NON-CONVEX UTILITY MAXIMIZATION IN GAUSSIAN MISO BROADCAST AND INTERFERENCE CHANNELS

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ABSTRACT

Utility (e.g., sum-rate) maximization for multiantenna broadcast and interference channels (with one antenna at the receivers) is known to be in general a non-convex problem, if one limits the scope to linear (beamforming) strategies at transmitter and receivers. In this paper, it is shown that, under some standard assumptions, most notably that the utility function is decreasing with the interference levels at the receivers, a global optimal solution can be found with reduced complexity via a suitably designed branch-and-bound method. Although infeasible for real-time implementation, this procedure enables a non-heuristic and systematic assessment of suboptimal techniques. In addition to the global optimal scheme, a real-time suboptimal algorithm, which generalizes the well-known distributed pricing techniques, is also proposed. Finally, numerical results are provided that compare global optimal solutions with suboptimal (pricing) techniques for sum-rate maximization problems, affording insight into issues such as the robustness against bad initializations in real-time suboptimal strategies.

Index Terms— Nonconvex optimization, branch-and-bound, interference channel, multiple-input single-output channel

1. INTRODUCTION

Precoding and power control are well studied strategies that support high spectral efficiency in wireless network with multiple antenna transceivers, when channel state information (CSI) is available at the transmitters. Several system-wide objective functions have been considered in the literature for precoding and power control optimization of broadcast channels (BCs) and interference channels (ICs). Some of these problems are convex, for example power minimization [1] or SINR balancing for the multiple-input single-output (MISO) BC [2], and thus solvable with standard techniques in reasonable (polynomial) time. However, in general, the problems at hand are non-convex. Unlike convex problems, non-convex problems typically do not afford efficient (i.e., polynomial-time) algorithms that are able to achieve global optimality [9]. For example, it is known that the weighted sum-rate maximization (WSRM) in parallel IC channels, where interference from other users is treated as noise (a non-convex problem) is NP-hard [3] (this result extends also to BC as a special case).

In this paper, we address the global minimization of a system-wide, in general non-convex, cost function with respect to the transmit covariance matrices $\{\mathbf{Q}_k\}$. Among global techniques, branch-and-bound (BB) algorithms are methods to solve general non-convex problems [5], producing an ε-suboptimal feasible point. BB methods have been already introduced to solve non-convex power control problems, although so far only multi-user single-input single-output systems have been addressed in [6] and references therein.

In this paper, we propose a novel BB framework for global optimization of a problem formulation that includes, for instance, MISO BC and IC WSRM with general convex power constraint. The proposed BB approach is based on the observation that a fairly general set of cost functions that arise in communication's problems, albeit non-convex, possess a *Partly Convex-Monotone* [7] structure. This structure is satisfied whenever one can identify a suitable set of interference functions $\mathbf{f_i}(\{\mathbf{Q}_k\})$, for which the following hold: (*i*) The cost function is *convex* in the transmit covariance matrices $\{\mathbf{Q}_k\}$ once the interference functions $\mathbf{f_i}(\{\mathbf{Q}_k\})$ are fixed; (*ii*) The cost function is *monotone* in the interference functions $\mathbf{f_i}(\{\mathbf{Q}_k\})$. We design the BB scheme to exploit the *Partly Convex-Monotone* structure of the problem. Branching is performed in a reduced space (of the size of the set of all feasible interference level vectors $\mathbf{f_i}(\{\mathbf{Q}_k\})$), instead of the original feasible space (of the size of the set of all feasible covariance matrices $\{\mathbf{Q}_k\}$). Bounding is efficiently carried out by solving only convex optimization problems.

In addition to the reduced-space BB method, we propose a suboptimal algorithm that attains quasi-optimal performance with polynomial complexity. This algorithm reduces to the distributed pricing scheme of [4], when applied to sum-rate maximization problems. Numerical results are provided to compare the global optimal solution based on BB, the suboptimal (pricing) technique and the non-linear dirty-paper coding scheme.

Notation: The Boldface is used to denote matrices (uppercase) and vectors (lowercase); $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and the Hermitian transpose, respectively; $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix; $E[\cdot]$ denotes the expectation operator. Moreover, given a vector $\mathbf{x}$ we address its l-th component as $[\mathbf{x}]_l$, and the vector inequality $\mathbf{x} \preccurlyeq \mathbf{y}$ means that $[\mathbf{x}]_l \leq [\mathbf{y}]_l\ \forall l$. Finally, unless otherwise specified, we address the set of covariance matrices $\{\mathbf{Q}_k\}_{k=1}^K$ as $\{\mathbf{Q}_k\}$.

2. SYSTEM MODEL

We model a multi-user communication system consisting of K transmitter-receiver pairs (or users). The k-th user has N_k transmit antennas and one receive antenna (MISO system). The signal at the

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k-th receiver is given by

$$y_k = \underbrace{\mathbf{h}_{kk}\mathbf{x}_k}_{signal} + \underbrace{\sum\nolimits_{j\neq k} \mathbf{h}_{jk}\mathbf{x}_j}_{interference} + w_k \quad (1)$$

where $\mathbf{x}_k \in \mathbb{C}^{N_k \times 1}$ is the k-th transmitter's signal, $\mathbf{h}_{kj} \in \mathbb{C}^{1\times N_k}$ accounts for the channel response of the MISO link between the k-th transmitter and the j-th receiver, and $w_k \in \mathbb{C}^{1\times 1}$ models the additive white Gaussian noise (AWGN) at k-th receiver: $w_k \sim \mathcal{CN}(0, \sigma_k^2)$. Assuming capacity-achieving Gaussian codebooks, we define the correlation matrix of the k-th transmitted signal as $\mathbf{Q}_k = E\left[\mathbf{x}_k\mathbf{x}_k^H\right]$. While model (1) accounts for an IC, a BC can be obtained as a special case by setting $\mathbf{h}_{jk} = \mathbf{h}_k \ \forall j, k$.

2.1. Problem Formulation

Due to multi-user interference, the system performance depends on the transmission strategy of every user, i.e., on the set of covariance matrices $\{\mathbf{Q}_k\}$. We consider the minimization with respect to $\{\mathbf{Q}_k\}$ of a system-wide cost function f (to be defined below) under a general convex set constraints $\mathcal{Q}$:

$$\min_{\{\mathbf{Q}_k\}\in\mathcal{Q}} f\left(\{\mathbf{Q}_k\}, \mathbf{f_i}\left(\{\mathbf{Q}_k\}\right)\right) \quad (2)$$

By defining a set of L auxiliary variables $\mathbf{i}$, problem (2) can be recast in the equivalent form

$$(\mathrm{P}) \quad \min_{\{\mathbf{Q}_k\}\in\mathcal{Q},\mathbf{i}} f\left(\{\mathbf{Q}_k\}, \mathbf{i}\right) \quad s.t. \ \ \mathbf{i} = \mathbf{f_i}\left(\{\mathbf{Q}_k\}\right) \quad (3)$$

The equivalence means that if $\{\mathbf{Q}_k^*\}$ is a solution to (2), then $\left(\{\mathbf{Q}_k^*\}, \mathbf{f_i}\left(\{\mathbf{Q}_k^*\}\right)\right)$ is a solution to (3). Conversely, if $\left(\{\mathbf{Q}_k^*\}, \mathbf{i}^*\right)$ is a solution to (3), then $\{\mathbf{Q}_k^*\}$ is a solution to (2).
We further make the following assumptions:

- A1 The L interference levels are given by the real vector function $\mathbf{f_i}\left(\{\mathbf{Q}_k\}\right)$, *affine* with respect to $\{\mathbf{Q}_k\}$, that is bounded in the L-dimensional rectangle $\left[\mathbf{i}^{\min}, \mathbf{i}^{\max}\right] \subset \mathbb{R}^L$ (i.e., the l-th component satisfies $i_l^{\min} \leq \left[\mathbf{f_i}\left(\{\mathbf{Q}_k\}\right)\right]_l \leq i_l^{\max}$ for $l = 1, \ldots, L$). For instance, we typically have $L = K$ and the interference level at the k-th receiver reads $\left[\mathbf{f_i}\left(\{\mathbf{Q}_k\}\right)\right]_k = \sum_{j=1,j\neq k}^{K} \mathbf{h}_{jk}\mathbf{Q}_j\mathbf{h}_{jk}^H$;
- A2 The cost function $f\left(\{\mathbf{Q}_k\}, \mathbf{i}\right)$ is a real scalar function that is: *continuous* in $\left(\{\mathbf{Q}_k\}, \mathbf{i}\right)$; *monotonic increasing*[1] with respect to $\mathbf{i} \in \left[\mathbf{i}^{\min}, \mathbf{i}^{\max}\right]$ for fixed $\{\mathbf{Q}_k\} \in \mathcal{Q}$; *convex* with respect to $\{\mathbf{Q}_k\}$ for fixed $\mathbf{i} \in \left[\mathbf{i}^{\min}, \mathbf{i}^{\max}\right]$;
- A3 The set $\mathcal{Q}$ is *closed* and *convex*. For example, $\mathcal{Q}$ may be the set of positive semidefinite covariance matrices $\{\mathbf{Q}_k \succcurlyeq \mathbf{0}\}$ satisfying the generalized power constraints $\sum_{k=1}^{K} \mathrm{Tr}\left(\mathbf{A}_{k,\ell}\mathbf{Q}_k\right) \leq P_\ell$ for $\ell = 1, \ldots, D$, where $\{\mathbf{A}_{k,\ell}\}$ are positive semidefinite matrices (possibly $\mathbf{A}_{k,\ell} = \mathbf{0}$ if k-th user doesn't belong to ℓ-th constraint) and $\{P_\ell\}$ are non-negative coefficients. This definition includes some important special cases studied in the literature, such as per-antenna, per-group of antennas, the classical sum-power or the interference constraints in cognitive radio scenarios.

Throughout the paper, we refer to problem (3) as (P). We next provide examples of problems that satisfy these assumptions.

[1] By suitably modifying the same arguments, the proposed framework can handle an analogous but more general case where $f\left(\{\mathbf{Q}_k\}, \mathbf{i}^+, \mathbf{i}^-\right)$ results *monotone increasing* in $\mathbf{i}^+$ and *monotone decreasing* in $\mathbf{i}^-$.

2.2. Examples

An example of cost function included in our framework is the α-fairness criterion [8]: $f\left(\{\mathbf{Q}_k\}\right) = \sum_{k=1}^{K} -w_k f_\alpha\left(r_k\left(\{\mathbf{Q}_k\}\right)\right)$, where w_k is a positive constant, f_α is an increasing strictly concave function defined as

$$f_\alpha(r) := \begin{cases} \log r & \text{if } \alpha = 1 \\ (1-\alpha)^{-1} r^{1-\alpha} & \text{otherwise} \end{cases}, \quad (4)$$

and $r_k\left(\{\mathbf{Q}_k\}\right)$ is the k-th user's rate, which depends on covariance matrices $\{\mathbf{Q}_k\}$ and on the *channel scenario*. The α-fairness criterion reduces, as special cases, to the WSRM problem ($\alpha = 0$) or the *proportional fairness* problem ($\alpha = 1$). Moreover, as α becomes large, it converges to the *max–min fairness* problem [8].
In the following we present some examples of channel scenario that can be addressed within our framework:

- *Parallel MISO IC*: The k-th transmitter operates over L_C parallel subcarriers, it has power constraint P_k and has knowledge of channels $\{\mathbf{h}_{jkl}\}$ for $j = 1, \ldots, K$ and $\forall l$.
  The minimization of the (p, α)-fairness cost function reads

$$\min_{\{r_k\},\{\mathbf{Q}_{kl}\succcurlyeq\mathbf{0}\}} \sum_{k=1}^{K} -w_k f_\alpha(r_k) \quad (5)$$
$$s.t. \begin{cases} r_k \leq \sum\limits_{l=1}^{L_C} \log\left(1+\dfrac{\mathbf{h}_{kkl}\mathbf{Q}_{kl}\mathbf{h}_{kkl}^H}{\sigma_{kl}^2+\sum_{j=1,j\neq k}^{K}\mathbf{h}_{jkl}\mathbf{Q}_{jl}\mathbf{h}_{jkl}^H}\right) & \forall k \\ \mathrm{Tr}\left(\sum_{l=1}^{L_C}\mathbf{Q}_{kl}\right) \leq P_k & \forall k \end{cases}$$

  Defining $\left[\mathbf{f_i}\left(\{\mathbf{Q}_{kl}\}\right)\right]_{l+L_C(k-1)} = \sum_{j=1,j\neq k}^{K} \mathbf{h}_{jkl}\mathbf{Q}_{jl}\mathbf{h}_{jkl}^H$ $\forall k, l$ and $\mathcal{Q} = \left\{\mathbf{Q}_{kl} \succcurlyeq \mathbf{0} \ \forall k, l \mid \mathrm{Tr}\left(\sum_{l=1}^{L_C}\mathbf{Q}_{kl}\right) \leq P_k \ \forall k\right\}$, problem (5) is recast into (P). Also, $\mathbf{f_i}\left(\{\mathbf{Q}_{kl}\}\right) \in \left[\mathbf{0}, \mathbf{i}^{\max}\right]$ where $\mathbf{i}^{\max}$ is a proper upper bound on interference, always available since $\mathcal{Q}$ is bounded (finite power constraints).
- *Parallel MISO BC*: This scenario is obtained from (5) by setting $\mathbf{h}_{jkl} = \mathbf{h}_{kl} \ \forall j$ and imposing a sum-power constraint $\mathrm{Tr}\left(\sum_{k=1}^{K}\sum_{l=1}^{L_C}\mathbf{Q}_{kl}\right) \leq P_{tot}$.

3. PROBLEM SOLUTION VIA BRANCH-AND-BOUND

In this section we show that, adopting standard BB techniques (see [5] and [7]), problem (P) can be optimally solved by means of an efficient BB that exploits the structure dictated by assumptions (A1-A3). The BB algorithm is fully characterized by two procedures: *branching* and *bounding*. These are iteratively performed until the solution's suboptimality falls below some prescribed accuracy ε. In the following we explicitly tailor those procedures to problem (P) and we show convergence of the proposed BB algorithm to the global optimal solution of (P). For readability's sake, we define $\mathbf{Q} := \{\mathbf{Q}_k\}$[2] and we address an interval as $\mathcal{M} := [\mathbf{a}, \mathbf{b}]$, meaning that $\mathbf{c} \in \mathcal{M} \Leftrightarrow [\mathbf{a}]_l \leq [\mathbf{c}]_l \leq [\mathbf{b}]_l$ for $l = 1, \ldots, L$.

3.1. Branching Procedure

A partition set $\mathcal{P}_t$ of rectangles $\{\mathcal{M}\}$ in the space $\mathbb{R}^L$, each labeled with a *lower* $\mathfrak{L}_{\mathfrak{B}}(\mathcal{M})$ and *upper* $\mathfrak{U}_{\mathfrak{B}}(\mathcal{M})$ bounds, is given. By splitting a rectangle that satisfies $\mathcal{M}_t \in \arg\min_{\mathcal{M}\in\mathcal{P}_t} \mathfrak{L}_{\mathfrak{B}}(\mathcal{M})$ in J non-overlapping sub-rectangles $\left\{\hat{\mathcal{M}}_t\right\}$ (i.e., $\bigcap_{j=1}^{J} \hat{\mathcal{M}}_t^{(j)} = \emptyset$ and

[2] Here and in the following, the expression $\mathbf{Q} \in \mathcal{Q}$ stands for $\{\mathbf{Q}_k\} \in \mathcal{Q}$.

$\bigcup_{j=1}^{J}\hat{\mathcal{M}}_t^{(j)} = \mathcal{M}_t$), the enhanced partition $\mathcal{P}_{t+1} \triangleq \{\mathcal{P}_t \backslash \mathcal{M}_t\} \cup \left\{\hat{\mathcal{M}}_t\right\}$ is obtained. Lower and upper bounds for each sub-rectangle in $\left\{\hat{\mathcal{M}}_t\right\}$ are then obtained via the following *bounding procedure*.

3.2. Bounding Procedure

Exploiting the *Partly Convex-Monotone* structure of problem (P), for every rectangle $\mathcal{M} = \left[\mathbf{i}^{\min}, \mathbf{i}^{\max}\right] \in \mathcal{P}_t$, a *lower bound* $\mathfrak{L}_{\mathfrak{B}}(\mathcal{M})$ is evaluated by solving the following problem:

$$\mathfrak{L}_{\mathfrak{B}}(\mathcal{M}) :\ = \min_{\mathbf{Q}\in\mathcal{Q}} f\left(\mathbf{Q}, \mathbf{i}^{\min}\right) \quad (6)$$
$$s.t.\ \mathbf{i}^{\min} \preccurlyeq \mathbf{f}_{\mathbf{i}}(\mathbf{Q}) \preccurlyeq \mathbf{i}^{\max}.$$

Thanks to assumptions (A1-A3) two fundamental results can be verified: (*i*) problem (6) is *convex* since the cost function $f(\mathbf{Q}, \mathbf{i})$ is convex for a fixed $\mathbf{i}$ and the constraints form a convex set, (*ii*) using standard convex optimization arguments, it can be shown that this bounding procedure satisfies the natural condition:

$$\mathcal{M}' \subset \mathcal{M} \Rightarrow \mathfrak{L}_{\mathfrak{B}}(\mathcal{M}') \geq \mathfrak{L}_{\mathfrak{B}}(\mathcal{M}). \quad (7)$$

Moreover, denoting with $\mathbf{Q}^{(\mathfrak{L}_{\mathfrak{B}})}$ the optimal solution of problem (6), a valid upper bound $\mathfrak{U}_{\mathfrak{B}}(\mathcal{M})$ is obtained by evaluating the function at $\mathbf{Q}^{(\mathfrak{L}_{\mathfrak{B}})}$, i.e., $\mathfrak{U}_{\mathfrak{B}}(\mathcal{M}) :\ = f\left(\mathbf{Q}^{(\mathfrak{L}_{\mathfrak{B}})}, \mathbf{f}_{\mathbf{i}}\left(\mathbf{Q}^{(\mathfrak{L}_{\mathfrak{B}})}\right)\right)$.
Finally, the algorithm checks if the prescribed accuracy is met (i.e., if $\min \mathfrak{U}_{\mathfrak{B}}(\mathcal{M}) - \min \mathfrak{L}_{\mathfrak{B}}(\mathcal{M}) \leq \varepsilon$) otherwise it goes back to the *branching procedure*.

3.3. Convergence Analysis

Here we proves convergence of the proposed BB algorithm.

Lemma 1 *The proposed BB algorithm (which is performed in the reduced space spanned by interference levels/variable* $\mathbf{i}$*), is convergent to a global optimal solution of problem* (P).

Proof. As explained above, since the chosen bounding procedure satisfyies (7), the BB algorithm generates a sequences of partition sets $\{\mathcal{M}_t\}$ collapsing to a point $\bigcap_{t\to\infty}\mathcal{M}_t = \mathbf{i}^*$ (recall that $\mathcal{M}_t$ is the rectangle selected for splitting at the t-th branching iteration). In order to prove convergence we need to show that, as the size of rectangle $\mathcal{M}_t$ gets smaller, $\mathfrak{U}_{\mathfrak{B}}(\mathcal{M}_t) - \mathfrak{L}_{\mathfrak{B}}(\mathcal{M}_t)$ is also sufficiently small. The proof follows standard arguments [5]. This is shown in Appendix. ■

3.4. Broadcast WSRM Example

Considering the BC WSRM scenario (see Sec.2.2), for a given interval $\mathcal{M} = \left[\mathbf{i}^{\min}, \mathbf{i}^{\max}\right]$, the evaluation of a lower bound $\mathfrak{L}_{\mathfrak{B}}$ results in the following convex problem

$$\mathfrak{L}_{\mathfrak{B}}(\mathcal{M}) :\ = \min_{\{\mathbf{Q}_k \succcurlyeq \mathbf{0}\}} \sum_{k=1}^{K} -w_k \log\left(1+\frac{\mathbf{h}_k\mathbf{Q}_k\mathbf{h}_k^H}{\sigma_k^2+i_k^{\min}}\right) \quad (8)$$
$$s.t.\ \begin{cases} \mathrm{Tr}\left(\sum_{k=1}^{K}\mathbf{Q}_k\right) \leq P_{tot} \\ i_k^{\min} \leq \mathbf{h}_k\left(\sum_{j\neq k}\mathbf{Q}_j\right)\mathbf{h}_k^H \leq i_k^{\max} \quad \forall k \end{cases}.$$

Defining $\{\mathbf{Q}_k^*\}$ the optimal solution of (8), a valid upper bound is given by $\mathfrak{U}_{\mathfrak{B}}(\mathcal{M}) :\ = \sum_{k=1}^{K} -w_k \log\left(1+\frac{\mathbf{h}_k\mathbf{Q}_k^*\mathbf{h}_k^H}{\sigma_k^2+i_k}\right)$ where $i_k = \mathbf{h}_k\left(\sum_{j\neq k}\mathbf{Q}_j^*\right)\mathbf{h}_k^H\ \forall k$.

4. SUBOPTIMAL SOLUTION

While the proposed BB algorithm always converges to the global optimal solution and has reduced complexity with respect to a general-purpose implementation of BB, it is still feasible only for offline simulation. In this section we propose a suboptimal algorithm with polynomial complexity that extends the distributed pricing schemes of [4] to the more general class of problem (P).
Exploiting the *Partly Convex-Monotone* structure, problem (3) can be equivalently reformulated as the non-convex problem:

$$\min_{\mathbf{i}\in\mathcal{M}_0} \sup_{\boldsymbol{\lambda}} \left[\min_{\mathbf{Q}\in\mathcal{Q}}\left[f(\mathbf{Q}, \mathbf{i}) + \boldsymbol{\lambda}^T\mathbf{f}_{\mathbf{i}}(\mathbf{Q})\right] - \boldsymbol{\lambda}^T\mathbf{i}\right]. \quad (9)$$

where $\boldsymbol{\lambda}$ is the Lagrange multiplier associated to the affine constraint $\mathbf{i} = \mathbf{f}_{\mathbf{i}}(\mathbf{Q})$. Building on (9), in the following table we formalize the proposed suboptimal algorithm.

Algorithm 1 - Suboptimally solve problem (P)

0: *Set* $\varepsilon_{\boldsymbol{\lambda}}, \varepsilon_{\mathbf{i}}$
1: *Initialize* $\boldsymbol{\lambda} = \widehat{\boldsymbol{\lambda}}$
2: *Initialize* $\mathbf{i} = \widehat{\mathbf{i}}$
3: *Evaluate* $\mathbf{Q}^* = \arg\min_{\mathbf{Q}\in\mathcal{Q}} f\left(\mathbf{Q}, \widehat{\mathbf{i}}\right) + \widehat{\boldsymbol{\lambda}}^T\mathbf{f}_{\mathbf{i}}(\mathbf{Q})$
4: **If** $\left\|\widehat{\mathbf{i}} - \mathbf{f}_{\mathbf{i}}(\mathbf{Q}^*)\right\| > \varepsilon_{\mathbf{i}}$
5: *Update* $\widehat{\mathbf{i}} = \mathbf{f}_{\mathbf{i}}(\mathbf{Q}^*)$
6: *Go back to step 3*
7: **elseIf** $\left\|\widehat{\boldsymbol{\lambda}} - \left.\frac{\partial f(\mathbf{Q},\mathbf{i})}{\partial \mathbf{i}}\right|_{\mathbf{Q}=\mathbf{Q}^*, \mathbf{i}=\widehat{\mathbf{i}}}\right\| > \varepsilon_{\boldsymbol{\lambda}}$
8: *Update* $\widehat{\boldsymbol{\lambda}} = \left.\frac{\partial f(\mathbf{Q},\mathbf{i})}{\partial \mathbf{i}}\right|_{\mathbf{Q}=\mathbf{Q}^*, \mathbf{i}=\widehat{\mathbf{i}}}$
9: *Go back to step 2*
10: **end**

Since a stationary point of this algorithm fulfills the *necessary* Karush-Kuhn-Tucker (KKT) conditions of problem (3), if the algorithm converges, it attains a local optimal point of problem (3).
It is worth noticing that, by specializing our framework to the case when the cost function $f(\mathbf{Q}, \mathbf{i})$ is the WSRM (i.e., $f_\alpha(r_k) = r_k$ $\forall k$ in (5)) the Lagrangian multiplier $\boldsymbol{\lambda}$ plays the role of the *interference prices* defined in the distributed pricing algorithm [4]. Thus Algorithm 1 can be seen as a generalization of distributed pricing technique with an arbitrarily cost function and arbitrary *interference* functions (satisfying assumptions A1-A3).
Finally, since the problem at hand is non-convex, initialization of the parameters $\boldsymbol{\lambda}$ and $\mathbf{i}$ results crucial for performances and convergence. In Sec.5 we assess the performances of this technique in relation to the global optimal solution evaluated via BB algorithm.

5. NUMERICAL RESULTS

We assess the performance of the two proposed techniques: a multi-user linear precoder optimized via (*i*) the efficient BB algorithm (BB - LB and UB); (*ii*) the suboptimal Algorithm 1.
We consider the sum-rate utility function (i.e., in (5), $f_\alpha(r_k) = r_k$ $\forall k$ and $w_k = 1\ \forall k$). In BB algorithm, the solution's accuracy is $\varepsilon = 10^{-3}$, while, in Algorithm 1, we run two different price initializations ($\lambda_k = 10^{-5}\ \forall k$ and $\lambda_k = 1\ \forall k$) and for both we initialize $i_k = \sigma_k^2\ \forall k$, selecting $\sigma_k^2 = 1$ at each receive antenna.
Fig.1 shows the sum-rate versus the transmitting power for a single-

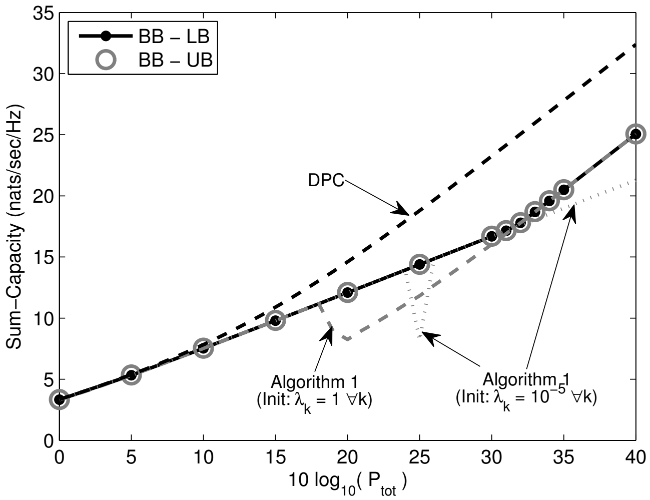

Fig. 1. Sum-Rate versus transmitting power for the single-carrier ($L_C = 1$) BC scenario. The figure compares the two proposed linear precoding algorithms: the optimal BB algorithm (BB - LB and UB) and the suboptimal Algorithm 1 considering two different initializations ($\lambda_k = 10^{-5}\ \forall k$ and $\lambda_k = 1\ \forall k$). The optimal *non-linear* DPC technique is also plotted as a reference.

carrier ($L_C = 1$) BC channel[3] where a $N = 4$ transmit antennas base-station serves $K = 4$ single-antennas users, subject to a sum-power constraint, $\mathrm{Tr}\left(\sum_{k=1}^{K} \mathbf{Q}_k\right) \leq P_{tot}$. The sum-capacity achieving *non-linear* technique Dirty Paper Coding (DPC) is also plotted as a reference.

It can be noticed that the suboptimal Algorithm 1, while showing near-optimal performance at several power levels, happens to be quite sensitive to initialization. For instance, initialization $\lambda_k = 10^{-5}\ \forall k$ yields a suboptimal slope in high power regime, as observed for $P_{tot}|_{dB} > 31$, and, at $P_{tot}|_{dB} = 25dB$ both initializations lead to highly suboptimal performances. A last observation pertains to the significant gains of *non-linear* DPC with respect to *linear* precoding at high power regime.

Finally, not to confuse the reader, since a utility function (sum-rate) instead of a cost function is plotted, in fig.1, the lower bound results as the maximum feasible value while the upper bound is the maximum upper bound among BB partitions.

6. CONCLUSIONS

This work presents a global optimization framework for the minimization of non-convex cost functions in MISO BC and IC channels. Examples are given for the general α-fairness optimization considering parallel IC and BC channels. Knowing the global optimal solution, even if impractical for real-time implementation, allows to assess the quality and to fine-tune (e.g., initialize) suboptimal schemes. In addition to the global optimal BB, we have proposed a real-time, hence suboptimal, algorithm that generalizes the pricing scheme of [4]. Extensions to MIMO networks are the subject of future work.

[3]Due to space limitation, Fig.1 channels realization is available at: `http://web.njit.edu/~mr227/papers/paper_BB_H_BC.mat`

7. APPENDIX

We need to prove that, as the maximum length of the edges of $\mathcal{M}_t$, denoted by $size(\mathcal{M}_t)$, goes to zero, the difference between upper and lower bounds uniformly converges to zero, i.e., $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \mathcal{M}_t \subseteq \mathcal{M}_0\ \ size(\mathcal{M}_t) \leq \delta \Longrightarrow \mathfrak{U}_{\mathfrak{B}}(\mathcal{M}_t) - \mathfrak{L}_{\mathfrak{B}}(\mathcal{M}_t) \leq \varepsilon$.

For each $\widehat{\mathbf{i}} \in \mathcal{M}_t = \left[\mathbf{i}^{\min}, \mathbf{i}^{\max}\right]$, we define the function $\mathfrak{F}\left(\widehat{\mathbf{i}}\right)$ as the result of the following constraint optimization problem:

$$\min_{\mathbf{Q} \in \mathcal{Q}} f\left(\mathbf{Q}, \widehat{\mathbf{i}}\right)$$
$$s.t.\ \ \mathbf{i}^{\min} \preccurlyeq \mathbf{f}_{\mathbf{i}}\left(\mathbf{Q}\right) \preccurlyeq \mathbf{i}^{\max}.$$

Using this notation, the lower bound in (6) is given by $\mathfrak{L}_{\mathfrak{B}} = \mathfrak{F}\left(\mathbf{i}^{\min}\right)$, while an upper bound is given by $\mathfrak{U}_{\mathfrak{B}} = \mathfrak{F}\left(\mathbf{i}^{\max}\right)$.

From jointly-continuity of the function $f\left(\mathbf{Q}, \widehat{\mathbf{i}}\right)$ with respect to $\left(\mathbf{Q}, \widehat{\mathbf{i}}\right)$ (assumption A2) and from the definition of $\mathfrak{F}\left(\widehat{\mathbf{i}}\right)$, we have that $\mathfrak{F}\left(\widehat{\mathbf{i}}\right)$ is continuous in the norm of $\widehat{\mathbf{i}}$ (i.e., $\left\|\widehat{\mathbf{i}}\right\|$). It follows that also $\mathfrak{L}_{\mathfrak{B}}$ and $\mathfrak{U}_{\mathfrak{B}}$ will result continue in $\left\|\widehat{\mathbf{i}}\right\|$, thus it holds

$$\forall \varepsilon\ \exists \delta \quad \left\|\mathbf{i}^{\max} - \mathbf{i}^{\min}\right\| \leq \delta \Longrightarrow \left|\mathfrak{F}\left(\mathbf{i}^{\max}\right) - \mathfrak{F}\left(\mathbf{i}^{\min}\right)\right| \leq \varepsilon$$

concluding the proof.

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