Types in special linear groups and the inertial Langlands correspondence

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Basic objects

- F a p-adic field: a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.
- $\mathcal{O}$ the ring of integers of F.
- W_F the Weil group of F: this is certain subgroup of the absolute Galois group of F which arises naturally in local class field theory.
- I_F the inertia subgroup of W_F. This is the maximal compact subgroup of W_F.
- $G = \mathbf{G}(F)$ the group of points of a connected reductive group over F.

The local Langlands correspondence

Conjecture (Local Langlands conjecture). *The smooth finite-dimensional (complex) representations of W_F are naturally parametrized by the smooth irreducible (complex) representations of G, as G ranges over all such groups.*

This is useful as it can be viewed as a "uniformization" process: the theory on the Galois representation side of things is difficult and does not seem to admit a single approach which encompasses all Galois representations. On the p-adic group side of things, one is able to study representations in a uniform manner – one such way is the Bushnell–Kutzko theory of types.

Bushnell–Kutzko types

Rough definition: For π a supercuspidal representation of G, a *type* (J, λ) for π is an irreducible representation λ of a compact open subgroup J of G which identifies π as closely as it is possible to.

More formally: (J, λ) is a type if, for any smooth irreducible representation π' of G, one has $\mathrm{Hom}_J(\pi' \downarrow_J, \lambda) \neq 0 \Leftrightarrow \pi' \simeq \pi \otimes \omega$, for some unramified character ω. (A character is unramified if it is trivial on all compact subgroups).

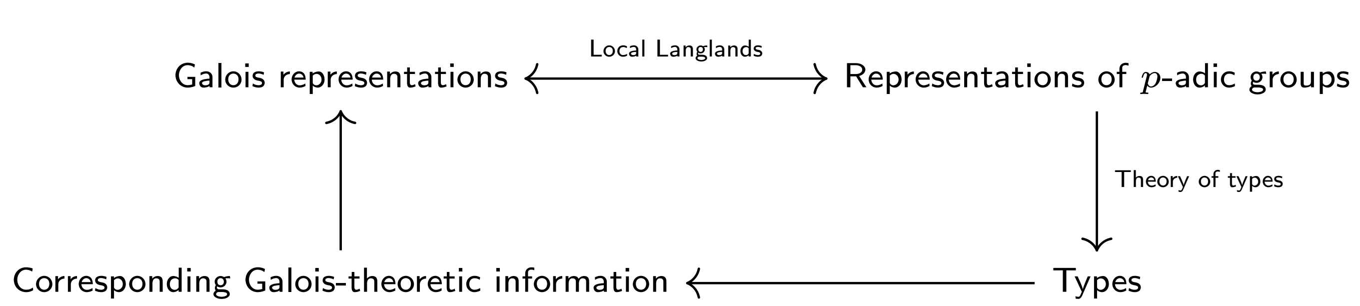
Theorem (Bushnell–Kutzko). *Let π be a supercuspidal representation of either $\mathbf{GL}_N(F)$ or $\mathbf{SL}_N(F)$. Then there exists a type for π.*

These types are given an *explicit* construction, and are called *(maximal) simple types*. Two simple types (J, λ) and (J', λ') for π and π', respectively, are then conjugate if and only if $\pi' \simeq \pi \otimes \omega$, for some unramified character ω.

Corollary. *Supercuspidal representations of $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$ are compactly induced from open, compact-modulo-centre subgroups.*

The idea

Get new information about Galois representations by translating *explicit* results from the theory of types:



An obvious first question is what Galois-theoretic information should types correspond to?

Unicity of types

Morally, simultaneous restriction to all compact open subgroups of G corresponds to restriction to I_F (see, e.g. the two notions of unramified characters in class field theory).

Idea: For π supercuspidal, the set of types (K, τ) contained in π with K *maximal compact* should identify the Galois representation associated to π up to restriction to I_F. For this to work well, one needs the following:

Conjecture (Unicity of types in $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$). *If π is a supercuspidal representation of $\mathbf{GL}_N(F)$ or $\mathbf{SL}_N(F)$ and (K, τ) is a type for π with K maximal compact, then there is a simple type (J, λ) with $J \subset K$ such that $\tau \simeq \mathrm{Ind}_J^K\ \lambda$.*

In particular, for K fixed there exists a unique type for π of the form (K, τ).

This is a theorem for $\mathbf{GL}_N(F)$ due to Henniart and Paskunas. In this case, K is unique up to conjugacy.

In $\mathbf{SL}_N(F)$, there are now N conjugacy classes of maximal compact subgroups, which complicates matters. In particular, a representation can contain multiple conjugacy classes of types.

The method

We utilize the special relationship between $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$. The key step is:

Theorem. *Let π be a supercuspidal representation of $\mathbf{GL}_N(F)$ and $\bar{\pi}$ be an irreducible subquotient of $\pi \downarrow_{\mathbf{SL}_N(F)}$. If $\bar{\pi}$ contains a type $(\mathbf{SL}_N(\mathcal{O}), \bar{\tau})$, then $\mathrm{Ind}_{\mathbf{SL}_N(\mathcal{O})}^{\mathbf{GL}_N(\mathcal{O})}\ \bar{\tau}$ contains as a subrepresentation the unique type $(\mathbf{GL}_N(\mathcal{O}), \tau)$ contained in π.*

Proof sketch: One identifies the obvious candidate for τ in $\mathrm{Ind}_{\mathbf{SL}_N(\mathcal{O})}^{\mathbf{GL}_N(\mathcal{O})}\ \bar{\tau}$, and calls it Ψ, say. This is clearly close to being a type, but it is hard to rule out a few possibilities. Assume for contradiction that Ψ is not isomorphic to τ. Then there are a number of cases, but one can build on Paskunas' results for $\mathbf{GL}_N(F)$ to obtain a contradiction in each case. In most cases, the approach is to show that Ψ is either a type, or contained in a non-cuspidal representation of $\mathbf{GL}_N(F)$, which is easy to rule out.

Main results

From the above theorem, the following is essentially immediate, by using the relationships between simple types in $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$ established by Bushnell and Kutzko:

Corollary. *The unicity of types conjecture is true for $\mathbf{SL}_N(F)$.*

There is one remaining question: $\mathbf{SL}_N(F)$ has multiple conjugacy classes of maximal compact subgroups, how many conjugacy classes of types does a given supercuspidal representation π of $\mathbf{SL}_N(F)$ contain?

This has a very satisfying answer. To π we associate a natural constant e_π, called the *ramification degree* of π. This is the integer $1 \leq e_\pi \leq N$ such that, if $\tilde{\pi}$ is a supercuspidal representation of $\mathbf{GL}_N(F)$ containing π, then there are N/e_π characters χ of $F^\times$ such that $\tilde{\pi} \simeq \tilde{\pi} \otimes (\chi \circ \det)$.

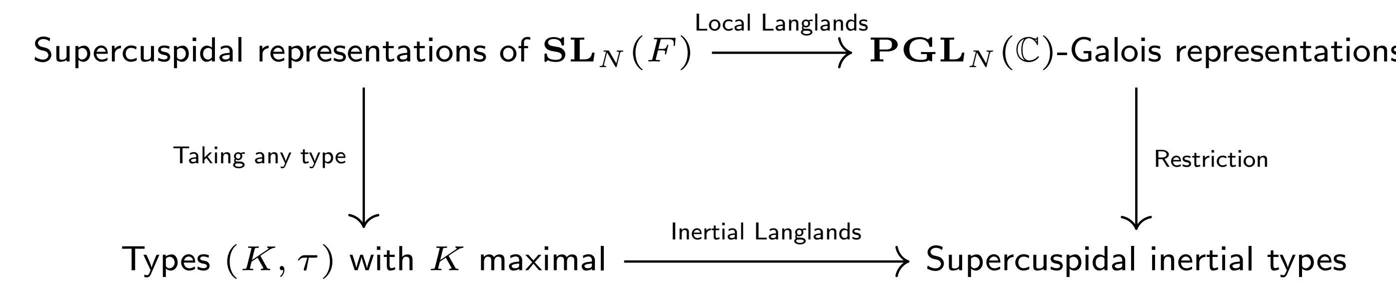
Theorem. *Up to conjugacy, the supercuspidal representation π contains precisely e_π types (K, τ) with K a maximal compact subgroup of $\mathbf{SL}_N(F)$.*

The proof of this result is based on an explicit evaluation of the embeddings of the groups J on which Bushnell and Kutzko's simple types are defined into the various maximal compact subgroups of $\mathbf{SL}_N(F)$.

Inertial Langlands correspondence

The unicity of types for $\mathbf{SL}_N(F)$ easily implies the existence of a unique inertial Langlands correspondence for supercuspidals. We say that a representation of I_F is a (supercuspidal) *inertial type* for $\mathbf{SL}_N(F)$ if it takes values in $\mathbf{PGL}_N(\mathbb{C})$ and extends irreducibly to a representation of W_F.

Theorem. *There exists a unique surjective map from the set of conjugacy classes of types (K, τ) contained in supercuspidal representations of $\mathbf{SL}_N(F)$ with K maximal compact to the set of supercuspidal inertial types for $\mathbf{SL}_N(F)$, compatible with the local Langlands correspondence in the sense that there is a commutative diagram*



Potential applications and future questions

Corresponding question for non-cuspidal representations: The methods used generalize to non-cuspidal representations, but there is a slight complication. I am hopeful that recent results of Nadimpalli will allow me to get around this.

$\mathbf{SL}_N(F)$-analogues of the Breuil–Mézard conjecture and related problems: The inertial Langlands correspondence seems to be essential to a number of problems in and around the Langlands program. It is possible that our results will have implications for these results, perhaps allowing for the $\mathbf{GL}_N(F)$ results to transfer over. This would, for example, avoid the issue of having to form deformation rings in $\mathbf{PGL}_N(\mathbb{C})$ if one wished to naively copy proofs.

Existence of automorphic representations of $\mathbf{SL}_N(F)$ with prescribed ramification: Corresponding problem for $\mathbf{GL}_N(F)$ was dealt with by Conley; our results should allow this to be transferred over.

Unicity for depth zero representations: Another class of representations for which there is a sufficiently well-developed theory of types are the depth zero supercuspidals – those with a vector fixed by the pro-unipotent radical of a maximal parahoric subgroup. I expect that such representations should contain a unique archetype, regardless of the group G. This is currently work in progress.

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