RESEARCH STATEMENT

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My research focuses on problems in the intersection of number theory, representation theory and algebraic geometry, in the broad area of the Langlands programme. I'm particularly interested in the representation theory of reductive p-adic groups and in the structure of Galois groups of p-adic fields, and the relationships between these two topics.

Specifically, I am interested in branching rules for admissible representations of reductive p-adic groups, with a particular focus on the Bushnell–Kutzko theory of types, and on applications of this towards the inertial local Langlands correspondence and the study of mod-ℓ congruences between Galois representations.

1. Introduction

Let F be a p-adic field, i.e. a finite algebraic extension of either the field $\mathbb{Q}_p$ of p-adic numbers, or of the field $\mathbb{F}_p((t))$ of formal Laurent series in one variable over $\mathbb{F}_p$. A central problem in algebraic number theory is to understand the structure of the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$ of F relative to some separable algebraic closure $\bar{F}/F$. To date the most fruitful approach to studying $\mathrm{Gal}(\bar{F}/F)$ has been by studying its representation theory over a field E, typically taken to be an algebraic extension of $\mathbb{Q}_\ell$, or of $\mathbb{F}_\ell$, for some prime ℓ. The case that $\ell = p$ comes with many additional analytic complications; I am largely interested in the case $\ell \neq p$, in which the representation theory may be approached purely algebraically. For the most part, we will restrict our attention to the case $E = \bar{\mathbb{Q}}_\ell$.

The local Langlands conjectures, now known in many cases, provide a dictionary between the study of such *Galois representations* and the representation theory of *reductive p-adic groups*. Very roughly, if $\mathbf{G}$ is a connected reductive algebraic group defined over F and $G = \mathbf{G}(F)$, then one expects that there should be a natural parametrization of the irreducible $\bar{\mathbb{Q}}_\ell$-representations of G in terms of *L-parameters*, which are homomorphisms $W_F' \to {}^LG$, where W_F' denotes the Weil–Deligne group of F and ${}^LG$ the Langlands dual group of G.

While the representation theory of p-adic groups is complicated, it appears to be more immediately tractable than the theory of Galois representations. One particularly fruitful approach has been via restriction to compact open subgroups. Given π an admissible representation of a p-adic group $G = \mathbf{G}(F)$, and $K \subset G$ a compact open subgroup, the restriction $\pi|_K = \bigoplus_i \tau_i$ is a semisimple representation, each of the irreducible components of which is finite-dimensional. It is most interesting to study this restriction when K is a *maximal* compact open subgroup of G; there are three obvious questions about this restriction which form the focus of my current research.

(a) Can one explicitly describe each of the components τ_i?
(b) Given a component τ_i, can one describe the set of irreducible representations π' of G such that τ_i occurs in $\pi'|_K$?
(c) Are there any components τ_i which may *only* occur in π'?

To date, the answer to (a) is largely negative, unless G is a particularly simple example of a p-adic group such as $\mathbf{GL}_2(F)$ or $\mathbf{SL}_2(F)$ [Cas73, Nev13], or the representation π itself admits a particularly simple description [Nev14], with the main obstruction being that describing the irreducible representations of K is in general extremely difficult. The answer to (b) is also currently negative, although some partial progress is made: we are at least often able to construct *some* $\pi' \not\simeq \pi$ which contains a given component τ_i.

It turns out that the answer to (c) is often positive, at least allowing for a minor correction: if π contains τ_i, then certainly so must $\pi \otimes \omega$, for any unramified character ω of G[1]. More generally, the best that one can hope for is to find a component τ_i such that whenever τ_i occurs in $\pi'|_K$, then π' must occur in the same block of the category $\mathrm{Rep}_E(G)$ of smooth E-representations of G; in the most pertinent case where $E = \bar{\mathbb{Q}}_\ell$, these blocks have been explicitly described by Bernstein in terms of the inertial supports of representations [Ber84]. This leads one to follow Bushnell and Kutzko [BK98] in making the following definition:

Definition 1. Let $\mathfrak{B}$ be a block of the category $\mathrm{Rep}(G)$. A *type* for $\mathfrak{B}$ is a pair (J, λ) consisting of a compact open subgroup J of G and a smooth irreducible representation λ of J such that whenever an irreducible representation π' of G is such that $\pi'|_J$ contains λ, it must be the case that π' is contained in $\mathfrak{B}$.

Given a type (J, λ) for a cuspidal block $\mathfrak{B}$, and a compact open subgroup $K \supset J$ of G, it is simple to obtain a type of the form (K, τ) for $\mathfrak{B}$. We will therefore often restrict our attention to types defined on maximal compact open subgroups of G.

Types are of particular interest for the blocks $\mathfrak{B}$ whose irreducible objects are cuspidal: an irreducible representation π of G is said to be cuspidal if whenever ρ is a non-zero quotient of π, every simple root subgroup of G acts non-trivially through ρ. Given such a block $\mathfrak{B}$, the existence of a type for $\mathfrak{B}$ is closely related (and, in practice, often essentially equivalent) to establishing the long-standing folklore conjecture that every irreducible object of $\mathfrak{B}$ should be compactly induced from an open, compact-modulo-centre subgroup of G.

The construction of types goes back to Howe [How77], who constructed types for what are now called *essentially tame*[2] cuspidal representations of $\mathbf{GL}_N(F)$. This was later generalized to all cuspidal representations of $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$ by Bushnell and Kutzko

[1] A character $\omega : G \to E^\times$ is *unramified* if it has trivial restriction to any compact open subgroup of G.

[2] A cuspidal representation of $\mathbf{GL}_N(F)$ is said to be essentially tame if its *ramification degree* is coprime to p; the ramification degree of π is equal to N/t where the *torsion number* t is the number of unramified characters ω of G such that $\pi \simeq \pi \otimes \omega$. More generally, we say that a cuspidal representation of an arbitrary group G is essentially tame if it contains one of the types constructed by Yu, as described below.

[BK93a, BK93b, BK94]. More recently, Stevens used the Bushnell–Kutzko construction to find types for all cuspidal representations of classical groups when p is odd [Ste08]. In another direction, there is an extremely general construction of essentially tame representations of arbitrary groups due to Yu [Yu01], and the theory for depth-zero representations[3] is completely understood due to Moy–Prasad and Morris [MP94, Mor99]. Most importantly, in each case where types are known to exist, one is able to give a completely explicit description of the types. Moreover, any two such types for a given block are seen to be obtained from one another through a simple process of representation theoretic renormalizations.

For the sake of consistency in language between the various constructions of types, let us say that each proceeds in the following way: one constructs a class of *data* Σ, and to each datum Σ one associates a type $(J_\Sigma, \lambda_\Sigma)$ for a cuspidal block $\mathfrak{B}_\Sigma$ whose irreducible objects are precisely the unramified twists of a given cuspidal representation π_Σ.

2. Classifying types for cuspidal Bernstein blocks

One is therefore led to, in a case where types are known to exist for a cuspidal block $\mathfrak{B}$, ask for a complete list of the possible types (K, τ), as K ranges over the (finitely many conjugacy classes of) maximal compact open subgroups of G. The natural conjecture is the following, which is generally referred to as *the unicity of types*:

Conjecture 2. *Let $\mathfrak{B}$ be a cuspidal block of $\mathrm{Rep}(G)$, and let K be a maximal compact subgroup of G. There exists a type for $\mathfrak{B}$ of the form (K, τ) if and only if there exists a datum Σ such that $J_\Sigma \subset K$ and $\pi_\Sigma \in \mathfrak{B}$. Conversely, whenever there is a type for $\mathfrak{B}$ of the form (K, τ), the restriction to J_Σ of τ must contain λ_Σ.*

It seems reasonable to expect that similar results might also hold for the non-cuspidal blocks of $\mathrm{Rep}(G)$. However, there is much less evidence in this direction than for the cuspidal case (and none beyond the case of split groups of type A), and our focus in the remainder of this discussion will be entirely on the cuspidal case. The first results towards the unicity of types were obtained by Henniart in the appendix of [BM02], where a positive answer was given for $G = \mathbf{GL}_2(F)$. Henniart's result was then generalized to $\mathbf{GL}_N(F)$ by Paskunas in [Pas05]. Further generalizations to groups other than $\mathbf{GL}_N(F)$ were the focus of my PhD thesis.

My first main result was an extension of Paskunas' result to $\mathbf{SL}_N(F)$, using the natural approach of going via Clifford theory. The main additional complication is that, unlike for $\mathbf{GL}_N(F)$, there are multiple conjugacy classes of maximal compact subgroups in $\mathbf{SL}_N(F)$.

Theorem 3 ([Lat15, Lat18])**.** *The unicity of types holds for the essentially tame cuspidal blocks of $G = \mathbf{SL}_N(F)$. Moreover, given a maximal compact subgroup K of G and an essentially tame cuspidal block $\mathfrak{B}$, the number of conjugacy classes of types for $\mathfrak{B}$ which*

[3] An irreducible cuspidal representation π of G is said to be of *depth-zero* if there exists a maximal compact subgroup K of G such that the space of vectors in π invariant under the pro-p radical K^+ of K is equal to the pullback to K of a cuspidal representation of the group K^+/K, whose connected component is a finite group of Lie type over the residue field of F.

are defined on K is at most 1, and the number of conjugacy classes of maximal compact subgroups K admitting a type for $\mathfrak{B}$ is equal to the ramification degree of the cuspidal representations in $\mathfrak{B}$.

The appearance of the hypothesis that the block $\mathfrak{B}$ be essentially tame is due to the much more complicated nature of the relationship between types for non-essentially tame blocks of $\mathrm{Rep}(\mathbf{GL}_N(F))$ and types for the corresponding blocks of $\mathrm{Rep}(\mathbf{SL}_N(F))$.[4]

The other main result of my thesis was to complete the classification of types for depth-zero cuspidal representations of arbitrary p-adic groups:

Theorem 4 ([Lat17])**.** *Let $\mathfrak{B}$ be a depth-zero cuspidal block of $\mathrm{Rep}(G)$, for G an arbitrary reductive p-adic group. Then the unicity of types holds for $\mathfrak{B}$. Moreover, there exists a unique conjugacy class of maximal compact subgroups K of G on which a type for $\mathfrak{B}$ may occur.*

The approach to obtaining this also provides a particularly satisfying partial answer to our question (b). Given a depth-zero cuspidal representation π of G, and a maximal compact subgroup K of G, one obtains a dichotomy: either a component τ of $\pi|_K$ is a type, or it is contained in some non-cuspidal irreducible representation of G (which is given a completely explicit description).

3. The inertial Langlands correspondence

The unicity of types has a close relationship to the existence of a *well-behaved* inertial Langlands correspondence. Suppose that we are in a situation where one has a construction of both the local Langlands correspondence for G, and of types for each of the cuspidal blocks of $\mathrm{Rep}(G)$. The local Langlands correspondence gives a natural finite-to-one surjection rec from $\mathrm{Cusp}(G)$, the set of isomorphism classes of cuspidal representations of G, to $\mathcal{L}(G)$, the set of *relevant*[5] cuspidal L-parameters for G. Let $\mathcal{A}(G) = \coprod_{\mathfrak{B}} \mathcal{A}^{\mathfrak{B}}(G)$, where for each cuspidal block $\mathfrak{B}$, $\mathcal{A}^{\mathfrak{B}}(G)$ denotes the set of conjugacy classes of types for $\mathfrak{B}$ which are defined on maximal compact subgroups of G. On the other hand, let $\mathcal{I}(G)$ denote the set of *inertial types*: these are the restrictions to the inertia subgroup I_F of the relevant L-parameters in $\mathcal{L}(G)$. It is then a formal matter to show the existence of a unique surjective map $\mathrm{iner} : \mathcal{A}(G) \to \mathcal{I}(G)$ such that, for any map T which assigns to each representation π in $\mathrm{Cusp}(G)$ a type contained in π, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Cusp}(G) & \xrightarrow{\mathrm{rec}} & \mathcal{L}(G) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Res}^{W_F'}_{I_F}} \\ \mathcal{A}(G) & \xrightarrow[\mathrm{iner}]{} & \mathcal{I}(G) \end{array}$$

This map iner is the *(cuspidal) inertial Langlands correspondence* for G. An alternative point of view on the unicity of types is that it guarantees that iner behaves in the best

[4]In particular, if Σ is a datum defining a type $(J_\Sigma, \lambda_\Sigma)$ for a cuspidal block $\mathfrak{B}$ of $\mathrm{Rep}(\mathbf{SL}_N(F))$ then if $\mathfrak{B}$ is not essentially tame then it no longer needs to be the case that J_Σ is equal to its own projective normalizer.

[5]This is a technical condition, which is vacuous if $\mathbf{G}$ is F-quasi split.

way possible, by providing a uniform relationship between the fibres of iner and the fibres of rec. The simplest case of this is for $\mathbf{GL}_N(F)$, where Paskunas' results show that, for any $\varphi \in \mathcal{L}(G)$, there is a canonical bijection $\mathrm{iner}^{-1}(\varphi|_{I_F}) \to \mathrm{rec}^{-1}(\varphi)$.

Theorem 5 ([Lat18])**.** *For $G = \mathbf{SL}_N(F)$, assuming that one restricts attention to the essentially tame cuspidal representations of G and their corresponding L-parameters, the inertial Langlands correspondence has finite fibres. Given an essentially tame L-parameter $\varphi \in \mathcal{L}(G)$, there is a canonical surjective map $\mathrm{iner}^{-1}(\varphi|_{I_F}) \to \mathrm{rec}^{-1}(\varphi)$, each of the fibres of which is of cardinality equal to the ramification degree of φ.*

Theorem 6 ([Lat17])**.** *Let G be an arbitrary reductive p-adic group, and restrict attention to the regular (in the sense of [DR09]) depth-zero cuspidal representations of G and their corresponding L-parameters, which are precisely the parameters in $\mathcal{L}(G)$ with trivial restriction to any pro-p subgroup of W_F. Then the inertial Langlands correspondence has finite fibres.*

If one denotes by $\mathcal{A}'(G)$ the set of conjugacy classes of minimal K-types (as defined in [MP94]) of depth-zero, then there are canonical surjective maps $R : \mathcal{A}(G) \to \mathcal{A}'(G)$ and $\mathrm{iner}' : \mathcal{A}'(G) \to \mathcal{I}(G)$ such that $\mathrm{iner} = \mathrm{iner}' \circ R$. For any $\varphi \in \mathcal{I}(G)$, one has a canonical bijection $\mathrm{iner}'^{-1}(\varphi|_{I_F}) \to \mathrm{rec}^{-1}(\varphi)$. Thus the order of the fibre $\mathrm{iner}^{-1}(\varphi|_{I_F})$ is equal to $\sum_{(G_x,\sigma)} S(G_x,\sigma)$, where (G_x,σ) ranges over the minimal K-types in the fibre $\mathrm{iner}'^{-1}(\varphi_{I_F})$ and $S(G_x,\sigma)$ denotes the number of isomorphism classes of irreducible representations in $\mathrm{Ind}_{G_x}^{K}\ \sigma$, where K denotes the maximal compact subgroup of the G-normalizer of G_x.

4. Current and future work

4.1. Further branching rules and cases of the unicity of types

Currently, my main project is joint work with Monica Nevins, aiming to generalize the results of my thesis to obtain branching rules for, and establish the unicity of types for, essentially tame cuspidal representations of arbitrary p-adic groups. We currently have substantial results in this direction, and are able to prove the following two results:

Theorem 7. *Suppose that $G = \mathbf{G}(F)$ is an arbitrary reductive p-adic group and $\mathfrak{B}$ is an essentially tame cuspidal block whose irreducible objects are toral representations[6]. Then the unicity of types holds for $\mathfrak{B}$*

Theorem 8. *Suppose that $\mathbf{G}$ is a semisimple and simply connected group of rank at most 2 and $\mathfrak{B}$ is an essentially tame cuspidal block of $\mathrm{Rep}(G)$. Then, for any maximal compact subgroup K of G, there exist at most finitely many isomorphism classes of types defined on K.*

Note that in the latter result we are not able to completely establish unicity; this is due to the problem that there are a small number of components of $\pi_\Sigma|_K$ which may only feasibly be studied in terms of the representation theory of the reductive quotient of K

[6] An essentially tame cuspidal representation $\pi = \pi_\Sigma$ arises from a datum Σ which, amongst other things, defines an increasing sequence of twisted Levi subgroups $\mathbf{G}^i$ of $\mathbf{G}$ which splits over a tamely ramified extension of F; we say that π is *toral* if the smallest subgroup $\mathbf{G}^0$ of these is a torus.

by its pro-p radical. For arbitrary G, this quotient can be an arbitrary finite group of Lie type, and our results reduce the problem of unicity to some explicit questions on the representation theory of finite groups of Lie type which we are currently unable to solve.

We expect the condition that the rank of $\mathbf{G}$ is at most 2 to be far less serious. Our approach is to parametrize the components of $\pi_\Sigma|_K$ in terms of the vertices of the Bruhat–Tits building of G. We then construct two regions of the building in which we may apply different arguments to rule out the possibility that the corresponding components are types; one region gives rise to components which must be contained in non-cuspidal representations, while the other gives rise to components which must be contained in some cuspidal representation $\pi' \not\simeq \pi$. While we are currently only able to show that these regions overlap to cover all but a compact subset of the building (which contains finitely many vertices) when the rank of $\mathbf{G}$ is at most 2, we expect that with slight generalizations of our constructions we should be able to show this for arbitrary $\mathbf{G}$. It is also likely that the condition that $\mathbf{G}$ be semisimple and simply connected is not required: in a way completely analogous to the one used in [Lat17] one should be able to use relationships between the affine and enlarged buildings of G to remove this assumption.

4.2. Failures of multiplicity 1 in the unicity of types

In the previous results on the unicity of types by Henniart, Paskunas, and myself, it was always the case that, for a given maximal compact open subgroup $K \subset G$, and a cuspidal block $\mathfrak{B}$ of $\mathrm{Rep}(G)$, there is at most one conjugacy class of types defined on K. My work with Monica Nevins has shown that this need not be the case: for example, if K denotes the non-hyperspecial maximal compact subgroup of $G = \mathbf{Sp}_4(F)$ then there exist toral essentially tame cuspidal representations $\pi = \pi_\Sigma$ of G whose restriction to K contains two non-conjugate types, one of which is the obvious type $\mathrm{Ind}_{J_\Sigma}^K\ \lambda_\Sigma$, while the other is $\mathrm{Ind}_{{}^gJ_\Sigma}^K\ {}^g\lambda_\Sigma$, where $g \in G\backslash N_G(J_\Sigma)$ preserves the containment of J_Σ in K. While this does not contradict the unicity of types (since still every known type arises from a renormalization of one of the known constructions of types), it is somewhat surprising. It appears that the existence of such failures of multiplicity 1 is closely related to a representation being contained in the image of the Langlands transfer from an endoscopic group for G. It would be very interesting to understand precisely how the theory of types interacts with endoscopy; and in particular to understand sufficient conditions on the datum Σ for the resulting representation π_Σ to lie in the image of an endoscopic transfer.

4.3. Breuil–Mézard conjectures

The other project on which I have been working is an attempt to generalize Shotton's proof of the mod-ℓ Breuil–Mézard conjecture for $\mathbf{GL}_N(F)$ [Sho16] to depth-zero representations of arbitrary unramified p-adic groups[7]. That is, one should be able to make use of my results on the inertial Langlands correspondence for tame L-parameters to set up an analogous conjecture relating congruences between tamely ramified Galois representations

[7] Or, more precisely, to TRSELPs for such groups, as defined by DeBacker and Reeder [DR09].

with congruences between representations of parahoric subgroups of G, and to perform the local calculations required for the proof of such a conjecture.

References

[Ber84] J. N. Bernstein, *Le "centre" de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32. MR 771671 (86e:22028)

[BK93a] Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993. MR 1204652 (94h:22007)

[BK93b] ______, *The admissible dual of* $\mathrm{SL}(N)$*. I*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 2, 261–280. MR 1209709 (94a:22033)

[BK94] ______, *The admissible dual of* $\mathrm{SL}(N)$*. II*, Proc. London Math. Soc. (3) **68** (1994), no. 2, 317–379. MR 1253507 (94k:22035)

[BK98] ______, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634. MR 1643417 (2000c:22014)

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR 1944572 (2004i:11052)

[Cas73] William Casselman, *The restriction of a representation of* $GL_2(k)$ *to* $\mathrm{GL}_2(\mathcal{O})$, Math. Ann. **206** (1973), 311–318. MR 0338274 (49 #3040)

[DR09] Stephen DeBacker and Mark Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901. MR 2480618

[How77] Roger E. Howe, *Tamely ramified supercuspidal representations of* Gl_n, Pacific J. Math. **73** (1977), no. 2, 437–460. MR 0492087

[Lat15] Peter Latham, *Unicity of types for supercuspidal representations of p-adic* $\mathbf{SL}_2$, J. Number Theory **162** (2015), 376–390.

[Lat17] ______, *The unicity of types for depth-zero supercuspidal representations*, Represent. Theory **21** (2017), 590–610. MR 3735454

[Lat18] ______, *On the unicity of types in special linear groups*, Manuscripta Math. **157** (2018), no. 3-4, 445–465. MR 3858412

[Mor99] Lawrence Morris, *Level zero* **G***-types*, Compositio Math. **118** (1999), no. 2, 135–157. MR 1713308 (2000g:22029)

[MP94] Allen Moy and Gopal Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1-3, 393–408. MR 1253198 (95f:22023)

[Nev13] Monica Nevins, *Branching rules for supercuspidal representations of* $\mathrm{SL}_2(k)$*, for k a p-adic field*, J. Algebra **377** (2013), 204–231. MR 3008903

[Nev14] ______, *On branching rules of depth-zero representations*, J. Algebra **408** (2014), 1–27. MR 3197168

[Pas05] Vytautas Paskunas, *Unicity of types for supercuspidal representations of* GL_N, Proc. London Math. Soc. (3) **91** (2005), no. 3, 623–654. MR 2180458 (2007b:22018)

[Sho16] Jack Shotton, *The breuil–Mézard conjecture when* $\ell \neq p$, Duke Math. J. (to appear) (2016).

[Ste08] Shaun Stevens, *The supercuspidal representations of p-adic classical groups*, Invent. Math. **172** (2008), no. 2, 289–352. MR 2390287 (2010e:22008)

[Yu01] Jiu-Kang Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622 (electronic). MR 1824988 (2002f:22033)