.2.4 in [**5**]. In fact, $L(f) = \int_E f d\mu$ and $B(f,g) = \int_E (f-L(f))(g-L(g))d\mu$, that is, in the case of a random variable A, we have $L(f) = \mathbb{E}(f\circ A)$ and $B(f,g) = \operatorname{cov}(f\circ A, g\circ A)$, i.e. $B(f,f) = \mathbb{V}(f\circ A)$ and $L = \mathbb{E}(A)$ in the Pettis sense.

It follows directly ([**5**] Corollary 2.2.6) that the product $\mu_1 \times \mu_2$ of Gaussian measures and the convolution $\mu_1 * \mu_2$ are Gaussian, as well.

■

With that prelude, we can state a central limit theorem which follows directly from Lindeberg's central limit theorem for real random variables.

Theorem A.34 (Central Limit Theorem). *Let $(A_k)_{k\in\mathbb{N}} \in L_2(\mu;E)^{\mathbb{N}}$ be independent with $\forall k \in \mathbb{N}:\ \mathbb{E}A_k = 0$ such that $A := \sum_{k\in\mathbb{N}} A_k$ converges in $L_2(\mu;E)$.*

Furthermore, let $\forall k \in \mathbb{N} : \; \mathbb{V}(t \circ A_k) > 0$, $s_n := \sqrt{\sum_{k=1}^{n} \mathbb{V}(t \circ A_k)} \to: s \in \mathbb{R}_{>0} \; (n \to \infty)$, *and let the Lindeberg condition*

$$\forall \varepsilon \in \mathbb{R}_{>0} : \; \lim_{n \to \infty} \frac{1}{s_n^2} \sum_{j=1}^{n} \int_{\{x \in \mathbb{R}; \; |x| > \varepsilon s_n\}} x^2 d\mu_{t \circ A_j}(x) = 0$$

hold for every $t \in L(E, \mathbb{R}) \setminus \{0\}$. *Then,* A *is Gaussian.*

PROOF. Since s_n and $\sum_{k=1}^{n} A_k$ are convergent, we obtain $\frac{\sum_{k=1}^{n} A_k}{s_n} \to \frac{1}{s} A \; (n \to \infty)$ and, hence,

$$\frac{\sum_{k=1}^{n} t \circ A_k}{s_n} \to \frac{1}{s} t \circ A \; (n \to \infty).$$

Thus, Lindeberg's Central Limit Theorem (cf. e.g. Theorem VIII.4.3 in [**22**]) implies that $\frac{1}{s} t \circ A$ is Gaussian which directly implies the assertion.

□

The central limit theorem for independent and identically distributed is a lot more involved. However, there are theorems in that direction like the following lemma (Lemma 7.6.9 in [**5**]).

LEMMA A.35. *Let* μ *be a probability Radon measure on* E *with mean zero and the sequence* $(\mu_n)_{n \in \mathbb{N}}$ *defined by* $\mu_1 := \mu$ *and* $\forall n \in \mathbb{N} : \; \mu_{n+1} := \mu * \mu_n$ *uniformly tight, that is,* $\forall \varepsilon \in \mathbb{R}_{>0} \; \exists K \subseteq_{\text{compact}} E \; \forall n \in \mathbb{N} : \; \mu_n(E \setminus K) < \varepsilon$. *Then,* μ_n *converges weakly to a Gaussian Radon measure.*

The glaring problem, however, is that we do not know whether or not there exists a random variable A satisfying $\mu_{\sum_{k=1}^{n} A_k} = \mu_n \rightharpoonup \mu_A$ with $(A_k)_{k \in \mathbb{N}}$ independent and identically distributed such that $(\mu_k)_{k \in \mathbb{N}}$ is uniformly tight, that is, whether or not the limit measure has a density; hence, raising the question whether the space or algebra at hand has the Radon-Nikodým property.

APPENDIX B

The gap topology and generalized convergence

In this appendix, we want to recall a few facts about the gap-topology. We will closely follow chapter IV in [**44**].

DEFINITION B.1. *Let E be a Banach space and $A, B \subseteq E$ (non-empty) closed linear subspaces. Then, we define*

$$\delta(A,B) := \begin{cases} 0 & , \ A = \{0\} \\ \sup\{\mathrm{dist}_E(u,B);\ u \in A \cap \partial B_E\} & , \ A \neq \{0\} \end{cases}$$

and

$$\hat{\delta}(A,B) := \max\{\delta(A,B), \delta(B,A)\}.$$

$\hat{\delta}$ is called the gap between A and B.

COROLLARY B.2. *Let E be a Banach space and $A, B \subseteq E$ (non-empty) closed linear subspaces. Then, the following are true.*

(i) $\delta(A,B) = 0 \ \Leftrightarrow \ A \subseteq B$

(ii) $\hat{\delta}(A,B) = 0 \ \Leftrightarrow \ A = B$

(iii) $\hat{\delta}(A,B) = \hat{\delta}(B,A)$

(iv) $\delta(A,B) \in [0,1]$

(v) $\hat{\delta}(A,B) \in [0,1]$

In other words, $\hat{\delta}$ is a semi-metric. Unfortunately, $\hat{\delta}$ does not satisfy the triangle inequality (in general). However, if E is a Hilbert space, then $\hat{\delta}$ is a metric; in fact, it is a metric that is nicer to work with than the metric we are about to define since it satisfies

$$\hat{\delta}(A,B) = \|\mathrm{pr}_A - \mathrm{pr}_B\|_{L(H)}$$

where pr_A and pr_B are the orthogonal projections onto A and B respectively (cf. footnote 1 p.198 in [**44**]). In order to obtain a metric in the general case, we will use the following definition.

Definition B.3. *Let E be a Banach space and $A,B \subseteq E$ (non-empty) closed linear subspaces. Then, we define*

$$d(A,B) := \begin{cases} 0 & , \ A = \{0\} \\ 2 & , \ A \neq \{0\} \ \wedge \ B = \{0\} \\ \sup\{\mathrm{dist}_E(u, B \cap \partial B_E);\ u \in A \cap \partial B_E\} & , \ A \neq \{0\} \ \wedge \ B \neq \{0\} \end{cases}$$

and

$$\hat{d}(A,B) := \max\{d(A,B), d(B,A)\}.$$

Theorem B.4. *Let E be a Banach space, $A,B,C \subseteq E$ (non-empty) closed linear subspaces, and $A^\perp$, $B^\perp$, $C^\perp$ their annihilators, i.e.*

$$A^\perp := \{\varphi \in E';\ \forall a \in A:\ \varphi(a) = 0\}.$$

Then, the following are true.

(i) $d(A,B) = 0 \ \Leftrightarrow \ A \subseteq B$

(ii) $\hat{d}(A,B) = 0 \ \Leftrightarrow \ A = B$

(iii) $\hat{d}(A,B) = \hat{d}(B,A)$

(iv) $d(A,B) \in [0,2]$

(v) $\hat{d}(A,B) \in [0,2]$

(vi) $d(A,C) \le d(A,B) + d(B,C)$

(vii) $\hat{d}(A,C) \le \hat{d}(A,B) + \hat{d}(B,C)$

(viii) $\delta(A,B) \le d(A,B) \le 2\delta(A,B)$

(ix) $\hat{\delta}(A,B) \le \hat{d}(A,B) \le 2\hat{\delta}(A,B)$

(x) $\delta(A,B) < 1 \;\Rightarrow\; \dim A \le \dim B$

(xi) $\hat{\delta}(A,B) < 1 \;\Rightarrow\; \dim A = \dim B$

(xii) $\delta(A,B) = \delta(B^\perp, A^\perp)$

(xiii) $\hat{\delta}(A,B) = \hat{\delta}(A^\perp, B^\perp)$

Proof. (i-v) are trivial.

"(vi)" If $\{0\} \in \{A,B,C\}$, then the assertion is trivial. Hence, let $\{0\} \notin \{A,B,C\}$. Then,

$$\begin{aligned}
\forall v \in B \cap \partial B_E:\; d(A,C) &= \sup\{\mathrm{dist}_E(u, C \cap \partial B_E);\; u \in A \cap \partial B_E\} \\
&= \sup_{u \in A \cap \partial B_E} \inf_{w \in C \cap \partial B_E} \mathrm{dist}_E(u,w) \\
&\le \sup_{u \in A \cap \partial B_E} \inf_{w \in C \cap \partial B_E} (\mathrm{dist}_E(u,v) + \mathrm{dist}_E(v,w)) \\
&= \sup_{u \in A \cap \partial B_E} \mathrm{dist}_E(u,v) + \inf_{w \in C \cap \partial B_E} \mathrm{dist}_E(v,w) \\
&\le \sup_{u \in A \cap \partial B_E} \mathrm{dist}_E(u,v) + \sup_{v' \in B \cap \partial B_E} \inf_{w \in C \cap \partial B_E} \mathrm{dist}_E(v',w) \\
&= \sup_{u \in A \cap \partial B_E} \mathrm{dist}_E(u,v) + d(B,C)
\end{aligned}$$

implies

$$d(A,C) \le \sup_{u \in A \cap \partial B_E} \inf_{v \in B \cap \partial B_E} \mathrm{dist}_E(u,v) + d(B,C) = d(A,B) + d(B,C).$$

"(vii)" Using $\forall x,y \in \mathbb{R}:\; |x| - |y| \le |x+y|$ yields

$$\hat{d}(A,C) = \max\{d(A,C), d(C,A)\}$$

$$\begin{aligned}&\leq \max\{d(A,B)+d(B,C), d(C,B)+d(B,A)\}\\&\leq \max\{d(A,B), d(B,A)\}+\max\{d(B,C), d(C,B)\}\\&=\hat{d}(A,B)+\hat{d}(B,C).\end{aligned}$$

"(viii)" $\delta(A,B) \leq d(A,B)$ is trivial. For $d(A,B) \leq 2\delta(A,B)$ it suffices to assume that $B \neq \{0\}$. Let $u \in A \cap \partial B_E$ and $\varepsilon \in \mathbb{R}_{>0}$. Then, there exists $v \in B \setminus \{0\}$ such that $\operatorname{dist}_E(u,v) \leq \operatorname{dist}_E(u,B)+\varepsilon$. Then, we obtain

$$\begin{aligned}\operatorname{dist}_E(u, B\cap \partial B_E) &\leq \left\| u - \frac{v}{\|v\|_E}\right\|_E\\&\leq \|u-v\|_E + \left\| v - \frac{v}{\|v\|_E}\right\|_E\\&= \|u-v\|_E + |\|v\|_E - 1| \left\| \frac{v}{\|v\|_E}\right\|_E\\&= \|u-v\|_E + |\|v\|_E - \|u\|_E|\\&\leq \|u-v\|_E + \|v-u\|_E\\&< 2\operatorname{dist}_E(u,B)+2\varepsilon.\end{aligned}$$

Since ε was arbitrary, we obtain $d(A,B) \leq 2\delta(A,B)$.

"(ix)"

$$\begin{aligned}\hat{\delta}(A,B) = \max\{\delta(A,B), \delta(B,A)\} &\leq \max\{d(A,B), d(B,A)\} = \hat{d}(A,B)\\&\leq \max\{2\delta(A,B), 2\delta(B,A)\} = 2\hat{\delta}(A,B).\end{aligned}$$

"(x-xi)" Corollary IV §2.6 in [**44**]

"(xii-xiii)" Theorem IV §2.9 in [**44**]

$\square$

Definition B.5. *Let E be a Banach space. Then, we call*

$$\mathrm{CLR}(E) := \{A \subseteq E;\ A \text{ is a (non-empty) closed linear space}\}$$

endowed with $\hat{d}$ the space of closed linear relation.

Convergence in CLR *is called gap-convergence, $\hat{\delta}$-convergence, or convergence in the generalized sense.*

Let F be another Banach space. Then, we will also write $\mathrm{CLR}(E, F) := \mathrm{CLR}(E \oplus F)$.

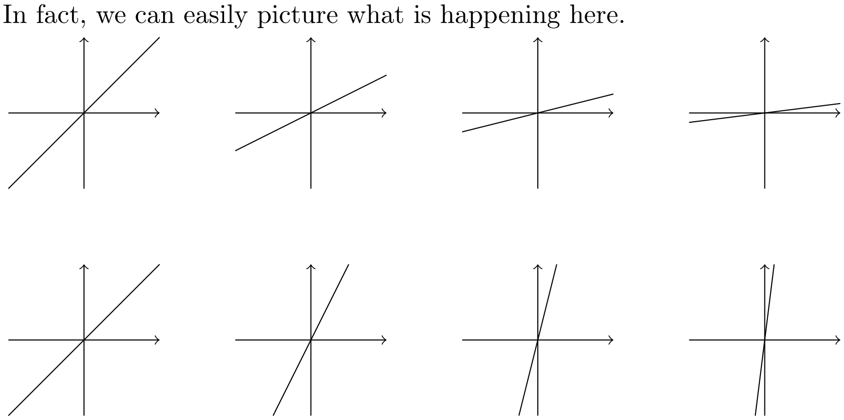
Furthermore, we will define the set of closed linear operators $\mathrm{CLO}(E, F)$ *as the set of all right-unique closed linear relations, i.e.*

$$\mathrm{CLO}(E, F) := \{A \in \mathrm{CLR}(E, F);\ \forall (x, y), (x, z) \in A:\ y = z\},$$

endowed with the topology induced by $\mathrm{CLR}(E, F)$.

Remark As remarked in Remark IV §2.1 in [**44**], it can be shown that $\mathrm{CLR}(E)$ is a complete metric space. However, for most applications, we are interested in $\mathrm{CLO}(E, F)$ which, in general, is not complete. To see that, we may choose $E = F$ and consider the sequence $(n\,\mathrm{id})_{n\in\mathbb{N}} \in L(E)^{\mathbb{N}}$. Then, we obtain $\forall x \in E:\ (n^{-1}x, x) \in n\,\mathrm{id}$, i.e. $\{0\} \times E \subseteq \lim_{n\to\infty} n\,\mathrm{id}$; but that is not a closed linear operator.

In fact, we can easily picture what is happening here.



Just as the sequence of linear operators $(x \mapsto n^{-1}x)_{n\in\mathbb{N}}$ converges to zero in $\hat{d}$, the sequence $(x \mapsto nx)_{n\in\mathbb{N}}$ converges to the relation $\{0\} \times E$ because everything is completely symmetrical.

■

Let us now state a few important theorems regarding $\mathrm{CLO}(E, F)$.

THEOREM B.6. *Let $T \in L(E, F)$ and $S \in \mathrm{CLO}(E, F)$ such that $\hat{\delta}(S, T) \le \sqrt{1 + \|T\|^2_{L(E,F)}}$. Then, $S \in L(E, F)$ and*

$$\|S - T\|_{L(E,F)} \le \frac{\left(1 + \|T\|^2_{L(E,F)}\right)\delta(S, T)}{1 - \sqrt{1 + \|T\|^2_{L(E,F)}}\delta(S, T)}.$$

PROOF. Theorem IV §2.13 in [**44**].

□

THEOREM B.7. *Let $T \in \mathrm{CLO}(E, F)$ and A T-bounded with relative bound less than 1, i.e. $[F]T \subseteq [F]A$ and*

$$\forall x \in [F]T : \; \|Ax\|_F \le a\,\|x\|_E + b\,\|Tx\|_F$$

with $b < 1$. Then, $S := T + A \in \mathrm{CLO}(E, F)$ with

$$\hat{\delta}(S, T) \le (1 - b)^{-1}\sqrt{a^2 + b^2}.$$

In particular, if $A \in L(E, F)$, then

$$\hat{\delta}(S, T) \le \|A\|_{L(E,F)}.$$

PROOF. Theorem IV §2.14 in [**44**].

□

THEOREM B.8. *Let $S, T \in \mathrm{CLO}(E, F)$ and $A \in L(E, F)$. Then,*

$$\hat{\delta}(S + A, T + A) \le 2\left(1 + \|A\|^2_{L(E,F)}\right)\hat{\delta}(S, T).$$

Proof. Theorem IV §2.17 in [**44**].

□

Theorem B.9. *Let* $S,T \in \mathrm{CLO}(E,F)$ *be densely defined. Then,*

$$\delta(S,T) = \delta(T^*, S^*)$$

and

$$\hat{\delta}(S,T) = \hat{\delta}(T^*, S^*).$$

Proof. Theorem IV §2.18 in [**44**].

□

Theorem B.10. *Let* $T \in \mathrm{CLO}(E,F)$. *Then, the following are true.*

(i) T *is bounded in the sense* $\exists c \in \mathbb{R}_{>0}\ \forall x \in [F]T:\ \|Tx\|_F \le c\,\|x\|_E$ *if and only if* $\delta(T,0) < 1$.

(ii) $T \in L(E,F) \iff \hat{\delta}(T,0) < 1$.

Proof. Problem IV §2.19 in [**44**].

□

Theorem B.11. *Let* $S,T \in \mathrm{CLO}(E,F)$ *be invertible. Then,*

$$\delta(S,T) = \delta(S^{-1}, T^{-1})$$

and

$$\hat{\delta}(S,T) = \hat{\delta}(S^{-1}, T^{-1}).$$

Proof. Problem IV §2.20 in [**44**].

□

THEOREM B.12. *Let $S, T \in \mathrm{CLO}(E,F)$, T boundedly invertible, and*

$$\hat{\delta}(S,T) < \sqrt{1 + \|T^{-1}\|^2_{L(E,F)}}.$$

Then, S is boundedly invertible.

PROOF. Theorem IV §2.21 in [**44**].

□

THEOREM B.13. *Let $T \in \mathrm{CLO}(E,F)$ and $(T_n)_{n\in\mathbb{N}} \in \mathrm{CLO}(E,F)^{\mathbb{N}}$.*

(i) *Let $T \in L(E,F)$. Then, $T_n \to T$ in the generalized sense if and only if $\exists N \in \mathbb{N}\ \forall n \in \mathbb{N}_{\geq N}:\ T_n \in L(E,F)$ and $\|T_n - T\|_{L(E,F)} \to 0$.*

(ii) *Let T be invertible with $T^{-1} \in L(F,E)$. Then, $T_n \to T$ in the generalized sense if and only if $\exists N \in \mathbb{N}\ \forall n \in \mathbb{N}_{\geq N}:\ T_n$ is invertible with $T_n^{-1} \in L(F,E)$ and $\left\|T_n^{-1} - T^{-1}\right\|_{L(F,E)} \to 0$.*

(iii) *Let $T_n \to T$ in the generalized sense and $A \in L(E,F)$. Then, $T_n + A \to T + A$ in the generalized sense.*

(iv) *Let the T_n and T be densely defined. Then, $T_n \to T$ in the generalized sense if and only if $T_n^* \to T^*$ in the generalized sense.*

PROOF. Theorem IV §2.23 in [**44**].

□

THEOREM B.14. *Let $T \in \mathrm{CLO}(E,F)$ and $(A_n)_{n\in\mathbb{N}} \in \mathrm{CLO}(E,F)^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}:\ [F]T \subseteq [F]A_n$ and*

$$\forall n \in \mathbb{N}\ \exists a_n, b_n \in \mathbb{R}_{>0}\ \forall x \in [F]T:\ \|A_n x\|_F \leq a_n \|x\|_E + b_n \|Tx\|_F.$$

If $a_n \to 0$ and $b_n \to 0$, then $\exists N \in \mathbb{N}\ n \in \mathbb{N}_{\geq N}:\ T + A_n \in \mathrm{CLO}(E,F)$ and $T + A_n \to T$ in the generalized sense.

Proof. Theorem IV §2.24 in [**44**].

□

Theorem B.15. *Let* $T \in \mathrm{CLO}(E,E)$ *and* $(T_n)_{n\in\mathbb{N}} \in \mathrm{CLO}(E,E)^{\mathbb{N}}$ *such that* $T_n \to T$ *in the generalized sense. If all* T_n *have compact resolvent and* T *has non-empty resolvent set, then* T *has compact resolvent.*

Proof. Theorem IV §2.26 in [**44**].

□

Theorem B.16. *Let* G *be another Banach space,* $(T_n)_{n\in\mathbb{N}_0} \in \mathrm{CLO}(E,F)^{\mathbb{N}_0}$*,* $(U_n)_{n\in\mathbb{N}_0} \in L(G,E)^{\mathbb{N}_0}$*, and* $(V_n)_{n\in\mathbb{N}_0} \in L(G,F)^{\mathbb{N}_0}$*, such that* $\forall n \in \mathbb{N}_0 :\ U_n|_G^{[F]T_n}$ *is a bijection,* $\forall n \in \mathbb{N}_0 :\ T_n U_n = V_n$*,* $\|U_n - U_0\|_{L(G,E)} \to 0$*, and* $\|V_n - V_0\|_{L(G,F)} \to 0$*. Then,* $T_n \to T_0$ *in the generalized sense.*

Proof. Theorem IV §2.29 in [**44**].

□

Having stated the most important properties of the gap topology and generalized convergence, we will now continue with the more important features regarding this thesis. In chapter 11, the following result is very important (cf. Lemma 6.1.1 in [**76**]).

Lemma B.17. *Let* H *be Hilbert space, and* $A, B \subseteq H$ *non-empty closed linear subspaces with* $\hat{\delta}(A,B) < \frac{1}{3}$*. Then,*

$$\mathrm{pr}_A|_B^A :\ B \to A$$

is an isomorphism. Furthermore,

$$B_{\hat{\delta}}\left(A, \frac{1}{3}\right) = \left\{C \in \mathrm{CLR}(H,H);\ \hat{\delta}(A,C) < \frac{1}{3}\right\} \ni C \mapsto \left(\mathrm{pr}_A|_C^A\right)^{-1} \in L(A,H)$$

is continuous in $\hat{\delta}$ *and norm.*

PROOF. From

$$
\begin{aligned}
1-(\mathrm{pr}_B-\mathrm{pr}_A)(\mathrm{pr}_B-\mathrm{pr}_{B^\perp}) &= 1-\mathrm{pr}_B+\mathrm{pr}_B\,\mathrm{pr}_{B^\perp}+\mathrm{pr}_A\,\mathrm{pr}_B-\mathrm{pr}_A\,\mathrm{pr}_{B^\perp}\\
&= 1-\mathrm{pr}_B+\mathrm{pr}_A\,\mathrm{pr}_B-\mathrm{pr}_A\,\mathrm{pr}_{B^\perp}\\
&= \mathrm{pr}_{B^\perp}+\mathrm{pr}_A\,\mathrm{pr}_B-\mathrm{pr}_A\,\mathrm{pr}_{B^\perp}\\
&= \mathrm{pr}_{A^\perp}\,\mathrm{pr}_{B^\perp}+\mathrm{pr}_A\,\mathrm{pr}_B,
\end{aligned}
$$

$$
\|\mathrm{pr}_B-\mathrm{pr}_{B^\perp}\|_{L(H)} = \|\mathrm{pr}_B-(1-\mathrm{pr}_B)\|_{L(H)} \le 2\,\|\mathrm{pr}_B\|_{L(H)}+1 = 3,
$$

and

$$
\|\mathrm{pr}_B-\mathrm{pr}_A\|_{L(H)} = \hat{\delta}(A,B) < \frac{1}{3}
$$

we obtain (using the Neumann series) that

$$
P(A,B) \coloneqq \mathrm{pr}_{A^\perp}\,\mathrm{pr}_{B^\perp}+\mathrm{pr}_A\,\mathrm{pr}_B :\ H \to H
$$

is an isomorphism. Furthermore, $H = A + A^\perp = B + B^\perp$, $P(A,B)[B] \subseteq A$, and $P(A,B)[B^\perp] \subseteq A^\perp$ show $P(A,B)[B] = A$ and $P(A,B)[B^\perp] = A^\perp$ because $P(A,B)$ is surjective. Hence,

$$
\mathrm{pr}_A|_B^A = P(A,B)|_B^A :\ B \to A
$$

and

$$
P(A,B)|_{B^\perp}^{A^\perp} :\ B^\perp \to A^\perp
$$

are isomorphisms, as well.

Since $\hat{\delta}$-continuity and norm-continuity are equivalent, we can use both notions interchangeably when showing continuity of

$$B_{\hat{\delta}}\left(A,\frac{1}{3}\right)=\left\{C\in\mathrm{CLR}(H,H);\ \hat{\delta}(A,C)<\frac{1}{3}\right\}\ni C\mapsto\left(\mathrm{pr}_A|_C^A\right)^{-1}\in L(A,H).$$

First, we note that $(H_0\oplus H_1)^2\ni(x,y)\mapsto(y,x)\in(H_1\oplus H_0)^2$ is an isometry for any two Hilbert spaces H_0 and H_1, i.e. $GL(H_0,H_1)\ni T\mapsto T^{-1}\in GL(H_1,H_0)$ is $\hat{\delta}$-continuous. By assumption of $\hat{\delta}$-continuity, we have norm-continuity of $B_{\hat{\delta}}\left(A,\frac{1}{3}\right)\ni C\mapsto\mathrm{pr}_C\in L(H)$. Furthermore, since $\hat{\delta}(A,B)=\hat{\delta}(A^\perp,B^\perp)$ (Theorem B.4 (xiii)), this implies norm-continuity of $B_{\hat{\delta}}\left(A,\frac{1}{3}\right)\ni C\mapsto\mathrm{pr}_{C^\perp}\in L(H)$. Hence, $B_{\hat{\delta}}\left(A,\frac{1}{3}\right)\ni C\mapsto P(A,C)\in L(H)$ is continuous, as well, and by continuity of the inversion

$$B_{\hat{\delta}}\left(A,\frac{1}{3}\right)\ni C\mapsto P(A,C)^{-1}\in L(H)$$

is continuous. Finally, for $C,D\in B_{\hat{\delta}}\left(A,\frac{1}{3}\right)$,

$$\begin{aligned}\left\|\left(\mathrm{pr}_A|_C^A\right)^{-1}-\left(\mathrm{pr}_A|_D^A\right)^{-1}\right\|_{L(A,H)}&=\left\|P(A,C)^{-1}|_A-P(A,D)^{-1}|_A\right\|_{L(A,H)}\\&\le\left\|P(A,C)^{-1}-P(A,D)^{-1}\right\|_{L(H)}\\&\to0\quad(D\to C)\end{aligned}$$

shows continuity of

$$B_{\hat{\delta}}\left(A,\frac{1}{3}\right)=\left\{C\in\mathrm{CLR}(H,H);\ \hat{\delta}(A,C)<\frac{1}{3}\right\}\ni C\mapsto\left(\mathrm{pr}_A|_C^A\right)^{-1}\in L(A,H).$$

in norm and, thus, in $\hat{\delta}$.

□

We will also need the closely related Lemma B.19 which needs the following lemma in preparation.

LEMMA B.18. *Let X, Y be Banach spaces, $A \in L(X,Y)$, and $B \in L(Y,X)$ such that AB is boundedly invertible in $L(X)$ and BA is boundedly invertible in $L(Y)$. Then, A and B are isomorphisms.*

PROOF. Note, A has the right-inverse $B(AB)^{-1}$ and the left-inverse $(BA)^{-1}B$. Similarly, B has the right-inverse $A(BA)^{-1}$ and the left-inverse $(AB)^{-1}A$. Since the existence of a left-inverse implies injectivity and the existence of a right-inverse implies surjectivity, both A and B are bijective, i.e. the bounded inverse theorem yields the assertion.

□

LEMMA B.19. *Let $P, Q \in L(E)$ be projections with $\|(P-Q)^2\|_{L(E)} < 1$. Then, $P\colon\ Q[E] \to P[E]$ and $Q\colon\ P[E] \to Q[E]$ are isomorphisms.*

PROOF. Let $S := (P-Q)^2 = P + Q - PQ - QP$. Then,

$$SP = P - PQP = PS$$

and

$$SQ = Q - QPQ = QS$$

hold. Hence, $P[E]$ and $Q[E]$ are invariant under S. Since $1 - S$ is boundedly invertible (Neumann series),

$$\begin{aligned}(1-S)|_{Q[E]} &= (1 - P - Q + PQ + QP)|_{Q[E]} \\ &= (Q - PQ - Q + PQ + QPQ)|_{Q[E]} \\ &= QP|_{Q[E]}\end{aligned}$$

shows that QP is boundedly invertible on $Q[E]$ and

$$(1-S)|_{P[E]} = (1 - P - Q + PQ + QP)|_{P[E]}$$

$$=(P - P - QP + PQP + QP)|_{P[E]}$$

$$=PQ|_{P[E]}$$

shows that PQ is boundedly invertible on $P[E]$. Hence, the assertion follows from Lemma B.18 with $A = P$, $B = Q$, $X = Q[E]$, and $Y = P[E]$.

□

The other main application of the gap topology appears in chapter 6. There, we are particularly interested in the perturbation of eigenvalues with respect to the gap topology.

THEOREM B.20. *Let* $T \in \mathrm{CLO}(E, E)$ *and* K *a compact subset of the resolvent set* $\varrho(T)$. *Then,* $\exists \delta \in \mathbb{R}_{>0}\ \forall S \in B_{\hat{\delta}}(T, \delta) : K \subseteq \varrho(S)$.

PROOF. Theorem IV §3.1 in [**44**].

□

THEOREM B.21. *Let* $T \in \mathrm{CLO}(E, E)$ *such that the spectrum* $\sigma(T)$ *is separated into* σ_1 *and* σ_2 *by a rectifiable simple cycle*[1] γ. *Then, there are subspaces* $E_1, E_2 \subseteq E$ *such that* $E = E_1 + E_2$, $E_1 \cap E_2 = \{0\}$, $E_1 \oplus E_2 \ni (x_1, x_2) \mapsto x_1 + x_2 \in E$ *is a homeomorphism, and* $TP \supseteq PT$ *where* P *is the projection onto* E_1 *along* E_2, *that is,* $P[E] = E_1$ *and* $(1 - P)[E] = E_2$; *more precisely, since every* $x \in E$ *is uniquely decomposed as* $x_1 + x_2$ *with* $x_i \in E_i$, *we have* $Px = x_1$ *and* $(1 - P)x = x_2$. *Furthermore, there are operators* $T_i : E_i \to E_i$ *with* $[E_i]T_i = [E]T \cap E_i$, $T_i = T|_{E_i}$,[2] *and* $\sigma(T_i) = \sigma_i$. *If* σ_1 *is bounded (that is,* σ_1 *is the part of the spectrum with winding number* 1 *with respect to* γ*), then* $T_1 \in L(E_1)$.

[1] A cycle is a finite collection of closed curves with disjoint images. It is called simple if and only if every point that is not in the image of any of the curves has winding number in $\{0, 1\}$.

[2] In this case, we call (E_1, E_2) reducing for T.

Furthermore, there exists $\delta \in \mathbb{R}_{>0}$ *such that the following properties hold.* γ *separates the spectrum of any* $S \in \mathrm{CLO}(E,E)$ *with* $\hat{\delta}(S,T) < \delta$*. Let* $E = F_1 \oplus F_2$ *be the corresponding decomposition for* S*,* S_1 *and* S_2 *the corresponding operators, and* P_S *the projection onto* F_1 *along* F_2*. Then,* F_1 *and* F_2 *are isomorphic to* E_1 *and* E_2*, respectively. In particular,* $\dim E_1 = \dim F_1$ *and* $\dim E_2 = \dim F_2$*. Furthermore,* $\sigma(S_1)$ *and* $\sigma(S_2)$ *are non-empty if this is true for* $\sigma(T_1)$ *and* $\sigma(T_2)$*, and the decomposition* $E = F_1 \oplus F_2$ *is continuous with respect to* S*, that is,* $\|P_S - P\|_{L(E)} \to 0$ $(\hat{\delta}(S,T) \to 0)$.

PROOF. Theorem IV §3.16

□

REMARK Using the holomorphic functional calculus, we obtain

$$P = \frac{1}{2\pi i} \int_{\gamma} (\lambda - T)^{-1} d\lambda$$

and

$$T_1 = \frac{1}{2\pi i} \int_{\gamma} \lambda(\lambda - T)^{-1} d\lambda.$$

■

This last theorem is very interesting if we assume that σ_1 is a finite set of eigenvalues. Then, $\dim E_1 = \sum_{\lambda \in \sigma_1} \mu_\lambda < \infty$ where μ_λ is the multiplicity of $\lambda \in \sigma(T)$. In particular, if $T_n \to T$ in the generalized sense, then each T_n has (eventually) a separated spectrum and $\dim E_1(T_n) = \dim E_1$, i.e. $(T_n)_1$ is a matrix and $\sigma((T_n)_1)$ contains only eigenvalues whose multiplicities add up to the total multiplicity of eigenvalues of T in σ_1. Choosing a sequence of cycles $(\gamma_n)_{n \in \mathbb{N}}$ such that the images converge to σ_1, i.e. the encirclement of σ_1 is getting tighter and tighter, we obtain that systems of finitely many eigenvalues behave continuously under small perturbations in $\hat{\delta}$; very similar to the behavior of perturbations of eigenvalues of matrices.

For infinitely many eigenvalues, however, there is no uniform bound on the perturbation. Consider an operator T with unbounded and purely discrete spectrum. Given $\delta \in \mathbb{R}_{>0}$ it is possible to find $\varepsilon \in \mathbb{R}_{>0}$ such that $\hat{\delta}((1+\varepsilon)T, T) < \delta$. However, the eigenvalue λ of T is perturbed by $\varepsilon\lambda$ in $(1+\varepsilon)T$. Since $\sigma(T)$ is unbounded, there is no uniform bound on the perturbation of infinite systems of eigenvalues.

For finitely many eigenvalues, on the other hand, we do know quite a lot about their perturbations; in particular, if we consider holomorphic perturbations. We will end this appendix with a theorem (Theorem B.24) on holomorphic perturbations which is very interesting for the spectral mollification discussed in chapter 6.

Definition B.22. *Let $\Omega \subseteq \mathbb{C}$ be open and $T \in \mathrm{CLO}(E,E)^\Omega$. T is called resolvent-holomorphic if and only if for every $z_0 \in \Omega$ there are $\lambda_0 \in \varrho(T(z_0))$ and an open neighborhood U of z_0 such that $\forall z \in U:\ \lambda_0 \in \varrho(T(z))$ and*

$$U \ni z \mapsto (\lambda_0 - T(z))^{-1} \in L(E)$$

is holomorphic.

Lemma B.23. *Let $\Omega \subseteq \mathbb{C}$ be open and $T \in L(E)^\Omega$ holomorphic. Then, T is resolvent-holomorphic. More precisely, for every $z_0 \in \Omega$ and $\lambda \in \varrho(T(z_0))$, there exists an open neighborhood U of z_0 such that $\forall z \in U:\ \lambda \in \varrho(T(z))$ and $U \ni z \mapsto (\lambda - T(z))^{-1} \in L(E)$ is holomorphic.*

Proof. Let $z_0 \in \Omega$ and $\lambda \in \varrho(T(z_0))$. Then, the Neumann series implies that

$$\lambda - T(z) = \Big(1 - (T(z) - T(z_0))\,(\lambda - T(z_0))^{-1}\Big)\,(\lambda - T(z_0))$$

is boundedly invertible for every

$$z \in U := \left\{ s \in \Omega;\ \|T(s) - T(z_0)\|_{L(E)} < \left\| (\lambda - T(z_0))^{-1} \right\|_{L(E)}^{-1} \right\}$$

and

$$(\lambda - T(z))^{-1} = (\lambda - T(z_0))^{-1} \sum_{j\in\mathbb{N}_0} \left((T(z) - T(z_0)) (\lambda - T(z_0))^{-1} \right)^j$$

converges uniformly on compact subsets of U since for every $K \subseteq_{\text{compact}} U$ there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\sup_{s\in K} \|T(s) - T(z_0)\|_{L(E)} \left\| (\lambda - T(z_0))^{-1} \right\|_{L(E)} < 1 - \varepsilon,$$

i.e.

$$\begin{aligned}
&\sup_{z\in K} \left\| (\lambda - T(z))^{-1} \right\|_{L(E)} \\
&\leq \sup_{z\in K} \left\| (\lambda - T(z_0))^{-1} \right\|_{L(E)} \sum_{j\in\mathbb{N}_0} \|(T(z) - T(z_0))\|_{L(E)}^j \left\| (\lambda - T(z_0))^{-1} \right\|_{L(E)}^j \\
&\leq \left\| (\lambda - T(z_0))^{-1} \right\|_{L(E)} \sum_{j\in\mathbb{N}_0} (1-\varepsilon)^j \\
&= \frac{\left\| (\lambda - T(z_0))^{-1} \right\|_{L(E)}}{\varepsilon}.
\end{aligned}$$

Hence, $z \mapsto (\lambda - T(z))^{-1}$ is holomorphic.

□

THEOREM B.24. *Let $\Omega \subseteq_{\text{open}} \mathbb{C}$, $T \in \mathrm{CLO}(E,E)^\Omega$ resolvent-holomorphic, $z_0 \in \Omega$, $\lambda_0 \in \sigma_d(T(z_0))$ where $\sigma_d(T(z_0))$ is the discrete spectrum, i.e. the set of eigenvalues with finite multiplicity, and m the algebraic multiplicity of λ_0.*

(i) *Then, there exist $\delta, \varepsilon \in \mathbb{R}_{>0}$ such that $\sigma(T(z)) \cap B(\lambda_0, \varepsilon) \subseteq \sigma_d(T(z))$ for every $z \in B(z_0, \delta)$ and the total multiplicity of eigenvalues of $T(z)$ in $B(\lambda_0, \varepsilon)$ is m. Furthermore, for the projections $P(z)$ corresponding to $T(z)$ and $\sigma(T(z)) \cap B(\lambda_0, \varepsilon)$, we obtain that $z \mapsto P(z)$ and $z \mapsto T(z)P(z)$ are holomorphic.*

(ii) *There exist* $\delta, \varepsilon \in \mathbb{R}_{>0}$ *such that we can write the eigenvalues of* $T(z)$ *in* $B(\lambda_0, \varepsilon)$ *as a Puiseux series*[3] *for* $z \in B(z_0, \delta)$. *If* $m = 1$, *then there exists a "holomorphic eigenvector".*

PROOF. "(i)" Since T is resolvent-holomorphic, we obtain $T(z) \to T(z_0)$ in the generalized sense by Theorem B.13 (ii). Hence, the assertion follows from Theorem B.21 and the fact that

$$z \mapsto P(z) = \frac{1}{2\pi i} \int_\gamma (\lambda - T(z))^{-1} d\lambda$$

and

$$z \mapsto T(z)P(z) = \frac{1}{2\pi i} \int_\gamma \lambda(\lambda - T(z))^{-1} d\lambda$$

are holomorphic (where γ is a suitable cycle).

"(ii)" Let $m = 1$ and x_0 an eigenvector of $T(z_0)$ corresponding to λ_0. Then, $x(z) := P(z)x_0$ is holomorphic and has no zero in a sufficiently small neighborhood of z_0. Furthermore, $x(z)$ is an eigenvector of $T(z)$ since $P(z)$ maps into the eigenspace.

For $m \geq 1$ and $\dim E < \infty$, i.e. $E = \mathbb{C}^n$ for some n, we obtain the eigenvalues of $T(z)$ from the roots of the Weierstrass polynomial $\det(\lambda - T(z))$. Hence, Theorem C.25 yields the assertion.

For $m \geq 1$ and $\dim E = \infty$, choose δ and ε as in (i). For $z \in B_{\mathbb{C}}(z_0, \delta)$, let $E_1(z) := P(z)[E]$ and $E_2(z) := (1 - P(z))[E]$. Without loss of generality, let δ and ε be sufficiently small such that $\|P(z) - P(z_0)\|_{L(E)} < 1$. Then, $P(z):\ E_1(z_0) \to E_1(z)$ and $1 - P(z):\ E_2(z_0) \to E_2(z)$ are isomorphisms by Lemma B.19. Hence,

[3]A Puiseux series is a "fractional power series", i.e. an expression of the form $\sum_{j \in \mathbb{N}_0} a_j (z - z_0)^{\frac{j}{n}}$ for some $n \in \mathbb{N}$.

each

$$U(z) := P(z)P(z_0) + (1 - P(z))(1 - P(z_0))$$

is an isomorphism of E and

$$U(z)P(z_0) = P(z)P(z_0) = P(z)U(z).$$

Let $S(z) := U(z)^{-1}T(z)U(z)$. Then,

$$P(z_0)U(z)^{-1} = U(z)^{-1}P(z)$$

implies

$$\begin{aligned} S(z)P(z_0) &= U(z)^{-1}T(z)U(z)P(z_0) \\ &= U(z)^{-1}T(z)P(z)U(z) \\ &\supseteq U(z)^{-1}P(z)T(z)U(z) \\ &= P(z_0)U(z)^{-1}T(z)U(z) \\ &= P(z_0)S(z), \end{aligned}$$

i.e. $(E_1(z_0), E_2(z_0))$ is reducing for S. Furthermore,

$$S(z)P(z_0) = U(z)^{-1}T(z)P(z)U(z)$$

shows that

$$B(z_0, \delta) \ni z \mapsto S(z)|_{E_1(z_0)} \in E_1(z_0)$$

is holomorphic. Hence, we obtain the assertion for the eigenvalues of $S|_{E_1(z_0)}^{E_1(z_0)}$ since $\dim E_1(z_0) < \infty$. However, the eigenvalues of $S(z)|_{E_1(z_0)}^{E_1(z_0)}$ and the eigenvalues of $T(z)$ in $B_{\mathbb{C}}(\lambda_0, \varepsilon)$ coincide by definition of S.

□

APPENDIX C

Puiseux series

In order to prove part (ii) of Theorem B.24, we need the notion of Puiseux series. In this appendix, we will, therefore, introduce all the necessary tools to prove Theorem B.24 (ii). The results in this appendix (just like Theorem B.24) have been introduced to me by Prof. Jürgen Voigt during a lecture series on operator theory in the fall term of 2011 at the Technische Universität Dresden.

DEFINITION C.1. *Let R be a commutative ring with 1 without zero divisors, i.e.* $\forall a, b \in R:\ (\ ab = 0\ \Rightarrow\ a = 0\ \vee\ b = 0\ )$. *Then, we call R an integral domain.*

Furthermore, we will define the following for $a, b \in R$.

(i) $a|b$ *(a divides b) if and only if* $\exists c \in R:\ ac = b$.

(ii) *a is called a unit if and only if $a|1$.*

(iii) $a \sim b$ *(a and b are associated) if and only if there exists a unit u such that* $a = ub$.

(iv) *a is called reducible if and only if $a \neq 0$ and there are non-units $b, c \in R$ such that $a = bc$.*

(v) *a is called irreducible if and only if $a \neq 0$, a is not a unit, and a is not reducible.*

(vi) *a is called a prime element if and only if* $\forall b, c \in R:\ (\ a|bc\ \Rightarrow\ a|b\ \vee\ a|c\ )$.

(vii) *$J \subseteq R$ is called co-prime if and only if there is no non-unit a such that* $\forall b \in J:\ a|b$.

DEFINITION C.2. *Let R be an integral domain. Then, we call R a unique factorization domain (UFD) if and only if every non-unit $a \neq 0$ there exist $n \in \mathbb{N}$*

and irreducible $c_1, \ldots, c_n \in R$ *such that the factorization* $a = \prod_{j=1}^{n} c_j$ *exists uniquely; that is, if* $a = \prod_{j=1}^{m} d_j$ *is another such factorization, then* $m = n$ *and there exists a permutation* $\sigma \in S_n$ *such that* $\forall j \in \mathbb{N}_{\leq n}: \ c_j \sim d_{\sigma(j)}$.

LEMMA C.3. *Let* R *be an integral domain. Then, every prime is irreducible. If* R *is furthermore a unique factorization domain, then every irreducible element is prime.*

PROOF. Let p be prime and $a, b \in R$ such that $p = ab$. Then, $p|ab$ implies $p|a \ \vee p|b$. But we also have $a|p$ and $b|p$. Without loss of generality, let $p|b$. Hence, there are $u, v \in R$ such that $b = up$ and $p = vb$, i.e. $b = uvb$ and $p = vup$. Since $p \neq 0$, we obtain $vu = 1$, i.e. u and v are units and $p \sim b$. Thus, $p = ab = aup$ implies $au = 1$, i.e. a is a unit and p irreducible.

Let R be a unique factorization domain, p irreducible, and $ab \in R$ such that $p|ab$. Hence, there exists $c \in R$ such that $ab = pc$. Factorizing a, b, and c into irreducibles implies that there must be an irreducible factor of a and b which is an associate of p, i.e. $p|a$ or $p|b$. Hence, p is prime.

□

DEFINITION C.4. *Let* R *be a commutative ring. Then, we call* $R[\tau]$ *the ring of polynomials in* τ *over* R.

More precisely, $R[\tau]$ *is isomorphic to* $c_c(\mathbb{N}_0, R) \subseteq R^{\mathbb{N}_0}$ *(*$c_c(\mathbb{N}_0, R)$ *is the set of finite sequences with values in* R*) since*

$$R[\tau] \ni p = \sum_{j \in \mathbb{N}_0} p_j \tau^j \mapsto (p_j)_{j \in \mathbb{N}_0} \in c_c(\mathbb{N}_0, R)$$

is a bijection and we endow $c_c(\mathbb{N}_0, R)$ with the component-wise addition and the multiplication

$$(p_j)_{j\in\mathbb{N}_0}(q_j)_{j\in\mathbb{N}_0} := \left(\sum_{k=0}^{j} p_{j-k}q_k\right)_{j\in\mathbb{N}_0}.$$

For $p \in R[\tau]$ we define $\deg p := \sup\{n \in \mathbb{N}_0;\ p_n \neq 0\}$ where $\sup\varnothing := -\infty$.

Let R be an integral domain and $p \in R[\tau]$. Then, p is called primitive if and only if $\{p_j;\ j \in \mathbb{N}_{0,\le \deg p}\}$ is co-prime.

LEMMA C.5. *Let R be an integral domain. Then, $R[\tau]$ is an integral domain.*

PROOF. It is easy to see that $R[\tau]$ is a commutative ring with 1. Let $p, q \in R[\tau] \setminus \{0\}$. Then, $p = \sum_{j=0}^{m} p_j\tau^j$ and $q = \sum_{j=0}^{n} q_j\tau^j$ with $p_m \neq 0$ and $q_n \neq 0$. Hence, the coefficient of τ^{m+n} in pq is given by $p_m q_n$ which is non-zero since R is an integral domain. Hence, $pq \neq 0$ and $R[\tau]$ is an integral domain.

□

From now on, let R be an integral domain and F its field of fractions, i.e. $F := (R \times (R \setminus \{0\}))/\hat{\sim}$ with

$$(a,b)\hat{\sim}(c,d) \;:\Leftrightarrow\; ad = cb$$

is endowed with the addition $(a,b) + (c,d) := (ad + cb, bd)$ and the multiplication $(a,b)\cdot(c,d) := (ac, bd)$. In other words, we interpret $(a,b) \in F$ as $\frac{a}{b}$.

LEMMA C.6. *Let $p, q \in R[\tau]$ be primitive. Then, pq is primitive.*

PROOF. Let $p = \sum_{j=0}^{m} p_j\tau^j$ and $q = \sum_{j=0}^{n} q_j\tau^j$. Let a be a prime, $k := \min\{j \in \mathbb{N}_{0,\le m};\ a \nmid p_j\}$, and $k := \min\{l \in \mathbb{N}_{0,\le m};\ a \nmid q_j\}$. Then, $a \nmid p_k$ and $a \nmid q_l$, i.e. $a \nmid p_k q_l$. However, the coefficient of τ^{k+l} is given by

$$\underbrace{p_0}_{a|\cdot} q_{k+l} + \underbrace{p_1}_{a|\cdot} q_{k+l-1} + \ldots + \underbrace{p_{k-1}}_{a|\cdot} q_{l+1} + \underbrace{p_k q_l}_{a\nmid\cdot} + p_{k+1}\underbrace{q_{l-1}}_{a|\cdot} + \ldots + p_{k+l}\underbrace{q_0}_{a|\cdot}.$$

Hence, the coefficient of τ^{k+l} is not divisible by a, i.e. pq is primitive.

□

LEMMA C.7. (i) *Let $J \subseteq R$ be co-prime, $a \in F$, and $aJ \subseteq R$. Then, $a \in R$.*

In particular, $p \in R[\tau]$ primitive, $a \in F$, and $ap \in R[\tau]$ imply $a \in R$.

(ii) *Let $\{\varnothing, \{0\}\} \not\ni J \subseteq_{\text{finite}} F$. Then, there exists $a \in F$ such that $aJ \subseteq R$ is co-prime. If $b \in F$ such that $bJ \subseteq R$ is co-prime as well, then $\frac{a}{b}$ is a unit in R.*

(iii) *Let $p \in F[\tau] \setminus \{0\}$. Then, there exists $a \in F$ such that $ap \in R[\tau]$ is primitive. If b is another element of F such that $bp \in R[\tau]$ is primitive, then $\frac{a}{b}$ is a unit in R. In other words, ap is unique up to multiplication with units and we call ap the primitive polynomial associated with p.*

PROOF. "(i)" Let $b \in R$ such that $ab \in R$ and $c \in R$ prime with $c|b$. Then, there exists $d \in J$ such that $c \nmid d$. However, $c|\overbrace{b\underbrace{(ad)}_{\in R}}^{=(ba)d}$ implies $c|ba$ and $\frac{b}{c}a \in R$. Dividing all prime factors of b implies $a \in R$.

"(ii)" Let $J = \{d_1, \dots, d_n\}$ and consider the factorization $d_i = \frac{\prod_{j=1}^{m_i} e_{ij}}{\prod_{j=1}^{n_i} f_{ij}}$. Let $a_1 := \prod_{i=1}^{n} \prod_{j=1}^{n_i} f_{ij}$ and a_2 the product of all common prime factors of all $a_1 d_i$. Then, $a := \frac{a_1}{a_2}$ satisfies the assertion.

Let b be as stated. Then, $\frac{b}{a}aJ = bJ$ implies $\frac{b}{a} \in R$ by (i). Similarly, $\frac{a}{b} \in R$ holds and we obtain $\frac{a}{b}\frac{b}{a} = 1$.

"(iii)" Apply (ii) to $J = \{p_j;\ j \in \mathbb{N}_{0,\leq \deg p}\}$ where $p = \sum_{j=0}^{\deg p} p_j \tau^j$.

□

PROPOSITION C.8. (i) *Let $r \in R[\tau]$, $p, q \in F[\tau]$, $r = pq$, $a \in F$, $\tilde{p} \in R[\tau]$ primitive, and $p = a\tilde{p}$. Then, $aq \in R[\tau]$ and $r = (aq)\tilde{p}$ is a decomposition of r in $R[\tau]$. If r is primitive, then so is aq.*

(ii) *Let* $p \in R[\tau]$ *be non-constant and irreducible in* $R[\tau]$*. Then,* p *is irreducible in* $F[\tau]$*.*

A primitive polynomial in $R[\tau]$ *is irreducible in* $R[\tau]$ *if and only if it is irreducible in* $F[\tau]$*.*

(iii) *Let* $P \subseteq R[\tau]$*. Then,* P *is co-prime in* $F[\tau]$ *if and only if the elements of* P *have no common, non-constant divisor in* $R[\tau]$*.*

Proof. "(i)" There exists $b \in F$ and a primitive $\tilde{q} \in R[\tau]$ such that $q = b\tilde{q}$. Hence, $r = ab\tilde{p}\tilde{q}$ and $\tilde{p}\tilde{q}$ is primitive which implies (Lemma C.7 (i)) that $ab \in R$ and, thus, $aq = ab\tilde{q} \in R[\tau]$.

If r is primitive, then ab is a unit and $aq = ab\tilde{q}$ is primitive.

"(ii)" Suppose p were reducible in $F[\tau]$. Choose a decomposition of $p = qr$ in $F[\tau]$ and (i) yields a decomposition of p in $R[\tau]$. (Note that constant polynomials are always reducible in $F[\tau]$ since there are no non-units in F.)

Let p is primitive and $p = qr$ with $q, r \in R[\tau]$. Then, $p = qr$ is also a factorization in $F[\tau]$. Hence, one of them is a unit. Without loss of generality, let q be the unit in F, i.e. of degree zero. Then, we have $p = qr$ with $q \in R$ and $r \in R[\tau]$. But, since p is primitive, q has to be a unit in R, that is, p is irreducible.

"(iii)" "$\Rightarrow$" is trivial. For "$\Leftarrow$" suppose P were not co-prime in $F[\tau]$. Then, there exists $p \in F[\tau]$ with $\deg p \geq 1$ (all constants are units) such that $\forall r \in P\ \exists q_r \in F[\tau]:\ r = pq_r$. Furthermore, let $a \in F$ and $\tilde{p} \in R[\tau]$ primitive such that $p = a\tilde{p}$. Then, (i) implies that $aq_r \in R[\tau]$ and $r = (aq_r)\tilde{p}$, i.e. all r have the common and non-constant divisor $\tilde{p}$.

□

THEOREM C.9 (Euclidean Algorithm). (i) *Let* $p, q \in F[\tau]$ *such that* $q \neq 0$*. Then, there are* $r, s \in F[\tau]$ *such that* $\deg s < \deg q$ *and* $p = rq + s$*.*

(ii) *Let* $p, q \in F[\tau]$ *be co-prime. Then, there are* $r, s \in F[\tau]$ *such that* $rp+sq = 1$*.*

(iii) *Let* $p \in F[\tau]$ *be irreducible. Then,* p *is prime.*

PROOF. "(i)" Polynomial division.

"(ii)" Let $t \in F[\tau] \setminus \{0\}$ be any element of $J := \{\varphi p + \psi q;\ \varphi, \psi \in F[\tau]\}$ of minimal degree. Then, there are $r, s \in F[\tau]$ such that $p = rt + s$ and $\deg s < \deg t$. Then, $s = p - rt = (1 - r\varphi)p - r\psi q$ for some $\varphi, \psi \in F[\tau]$ shows $s \in J$. Hence, $s = 0$ and $t|p$. Similarly, $t|q$ and $\{p, q\}$ being co-prime implies that t is a unit. Hence, $1 = \frac{\varphi}{t}p + \frac{\psi}{t}q$ for some $\varphi, \psi \in F[\tau]$.

"(iii)" Let $q, r \in F[\tau]$, $p \mid qr$, and $p \nmid q$. Then, $\{p, q\}$ is co-prime since p is irreducible. According to (ii), there are $\varphi, \psi \in F[\tau]$ such that $\varphi p + \psi q = 1$. Hence, $r = \varphi \underbrace{pr}_{p|\cdot} + \psi \underbrace{qr}_{p|\cdot}$ implies that p is prime.

□

PROPOSITION C.10. $F[\tau]$ *is a unique factorization domain.*

PROOF. Let $p \in F[\tau] \setminus \{0\}$ not be a unit, i.e. not a constant. If p is reducible, then we can write $p = qr$ with $\max\{\deg q, \deg r\} < \deg p$. Inductively, we obtain $p = \prod_{j=1}^{n} q_j$ where each q_j is irreducible and $\deg q_j \geq 1$. Then, Theorem C.9 (iii) implies that the q_j are prime.

Let $p = \prod_{j=1}^{m} q_j = \prod_{j=1}^{n} r_j$ be two factorization into primes. Then, each q_j divides $\prod_{j=1}^{n} r_j$, i.e. $n \geq m$ and there exists $\alpha :\ \mathbb{N}_{\leq m} \to \mathbb{N}_{\leq n}$ injective such that $\forall j \in \mathbb{N}_{\leq m} :\ q_j \sim r_{\alpha(j)}$. Similarly, $m \geq n$ and there exists $\beta :\ \mathbb{N}_{\leq n} \to \mathbb{N}_{\leq m}$ injective such that $\forall j \in \mathbb{N}_{\leq n} :\ r_j \sim q_{\beta(j)}$. In other words, the factorization is unique.

□

THEOREM C.11 (Gauss). *$R[\tau]$ is a unique factorization domain.*

PROOF. Let $p \in R[\tau] \setminus \{0\}$ not be a unit. Then, we can write $p = aq$ with $a \in R$ and $q \in R[\tau]$ primitive. Note that at most one of a and q can be a unit and we can factorize a and q separately into irreducibles. Since R is a unique factorization domain, the factorization $a = \prod_{j=1}^{m} a_j$ is unique and each a_j is irreducible in $R[\tau]$. Furthermore, we can factorize $q = \prod_{j=1}^{n} q_j$ where each q_j is a non-constant irreducible and primitive since q is primitive.

Let $p = \prod_{j=1}^{m} a_j \prod_{k=1}^{n} q_k = \prod_{j=1}^{m'} a'_j \prod_{k=1}^{n'} q'_k$. Then, the a_j and a'_j are irreducibles in R and the q_j and q'_j are non-constant irreducibles in $R[\tau]$. In particular, the q_j and q'_j are primitive. By Lemma C.6, we obtain that $\prod_{k=1}^{n} q_k$ and $\prod_{k=1}^{n'} q'_k$ are primitive, as well, i.e. there exists a unit u such that $\prod_{k=1}^{n} q_k = u \prod_{k=1}^{n'} q'_k$. Replacing q'_1 by uq'_1 implies

$$\prod_{k=1}^{n} q_k = u \prod_{k=1}^{n'} q'_k \text{ and } \prod_{j=1}^{m} a_j = \prod_{j=1}^{m'} a'_j.$$

Since R is a unique factorization domain, we directly obtain that $m = m'$ and there exists a bijection $\alpha : \mathbb{N}_{\le m} \to \mathbb{N}_{\le m}$ such that $\forall j \in \mathbb{N}_{\le m} : a_j \sim a'_{\alpha(j)}$. Hence, it remains to pair off the q_j and q'_j. Since they are non-constant irreducibles in $R[\tau]$, they are also irreducible in $F[\tau]$ (Proposition C.8) which is a unique factorization domain by Proposition C.10. Hence, $n = n'$ and there is a bijection $\beta : \mathbb{N}_{\le n} \to \mathbb{N}_{\le n}$ as well as units $u_j \in F$ such that $\forall j \in \mathbb{N}_{\le n} : q_j = u_j q'_{\beta(j)}$. However, Lemma C.7 (i) implies that $\forall j \in \mathbb{N}_{\le n} : u_j \in R$ and since the q_j are primitive, the u_j are units in R.

□

COROLLARY C.12. *Let $z_0 \in \mathbb{C}$ and $\mathcal{H}(z_0)$ be the ring of holomorphic germs at z_0. Then, $\mathcal{H}(z_0)$ and $\mathcal{H}(z_0)[\tau]$ are unique factorization domains.*

PROOF. By Theorem C.11, it suffices to prove the assertion for $\mathcal{H}(z_0)$.

Let $f, g \in \mathcal{H}(z_0)$ and $U \subseteq \mathbb{C}$ an open neighborhood of z_0 such that both f and g are defined on U. Let $fg = 0$ and $f \neq 0$. Then, the $[\{0\}]f$ has no accumulation point in U. Since $\mathbb{C}$ is an integral domain, this implies that $[\{0\}]g$ has the accumulation point z_0, i.e. $g = 0$. Hence, $\mathcal{H}(z_0)$ is an integral domain.

Let $f \in \mathcal{H}(z_0) \setminus \{0\}$ have the representation $f = \sum_{j \in \mathbb{N}_{\geq n}} a_j (z - z_0)^j$ with $a_n \neq 0$. Then, f is invertible (i.e. a unit) if and only if $n = 0$. Furthermore, f is reducible if and only if $n \geq 2$. Hence, $\mathcal{H}(z_0)$ is a unique factorization domain with only prime $(z - z_0)$ and f has the unique factorization $f = \underbrace{\sum_{j \in \mathbb{N}_0} a_{j+n} (z - z_0)^j}_{\text{unit}} (z - z_0)^n$.

□

DEFINITION C.13. *Let $p, q \in R[\tau]$, $\deg p \leq m$, $\deg q \leq n$, $p = \sum_{j=0}^{m} p_j \tau^j$, and $q = \sum_{j=0}^{n} q_j \tau^j$. Then, we call*

$$\Lambda_{mn}(p, q) := \det \begin{pmatrix} \Lambda_p & \Lambda_q \end{pmatrix}$$

the (m, n)-resultant of p and q where $\Lambda_p \in L(\mathbb{R}^n, \mathbb{R}^{m+n})$ and $\Lambda_q \in L(\mathbb{R}^m, \mathbb{R}^{m+n})$ such that

$$\Lambda_p := \begin{pmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & p_1 & \ddots & \\ p_m & \vdots & \ddots & p_0 \\ & p_m & \ddots & p_1 \\ & & \ddots & \vdots \\ & & & p_m \end{pmatrix} \text{ and } \Lambda_p := \begin{pmatrix} q_0 & & & \\ q_1 & q_0 & & \\ \vdots & q_1 & \ddots & \\ q_n & \vdots & \ddots & q_0 \\ & q_n & \ddots & q_1 \\ & & \ddots & \vdots \\ & & & q_n \end{pmatrix}.$$

If $m = n = 0$, then we define $\Lambda_{00} := 1$.

LEMMA C.14. *Let $m, n \in \mathbb{N}_0$, $p, q \in R[\tau]$, $\deg p \leq m$, and $\deg q \leq n$. Then, the following are equivalent.*

(i) *There are* $r, s \in R[\tau]$ *with* $(r,s) \neq (0,0)$, $\deg r < m$, *and* $\deg s < n$ *such that* $sp = rq$.

(ii) $\Lambda_{mn}(p,q) = 0$.

PROOF. Without loss of generality, $R = F$. Then, $sp = rq$ for $r, s \in F[\tau]$ with $(r,s) \neq (0,0)$, $\deg r < m$, and $\deg s < n$ is equivalent to

$$\forall j \in \mathbb{N}_{\leq m+n-1} : \sum_{k=0}^{j} s_k p_{j-k} - r_k q_{j-k} = 0.$$

Hence, there is a non-trivial solution $(s, -r)$ if and only if the matrix of coefficients

$$\begin{pmatrix} p_0 & & & & q_0 & & & \\ p_1 & p_0 & & & q_1 & q_0 & & \\ \vdots & p_1 & \ddots & & \vdots & q_1 & \ddots & \\ p_m & \vdots & \ddots & p_0 & q_n & \vdots & \ddots & q_0 \\ & p_m & \ddots & p_1 & & q_n & \ddots & q_1 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & p_m & & & & q_n \end{pmatrix}$$

has vanishing determinant, i.e. $\Lambda_{mn} = 0$.

$\square$

LEMMA C.15. *Let* $m, n \in \mathbb{N}_0$, $p, q \in R[\tau]$, $\deg p \leq m$, *and* $\deg q \leq n$. *Then, the following are equivalent.*

(i) $(p_m, q_n) \neq (0,0)$ *and* p *and* q *have no common, non-constant divisor.*

(ii) $\Lambda_{mn}(p,q) \neq 0$.

PROOF. "(i)⇒(ii)" Without loss of generality, let $p_m \neq 0$. Suppose $\Lambda_{mn}(p,q) = 0$. Then, there are $r, s \in R[\tau]$ with $(r,s) \neq (0,0)$, $\deg r < m$, and $\deg s < n$ such that $sp = rq$. Since $\deg p = m > \deg r$, there exists a non-constant prime factor of p which is not a prime factor of r. Hence, p and q have a common factor $\lightning$.

"(ii)⇒(i)" If $(p_m, q_n) = (0,0)$, then $\Lambda_{mn} = 0$ is trivial. Let $(p_m, q_n) \neq (0,0)$ and $t \in R[\tau]$ with $\deg t \geq 1$ be a common divisor of p and q. Then, there are $r, s \in R[\tau]$ with $(r,s) \neq (0,0)$ such that $p = tr$ and $q = ts$. In particular, $\deg r \leq \deg p - 1 < m$, $\deg s \leq \deg q - 1 < n$, and $sp = str = qr$, i.e. $\Lambda_{mn} = 0$.

$\square$

Definition C.16. *Let $n \in \mathbb{N}$, $p \in R[\tau]$ with $\deg p \leq n$, and $q := \sum_{j=1}^{n} j p_j \tau^{j-1}$. Then, we call*

$$\Delta_n(p) := \Lambda_{n,n-1}(p,q)$$

the n-discriminant of p. If $\deg p = n$, then we will also write $\Delta(p) := \Delta_n(p)$.

For $p_0, \ldots, p_n \in R$, we define $\Delta_n(p_0, \ldots, p_n) := \Delta_n(p)$ where $p := \sum_{j=0}^{n} p_j \tau^j$.

Corollary C.17. *Let $n \in \mathbb{N}$, $p \in R[\tau]$ with $\deg p \leq n$, and $q := \sum_{j=1}^{n} j p_j \tau^{j-1}$. Then, the following are equivalent.*

(i) *$\deg p = n$ and p and q have no common, non-constant divisor.*

(ii) *$\Delta_n(p) \neq 0$.*

Definition C.18. *Let $U \subseteq \mathbb{C}$ be open, $n \in \mathbb{N}$, and $a_j \colon U \to \mathbb{C}$ holomorphic for every $j \in \mathbb{N}_{0,<n}$. A function*

$$p \colon U \times \mathbb{C} \to \mathbb{C};\ (z,\lambda) \mapsto \lambda^n + \sum_{j=0}^{n-1} a_j(z)\lambda^j$$

is called Weierstrass polynomial.

Lemma C.19. *Let p be a Weierstrass polynomial on $U \times \mathbb{C}$, $z_0 \in U$, and λ_0 a simple zero of $p(z_0,\cdot)$. Then, there exist $\delta, \varepsilon \in \mathbb{R}_{>0}$ such that every every $p(z,\cdot)$ with $z \in B_{\mathbb{C}}(z_0,\delta) \subseteq U$ has exactly one zero $\lambda(z) \in B_{\mathbb{C}}(\lambda_0,\varepsilon)$ and $z \mapsto \lambda(z)$ is holomorphic.*

PROOF. Follows directly from the analytic implicit function theorem since $\partial_2 p(z_0,\lambda_0) \neq 0$.

□

LEMMA C.20. *Let* $\varepsilon \in \mathbb{R}_{>0}$*,* $n \in \mathbb{N}$*,* $\gamma = e^{\frac{2\pi i}{n}}$*,* $\varphi\colon B_{\mathbb{C}}(0,\varepsilon) \to \mathbb{C}$ *holomorphic, and*

$$\forall z \in B_{\mathbb{C}}(0,\varepsilon^n)\ \forall \lambda \in \mathbb{C}:\ p(z,\lambda) := \prod_{j=0}^{n-1} \left(\lambda - \varphi\left(\gamma^j z^{\frac{1}{n}}\right)\right)$$

where $z \mapsto z^{\frac{1}{n}}$ *is a holomorphic root. Note that this is independent of the explicit choice of* $z^{\frac{1}{n}}$ *since all choices are contained in* $\left\{\gamma^j z^{\frac{1}{n}};\ j \in \mathbb{N}_{0,<n}\right\}$*. Then,* p *is a Weierstrass polynomial on* $B_{\mathbb{C}}(0,\varepsilon^n) \times \mathbb{C}$ *and has the zeros* $\varphi\left(\gamma^j z^{\frac{1}{n}}\right)$ *for* $j \in \mathbb{N}_{0,<n}$ *including multiplicities.*

PROOF. Let $(-1)^{n-j} a_j(z)$ be the $(n-j)^{\text{th}}$ elementary symmetric polynomial with variables $\varphi\left(z^{\frac{1}{n}}\right)$, $\varphi\left(\gamma z^{\frac{1}{n}}\right)$, …, $\varphi\left(\gamma^{n-1} z^{\frac{1}{n}}\right)$, i.e.

$$a_j(z) = (-1)^{n-j} \sum_{1 \le k_1 < k_2 < \ldots < k_{n-j} \le n-1} \prod_{l=1}^{n-j} \varphi\left(\gamma^{k_l} z^{\frac{1}{n}}\right),$$

and for $z_1 \in B_{\mathbb{C}}(0,\varepsilon^n) \smallsetminus \{0\}$ choose a holomorphic root $z \mapsto z^{\frac{1}{n}}$ in a neighborhood U of z_1. Then, all a_j are holomorphic in U. Since z_1 and the holomorphic root were arbitrary, all a_j are holomorphic in $B_{\mathbb{C}}(0,\varepsilon^n) \smallsetminus \{0\}$. Since φ is continuous in zero, so are all a_j and Riemann's removable singularity theorem for holomorphic functions implies that all a_j are holomorphic in $B_{\mathbb{C}}(0,\varepsilon^n)$. This shows that p is a Weierstrass polynomial and the assertion about the zeros is trivial.

□

REMARK If φ is given by the power series $\varphi(z) = \sum_{j \in \mathbb{N}_0} c_j z^j$ near zero, then all roots of $p(z,\cdot)$ are of the form $\lambda_k(z) = \sum_{j \in \mathbb{N}_0} c_j \gamma^{kj} z^{\frac{j}{n}}$. Such a series is called a Puiseux series. Similarly, if we take the expansion near z_0, then the roots of $p(z_0,\cdot)$ are given by $\lambda_k(z) = \sum_{j \in \mathbb{N}_0} c_j \gamma^{kj} (z-z_0)^{\frac{j}{n}}$.

■

Definition C.21. *A functional element is a holomorphic function* $f: D(f) \subseteq \mathbb{C} \to \mathbb{C}$ *such that* $D(f) = B_{\mathbb{C}}(z, r)$ *for some* $z \in \mathbb{C}$ *and* $r \in \mathbb{R}_{>0}$.

Let $z_0 \in \mathbb{C}$, f_0 *a functional element with* $D(f_0) = B_{\mathbb{C}}(z_0, r_0)$, $\gamma \in C([0,1], \mathbb{C})$, *and* $\gamma(0) = z_0$. *A family* $(f_t)_{t \in [0,1]}$ *of functional elements is called an analytic continuation of* f_0 *along* γ *if and only if*

(i) $\forall t \in [0,1]\ \exists r_t \in \mathbb{R}_{>0}:\ D(f_t) = B_{\mathbb{C}}(\gamma(t), r_t)$ *and*

(ii) $\forall t \in [0,1]\ \exists \delta \in \mathbb{R}_{>0}\ \forall s \in B_{[0,1]}(t, \delta):\ \gamma(s) \in B_{\mathbb{C}}(\gamma(t), r_t) \wedge f_s|_{D(f_s) \cap D(f_t)} = f_t|_{D(f_s) \cap D(f_t)}$.

Note, condition (ii) implies that all analytic continuations of f_0 along γ are germ-equivalent along γ.

Corollary C.22. *Let* $z_0 \in \mathbb{C}$, f_0 *a functional element with* $D(f_0) = B_{\mathbb{C}}(z_0, r_0)$, $\gamma \in C([0,1], \mathbb{C})$, *and* $\gamma(0) = z_0$. *Furthermore, let there be* $0 = t_0 < t_1 < \ldots < t_n = 1$ *and functional elements* f_j *for* $j \in \mathbb{N}_{\leq n}$ *such that* $\forall j \in \mathbb{N}_{\leq n}:\ \gamma(t_j) \in D(f_j)$, $\gamma[[t_{j-1}, t_j]] \subseteq D(f_{j-1}) \cap D(f_j)$, *and* $f_{j-1}|_{D(f_{j-1}) \cap D(f_j)} = f_j|_{D(f_{j-1}) \cap D(f_j)}$. *Then, there exists an analytic continuation of* f_0 *along* γ.

Proof. For $t \in [0,1]$ choose $r_t \in \mathbb{R}_{>0}$ and $j \in \mathbb{N}_{0, \leq n}$ such that $B_{\mathbb{C}}(\gamma(t), r_t) \subseteq D(f_j)$. Then, we define $f_t := f_j|_{B_{\mathbb{C}}(\gamma(t), r_t)}$.

□

Lemma C.23. *Let* $\Omega \subseteq \mathbb{C}$ *be open,* $p:\ \Omega \times \mathbb{C} \to \mathbb{C}$ *a Weierstrass polynomial of degree* n, $\forall z \in \Omega:\ \Delta(p(z, \cdot)) \neq 0$, $z_0 \in \Omega$, $f_0:\ B_{\mathbb{C}}(z_0, r_0) \to \mathbb{C}$ *holomorphic,* $\forall z \in B_{\mathbb{C}}(z_0, r_0):\ p(z, f_0(z)) = 0$, $\gamma \in C([0,1], \Omega)$, *and* $\gamma(0) = z_0$.

Then, there exists an analytic continuation of f_0 *along* γ *and every analytic continuation* $(f_t)_{t \in [0,1]}$ *satisfies* $\forall t \in [0,1]\ \forall z \in D(f_t):\ p(z, f_t(z)) = 0$.

Proof. Since $\forall z \in \Omega: \ \Delta(p(z,\cdot)) \neq 0$, $p(z,\cdot)$ and $\partial_2 p(z,\cdot)$ have no common, non-constant divisor, i.e. all zeros of $p(z,\cdot)$ are simple. In particular, $p(z,\cdot)$ has n "holomorphic zeros" (Lemma C.19); more precisely, there exists $r: \ [0,1] \to \mathbb{R}_{>0}$ such that $B_{\mathbb{C}}(\gamma(t), r(t)) \subseteq \Omega$ and there are holomorphic functions $\lambda_1^t, \ldots, \lambda_n^t : B_{\mathbb{C}}(\gamma(t), r(t)) \to \mathbb{C}$ such that $\forall z \in B_{\mathbb{C}}(\gamma(t), r(t)) \ \forall j \in \mathbb{N}_{\leq n}: \ p(z, \lambda_j^t(z)) = 0$.

Furthermore, for each $t \in [0,1]$ let $\varepsilon_t \in \mathbb{R}_{>0}$ such that $\gamma[(t-\varepsilon_t, t+\varepsilon_t) \cap [0,1]] \subseteq B_{\mathbb{C}}(\gamma(t), r(t))$ and for $t \in (0,1)$ let $(t-\varepsilon_t, t+\varepsilon_t) \subseteq (0,1)$. Then, $((t-\varepsilon_t, t+\varepsilon_t))_{t\in[0,1]}$ is an open cover of $[0,1]$ and we can choose a minimal subcover $((t_j-\varepsilon_{t_j}, t_j+\varepsilon_{t_j}))_{j\in\mathbb{N}_{\leq k}}$ for some $k \in \mathbb{N}$. By definition of the ε_t, there are $j_0, j_1 \in \mathbb{N}_{\leq k}$ such that $t_{j_0} = 0$ and $t_{j_1} = 1$. Without loss of generality, let $0 = t_0 < t_1 < \ldots < t_k = 1$ and set $\delta_j := \varepsilon_{t_j}$ for $j \in \mathbb{N}_{\leq k}$. Note that $t_j - \delta_j < t_{j-1} + \delta_{j-1}$ has to hold (otherwise, $t_j - \delta_j$ is contained in another interval with index $> j$ or $< j-1$, i.e. either the interval with index j or the interval with index $j-1$ is fully contained in another interval, thus, contradicting the assumption of a minimal cover).

Since

$$\gamma[(t_1 - \delta_1, t_0 + \delta_0)] \subseteq B_{\mathbb{C}}(\gamma(0), r_0) \cap B_{\mathbb{C}}(\gamma(t_1), r_{t_1}),$$

we obtain that $B_{\mathbb{C}}(\gamma(0), r_0) \cap B_{\mathbb{C}}(\gamma(t_1), r_{t_1})$ is non-empty and simply connected[1]. Hence, there exists $j \in \mathbb{N}_{\leq n}$ such that

$$\lambda_j^{t_1}|_{B_{\mathbb{C}}(\gamma(0),r_0)\cap B_{\mathbb{C}}(\gamma(t_1),r_{t_1})} = f_0|_{B_{\mathbb{C}}(\gamma(0),r_0)\cap B_{\mathbb{C}}(\gamma(t_1),r_{t_1})}.$$

Let $f_{t_1} := \lambda_j^{t_1}$. Inductively, we can define $f_{t_m} := \lambda_{j_m}^{t_m}$ for $m \in \mathbb{N}_{0,\leq k}$ and some $j_m \in \mathbb{N}_{\leq n}$ depending on $f_{t_{m-1}}$. Thence, Corollary C.22 yields that there exists an analytic continuation $(f_t)_{t\in[0,1]}$ of f_0 along γ.

[1]A space is simply connected if and only if it is path connected and every two paths with the same endpoints are homotopic relative to $\{0,1\}$.

Finally, $\forall t \in [0,1]\ \forall z \in D(f_t):\ p(z, f_t(z)) = 0$ follows since

$$(D(f_t) \ni z \mapsto p(z, f_t(z)) \in \mathbb{C})_{t\in[0,1]}$$

is an analytic continuation of $D(f_0) \ni z \mapsto p(z, f_0(z)) \in \mathbb{C}$ which vanishes identically.

□

PROPOSITION C.24. *Let $\mathcal{H}(0)$ be the set of holomorphic germs in zero, $n \in \mathbb{N}$, $U \subseteq \mathbb{C}$ an open neighborhood of zero, and*

$$p(z,\lambda) := \lambda^n + \sum_{j=0}^{n-1} a_j(z)\lambda^j$$

a Weierstrass polynomial on $U \times \mathbb{C}$ irreducible in $\mathcal{H}(0)[\lambda]$.

Then, $p(0,\cdot)$ has only one root λ_0 of multiplicity n. Furthermore, there exists $\varepsilon \in \mathbb{R}_{>0}$ with $B_{\mathbb{C}}(0,\varepsilon) \subseteq U$ and a holomorphic function $\varphi:\ B_{\mathbb{C}}\left(0,\varepsilon^{\frac{1}{n}}\right) \to \mathbb{C}$ such that $\forall u \in B_{\mathbb{C}}\left(0,\varepsilon^{\frac{1}{n}}\right):\ p(u^n, \varphi(u)) = 0$ and this contains all roots. In other words, the zeros of $p(z,\cdot)$ are given by the Puiseux series $\lambda(z) = \varphi\left(z^{\frac{1}{n}}\right)$.

PROOF. Since p is irreducible in $\mathcal{H}(0)[\lambda]$ it has no non-constant divisor of strictly lesser degree. In particular, p and $\partial_2 p$ cannot have a common, non-constant divisor. Hence,

$$0 \neq \Delta(p) = \Delta_n(a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 1) \in \mathcal{H}(0).$$

Thus, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $\Delta(p)$ is defined on $B_{\mathbb{C}}(0,\varepsilon)$, $\forall z \in B_{\mathbb{C}}(0,\varepsilon)\setminus\{0\}:$ $\Delta(p)(z) \neq 0$, and $M := \sup\{|a_j(z)|;\ z \in B_{\mathbb{C}}(0,\varepsilon),\ j \in \mathbb{N}_{0,<n}\} < \infty$. Note that we can choose a smaller ε if $\Delta(p)(z) = 0$ for $z \neq 0$ and zero cannot be an accumulation point of zeros since that would imply $\Delta(p) = 0$. Furthermore, note that

$$\begin{aligned}\Delta(p(z,\cdot)) =& \Delta_n(a_0(z), a_1(z), \ldots, a_{n-2}(z), a_{n-1}(z), 1)\\ =& \Delta_n(a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 1)(z)\end{aligned}$$

$$=\Delta(p)(z).$$

Then,

$$\begin{aligned}\forall z\in B_{\mathbb{C}}(0,\varepsilon)\ \forall\lambda\in\mathbb{C}\smallsetminus B_{\mathbb{C}}(0,M+1):\ |p(z,\lambda)|\geq&|\lambda|^n\left(1-\sum_{j=0}^{n-1}\frac{M}{\lambda^{n-j}}\right)\\ \geq&|\lambda|^n\left(1-\sum_{j=0}^{n-1}\frac{M}{(M+1)^{n-j}}\right)\\ =&|\lambda|^n\left(1-M\sum_{j=1}^{n}\left(\frac{1}{M+1}\right)^j\right)\\ =&|\lambda|^n\left(1-M\frac{\frac{1}{M+1}-\left(\frac{1}{M+1}\right)^{n+1}}{1-\frac{1}{M+1}}\right)\\ =&|\lambda|^n\left(1-M\frac{1-\left(\frac{1}{M+1}\right)^{n}}{M}\right)\\ =&|\lambda|^n\left(\frac{1}{M+1}\right)^n\\ \geq&1\end{aligned}$$

shows that all zeros of $p(z,\cdot)$ are in $B_{\mathbb{C}}(0,M+1)$ provided that $z\in B_{\mathbb{C}}(0,\varepsilon)$.

Let $\hat{\Omega}:=B_{\mathbb{C}}(0,\varepsilon)\smallsetminus\mathbb{R}_{\leq 0}$ and $\check{\Omega}:=B_{\mathbb{C}}(0,\varepsilon)\smallsetminus\mathbb{R}_{\geq 0}$. Since $\hat{\Omega}$ and $\check{\Omega}$ are simply connected and $\Delta(p)(z)\neq 0$ for every $z\in B_{\mathbb{C}}(0,\varepsilon)\smallsetminus\{0\}$, that is, all zeros are simple, there are holomorphic functions $\hat{\lambda}_1,\dots,\hat{\lambda}_n:\ \hat{\Omega}\to\mathbb{C}$ and $\check{\lambda}_1,\dots,\check{\lambda}_n:\ \check{\Omega}\to\mathbb{C}$ such that each $\hat{\lambda}_j(z)$ and $\check{\lambda}_j(z)$ is a zero of $p(z,\cdot)$ for $z\in\hat{\Omega}$ or $z\in\check{\Omega}$, respectively (n "holomorphic zeros"; Lemma C.19). Without loss of generality, let $\hat{\lambda}_j=\check{\lambda}_j$ on $B_{\mathbb{C}}(0,\varepsilon)\cap\mathbb{C}_{\Im(\cdot)>0}$ for every $j\in\mathbb{N}_{\leq n}$. Then, there exists a permutation $\pi\in S_n$ such that $\hat{\lambda}_{\pi(j)}=\check{\lambda}_j$ on $B_{\mathbb{C}}(0,\varepsilon)\cap\mathbb{C}_{\Im(\cdot)<0}$ for every $j\in\mathbb{N}_{\leq n}$. Let n_0 be the length of the trajectory of 1 under the action of π; without loss of generality, let the trajectory be $(1,2,\dots,n_0)$, i.e. $\pi^{k-1}(1)=k+1-\left\lfloor\frac{k}{n_0}\right\rfloor n_0$.

Let $\varphi\colon\ B_{\mathbb{C}}\left(0,\varepsilon^{\frac{1}{n_0}}\right)\smallsetminus\{0\}\to\mathbb{C}$ be holomorphic and defined as follows.

$$\forall z\in B_{\mathbb{C}}\left(0,\varepsilon^{\frac{1}{n_0}}\right)\smallsetminus\{0\}:\ \varphi(u):=\begin{cases}\hat{\lambda}_{j+1}\left(z^{n_0}\right) & ,\ \left|\arg z-\frac{2j\pi}{n_0}\right|<\frac{\pi}{n_0}\\ \check{\lambda}_{j+1}\left(z^{n_0}\right) & ,\ \frac{2j\pi}{n_0}<\arg z<\frac{2(j+1)\pi}{n_0}\end{cases}.$$

Since

$$\forall j\in\mathbb{N}_{\le n}\ \forall\hat{z}\in\hat{\Omega}\ \forall\check{z}\in\check{\Omega}:\ \left|\hat{\lambda}_j\left(\hat{z}\right)\right|<M+1\ \wedge\ \left|\check{\lambda}_j\left(\check{z}\right)\right|<M+1$$

holds, we obtain

$$\lim_{z\to0}z\varphi(z)=0.$$

In other words, Riemann's removable singularity theorem for holomorphic functions implies that φ admits a holomorphic extension to $B_{\mathbb{C}}\left(0,\varepsilon^{\frac{1}{n_0}}\right)$. By Lemma C.20,

$$p_0(z,\lambda):=\prod_{j=0}^{n_0-1}\left(\lambda-\varphi\left(\gamma^j z^{\frac{1}{n_0}}\right)\right)$$

with $\gamma:=e^{\frac{2\pi i}{n_0}}$ is a Weierstrass polynomial on $B_{\mathbb{C}}(0,\varepsilon)\times\mathbb{C}$ with the roots

$$\left\{\varphi\left(\gamma^j z^{\frac{1}{n_0}}\right);\ j\in\mathbb{N}_{\le n_0}\right\}=\begin{cases}\left\{\hat{\lambda}_j(z);\ j\in\mathbb{N}_{\le n_0}\right\} & ,\ z\in\hat{\Omega}\\ \left\{\check{\lambda}_j(z);\ j\in\mathbb{N}_{\le n_0}\right\} & ,\ z\in\check{\Omega}\end{cases}.$$

Let

$$\forall z\in B_{\mathbb{C}}(0,\varepsilon)\smallsetminus\{0\}:\ \tilde{q}(z,\lambda):=\begin{cases}\prod_{j=n_0+1}^{n}\left(\lambda-\hat{\lambda}_j(z)\right) & ,\ z\in\hat{\Omega}\\ \prod_{j=n_0+1}^{n}\left(\lambda-\check{\lambda}_j(z)\right) & ,\ z\in\check{\Omega}\end{cases}.$$

Then, $\tilde{q}$ is well-defined and a Weierstrass polynomial on $(B_{\mathbb{C}}(0,\varepsilon)\smallsetminus\{0\})\times\mathbb{C}$ with bounded coefficients, i.e. $\tilde{q}$ can be extended to a Weierstrass polynomial q on $B_{\mathbb{C}}(0,\varepsilon)\times\mathbb{C}$. Since

$$\forall z\in B_{\mathbb{C}}(0,\varepsilon)\smallsetminus\{0\}:\ p(z,\cdot)=p_0(z,\cdot)q(z,\cdot)$$

holds, we obtain $p=p_0q$.

However, p was assumed irreducible, i.e. $p = p_0$, $n = n_0$, and $\lambda_0 = \varphi(0)$ is an n-fold zero of $p(0,\cdot)$.

□

THEOREM C.25. *Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $p: U \times \mathbb{C} \to \mathbb{C}$ a Weierstrass polynomial of degree m, and λ_0 a zero of $p(z_0,\cdot)$ of algebraic multiplicity n. Then, there are $\delta, \varepsilon \in \mathbb{R}_{>0}$ such that $B_{\mathbb{C}}(z_0,\varepsilon) \subseteq U$ and $p(z,\cdot)$ has exactly n roots (including multiplicities) in $B_{\mathbb{C}}(\lambda_0,\delta)$ provided that $z \in B_{\mathbb{C}}(z_0,\varepsilon)$. Furthermore, there are $n_1,\ldots,n_k \in \mathbb{N}$ with $\sum_{j=1}^{k} n_j = n$ and $\varphi_j: B_{\mathbb{C}}\left(z_0, \varepsilon^{\frac{1}{n_j}}\right) \to \mathbb{C}$ holomorphic ($j \in \mathbb{N}_{\leq k}$) such that the zeros of $p(z,\cdot)$ in $B_{\mathbb{C}}(\lambda_0,\delta)$ for $z \in B_{\mathbb{C}}(z_0,\varepsilon)$ are given by*

$$\forall l \in \mathbb{N}_{\leq k} \ \forall j \in \mathbb{N}_{0,<n_l}: \ \lambda_{l,j}(z) := \varphi_l\left(\gamma_l^j (z - z_0)^{\frac{1}{n_l}}\right)$$

where $\gamma_l := e^{\frac{2\pi i}{n_l}}$.

PROOF. Without loss of generality, let $z_0 = 0$ and consider p as an element of $\mathcal{H}(0)[\lambda]$. Since $\mathcal{H}(0)[\lambda]$ is a unique factorization domain (Corollary C.12), we can factorize p into $p = \prod_{j=1}^{k'} p_j$ where each p_j is prime. Without loss of generality, let all p_j be normalized, that is, they have leading coefficient 1, i.e. they are Weierstrass polynomials. For $l \in \mathbb{N}_{\leq k'}$, let $n_l := \deg p_l$. Then, each $p_l(0,\cdot)$ has a zero λ_l of multiplicity n_l according to Proposition C.24. Without loss of generality, let $k \in \mathbb{N}$ be such that $\forall l \in \mathbb{N}_{\leq k}: \ \lambda_l = \lambda_0$ and $\forall l \in \mathbb{N}_{>k,\leq k'}: \ \lambda_l \neq \lambda_0$. Then, Proposition C.24 yields holomorphic functions φ_l and $\varepsilon_l \in \mathbb{R}_{>0}$ as in the statement of Proposition C.24 for every $l \in \mathbb{N}_{\leq k'}$. Let $\delta := \frac{1}{2}\min\{|\lambda_l - \lambda_0|; \ l \in \mathbb{N}_{>k,\leq k'}\}$. Then, there exists $\varepsilon \in (0, \min\{\varepsilon_l; \ l \in \mathbb{N}_{\leq k'}\})$ such that $|\varphi_l(z) - \lambda_l| < \delta$ for every $z \in B_{\mathbb{C}}\left(0, \varepsilon^{\frac{1}{n_l}}\right)$ and $l \in \mathbb{N}_{\leq k'}$. Furthermore, the roots of $p(z,\cdot)$ in $B_{\mathbb{C}}(\lambda_0,\delta)$ for $z \in B_{\mathbb{C}}(0,\varepsilon)$ are precisely the roots of the $p_l(z,\cdot)$ ($l \in \mathbb{N}_{\leq k'}$).

□

Bibliography

[1] C. D. ALIPRANTIS and O. BURKINSHAW, *Locally Solid Riesz Spaces with Applications to Economics*, 2nd edition, AMS, Providence, RI, 2003.

[2] R. ARENS, *Duality in linear spaces*, Duke Mathematical Journal **14** (1947), 787-794.

[3] M. F. ATIYAH, *Algebraic Topology and Operators in Hilbert Space*, Lectures in Modern Analysis and Applications I, Lecture Notes in Mathematics **103** (1969), 101-121.

[4] ———, *K-Theory*, Addison-Wesley, 1989.

[5] V. I. BOGACHEV, *Gaussian Measures*, AMS, Providence, RI, 1998.

[6] B. BOOSS-BAVNBEK, M. LESCH, and J. PHILLIPS, *Unbounded Fredholm Operators and Spectral Flow*, Canadian Journal of Mathematics **57** (2005), no. 2, 225-250.

[7] L. BOUTET DE MONVEL, *Vanishing of the Logarithmic Trace of Generalized Szegő Projectors*, arXiv:math/0604166v1, Proceedings of the Conference "Algebraic Analysis of Differential Equations: From Microlocal Analysis to Exponential Asymptotics" Festschrift in honor of Prof. Takahiro Kawai (2008), 67-78.

[8] L. BOUTET DE MONVEL and V. GUILLEMIN, *The Spectral Theory of Toeplitz Operators*, Annals of Mathematics Studies no. 99, Princeton University Press and University of Tokyo Press, Princeton, NJ, 1981.

[9] C. BROUDER, N. V. DANG, and F. HÉLEIN, *A smooth introduction to the wavefront set*, Journal of Physics A: Mathematical and Theory **47** (2014).

[10] B. CASCALES, J. KĄKOL, and S. A. SAXON, *Metrizability vs. Fréchet-Urysohn property*, Proceedings of the American Mathematical Society **131** (2003), no. 11, 3623-3631.

[11] B. CASSELMAN, *Essays in analysis: Quasi-complete spaces* (2013).

[12] J. CIMA and G. SCHOBER, *On Spaces of Meromorphic Functions*, Rocky Mountain Journal of Mathematics **9** (Summer 1979), no. 3.

[13] J. F. COLOMBEAU, *Elementary Introduction to New Generalized Functions*, North Holland Mathematics Studies **113**, Elsevier Science Publishers, Amsterdam/New York, NY/Oxford, 1985.

[14] J. B. CONWAY, *Functions of one complex variable*, 2nd ed., Springer, New York, NY, 1978.

[15] Y. DABROWSKI, *Functional Properties of Generalized Hörmander Spaces of Distributions I: Duality theory, Completions and Bornologifications*, arXiv:1411.3012v1 (2014).

[16] ———, *Functional Properties of Generalized Hörmander Spaces of Distributions II: Multilinear Maps and Applications to Spaces of Functionals with Wave Front Set Condition*, arXiv:1412.1749v1 (2014).

[17] Y. DABROWSKI and C. BROUDER, *Functional Properties of Hörmander's Space of Distributions Having a Specified Wavefront Set*, Communications in Mathematical Physics **332** (2014), 1345-1380.

[18] A. DEFANT and K. FLORET, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies 176, Amsterdam/London/New York, NY/Tokyo, 1993.

[19] J. DELGADO and M. RUZHANSKY, *Schatten classes on compact manifolds: Kernel conditions*, Journal of Functional Analysis **267** (2014), 772-798.

[20] J. J. DUISTERMAAT, *Fourier Integral Operators*, Birkhäuser, Boston, MA/Basel/Berlin, 1996.

[21] J. J. DUISTERMAAT and L. HÖRMANDER, *Fourier Integral Operators II*, Acta Mathematica **128** (1972), 183-269.

[22] W. FELLER, *An Introduction to Probability Theory and Its Applications: volume II*, Wiley, 1971.

[23] S. P. FRANKLIN, *Spaces in which sequences suffice*, Fundamenta Mathematicae **57** (1965), 107-115.

[24] D. H. FREMLIN, *Measurable functions and almost continuous functions*, Manuscripta Mathematica **33** (1981), 387-405.

[25] ———, *Measure Theory: volume 2*, 2010.

[26] C. GARETTO, *Generalized Fourier Integral Operators on Spaces of Colombeau Type*, Operator Theory: Advances and Applications **189** (2009), 137-184.

[27] C. GARETTO, G. HÖRMANN, and M. OBERGUGGENBERGER, *Generalized Oscillatory Integrals and Fourier Integral Operators*, Proceedings of the Edinburgh Mathematical Society **52** (2009), 351-386.

[28] C. GARETTO and M. OBERGUGGENBERGER, *Generalized Fourier Integral Operators Methods for Hyperbolic Equations with Singularities*, Proceedings of the Edinburgh Mathematical Society **57** (2014), 423-463.

[29] V. D. GOLOVIN, *Duality in spaces of holomorphic functions with singularities*, Soviet Mathematics Doklady **7** (1966), 571-574.

[30] ———, *On some spaces of holomorphic functions with isolated singularities*, Mathematics of the USSR - Sbornik **2** (1967), 17-33.

[31] E. GRIEGER, *The local Atiyah Singer Index Formula: an elementary approach*, preprint, http://www.mth.kcl.ac.uk/ griegere/ASNonSing.pdf (2013).

[32] K.-G. GROSSE-ERDMANN, *The Locally Convex Topology on the Space of Meromorphic Functions*, Journal of the Australian Mathematical Society (Series A) **59** (1995), 287-303.

[33] V. GUILLEMIN, *A New Proof of Weyl's Formula on the Asymptotic Distribution of Eigenvalues*, Advances in Mathematics **55** (1985), 131-160.

[34] ———, *Gauged Lagrangian Distributions*, Advances in Mathematics **102** (1993), 184-201.

[35] ———, *Residue Traces for Certain Algebras of Fourier Integral Operators*, Journal of Functional Analysis **115** (1993), 391-417.

[36] ———, *Wave-trace invariants*, Duke Mathematical Journal **83** (1996), no. 2, 287-352.

[37] S. W. HAWKING, *Zeta Function Regularization of Path Integrals in Curved Spacetime*, Communications in Mathematical Physics **55** (1977), 133-148.

[38] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators.* I-IV, Springer, Berlin/Heidelberg, 1990.

[39] ———, *Fourier Integral Operators I*, Acta Mathematica **127** (1971), 79-183.

[40] H. S. HOLDGRÜN, *Fastautomorphe Funktionen auf komplexen Räumen*, Mathematische Annalen **203** (1973), 35-64.

[41] H. JARCHOW, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.

[42] R. JENTZSCH, *Untersuchungen zur Theorie der Folgen analytischer Funktionen*, Acta Mathematica **41** (1916), 219-251.

[43] M. I. KADETS and V. M. KADETS, *Series in Banach Spaces: Conditional and Unconditional Convergence*, Operator Theory: Advances and Applications, vol. 94, 1997.

[44] T. KATO, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin/Heidelberg, 1980.

[45] G. KÖTHE, *Topological Vector Spaces I*, Springer, New York, NY, 1969.

[46] ———, *Topological Vector Spaces II*, Springer, New York, NY, 1979.

[47] M. KONTSEVICH and S. VISHIK, *Determinants of elliptic pseudo-differential operators*, Max Planck Preprint, arXiv:hep-th/9404046 (1994).

[48] ———, *Geometry of determinants of elliptic operators*, Functional Analysis on the Eve of the XXI century, Vol. I, Progress in Mathematics **131** (1994), 173-197.

[49] A. LAPTEV, YU. SAFAROV, and D. VASSILIEV, *On Global Representation of Lagrangian Distributions and Solutions of Hyperbolic Equations*, Communications on Pure and Applied Mathematics **47** (1994), no. 11, 1411-1456.

[50] A. LAPTEV and I. M. SIGAL, *Global Fourier Integral Operators and Semiclassical Asymptotics*, Reviews in Mathematical Physics **12** (2000), no. 5, 749-766.

[51] M. LESCH, *On the Noncommutative Residue for Pseudodifferential Operators with* log-*Polyhomogeneous Symbols*, Annals of Global Analysis and Geometry **17** (1999), 151-187.

[52] I. MADSEN and J. TORNEHAVE, *From Calculus to Cohomology*, Cambridge University Press, Cambridge, 1997.

[53] A. NOWAK, *A note on the Fréchet theorem*, Annales Mathematiciae Silesianae **9** (1995), 43-45.

[54] A. OSTROWSKI, *Über Folgen analytischer Funktionen und einige Verschärfungen des Picardschen Satzes*, Mathematische Zeitschrift **24** (1926), 215-258.

[55] S. PAYCHA, *Zeta-regularized traces versus the Wodzicki residue as tools in quantum field theory and infinite dimensional geometry*, Proceedings of the International Conference on Stochastic Analysis and Applications (2004), 69-84.

[56] S. PAYCHA and S. G. SCOTT, *A Laurent expansion for regularized integrals of holomorphic symbols*, Geometric and Functional Analysis **17 (2)** (2007), 491-536.

[57] M. J. RADZIKOWSKI, *The Hadamard condition and Kay's conjecture in (axiomatic) quantum field theory on curved space-time*, Princeton University: Ph.D. thesis, 1992.

[58] ———, *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Communications in Mathematical Physics **179** (1996), 529-553.

[59] D. B. RAY, *Reidemeister torsion and the Laplacian on lens spaces*, Advances in Mathematics **4** (1970), 109-126.

[60] D. B. RAY and I. M. SINGER, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Mathematics **7** (1971), 145-210.

[61] M. REED and B. SIMON, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, San Diego,CA/ New York, NY/ Boston, MA/ London/ Sydney/ Tokyo/ Toronto, ON, 1975.

[62] YU. SAFAROV, *A Symbolic Calculus for Fourier Integral Operators*, Geometric and Spectral Analysis, Contemporary Mathematics 630, American Mathematical Society, Providence, RI, 2012.

[63] YU. SAFAROV and D. VASSILIEV, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, Translations of Mathematical Monographs **155**, American Mathematical Society, Providence, RI, 1997.

[64] S. A. SAXON and P. P. NARAYANASWAMI, *Metrizable [normable] (LF)-spaces and two classical problems in Fréchet [Banach] spaces*, Studia Mathematica **93(1)** (1989), 1-16.

[65] R. L. SCHILLING, *Measures, Integrals and Martingales*, Cambridge University Press, Cambridge, 2005.

[66] P. SCHNEIDER, *Nonarchimedian Functional Analysis*, Springer, Berlin/Heidelberg/New York, NY, 2002.

[67] S. G. SCOTT, *Traces and Determinants of Pseudodifferential Operators*, Oxford University Press, 2010.

[68] R. T. SEELEY, *Complex Powers of an Elliptic Operator*, Proceedings of Symposia in Pure Mathematics, American Mathematical Society **10** (1967), 288-307.

[69] M. A. SHUBIN, *Pseudodifferential Operators and Spectral Theory*, 2nd ed., Springer, Berlin/Heidelberg, 2001.

[70] L. A. STEEN and J. A. SEEBACH, *Counterexamples in Topology*, Springer, New York, NY/Heidelberg/Berlin, 1978.

[71] E. M. STEIN and G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.

[72] H. TIETZ, *Zur Klassifizierung meromorpher Funktionen auf Riemannschen Flächen*, Mathematische Annalen **142** (1961), 441-449.

[73] F. TREVES, *Locally Convex Spaces and Linear Partial Differential Equations*, Springer, New York, NY, 1967.

[74] J. VOIGT, *A course on topological vector spaces*, preprint, http://www.math.tu-dresden.de/~voigt/tvs/tvs.pdf, Aug. 2014.

[75] ——, *On the Convex Compactness Property for the Strong Operator Topology*, Note di Matematica **XII** (1992), 259-269.

[76] N. WATERSTRAAT, *The Index Bundle for Gap-Continuous Families, Morse-Type Index Theorems and Bifurcation*, Dr. rer. nat. thesis, Georg-August-Universität Göttingen, Göttingen, 2011.

[77] M. WODZICKI, *Noncommutative residue. I. Fundamentals. K-theory, arithmetic and geometry (Moscow, 1984-1986), 320-399, Lecture Notes in Math., 1289*, Springer, Berlin, 1987.

[78] ———, *Spectral asymmetry and noncommutative residue*, Ph.D. thesis, Moscow: Steklov Institute of Mathematics, 1984.

[79] S. ZELDITCH, *Wave invariants at elliptic closed geodesics*, Geometric and Functional Analysis **7** (1997), 145-213.

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