The Geometry of Risk

Can geometry tell us anything about finance? One hint that geometry may have something interesting to say comes from the relationship between risk and Pythagoras’s theorem.

To see this relationship first we need to decide what we mean by risk. Quantifying the risk of an investment is a complex and controversial subject. The first serious attempt at quantifying risk mathematically was made by Markowitz in 1952. We will follow his approach, and quantify the risk of an investment by using the standard deviation of its payoff (which we define to be the value of the investment at future time $T$).

Our question now becomes, how can we relate standard deviation and Pythagoras’s theorem? If $A$ and $B$ are uncorrelated random variables with finite standard deviation, then their variances are related by the formula

$$\text{Var}(A) + \text{Var}(B) = \text{Var}(A + B).$$

(1)

Therefore their standard deviations are related by the formula

$$\sigma(A)^2 + \sigma(B)^2 = \sigma(A + B)^2.$$  

(2)

We can now see that this looks very like Pythagoras’s theorem for the length, $\|a + b\|$ of the sum of two perpendicular vectors $a$ and $b$:

$$|a|^2 + |b|^2 = |a + b|^2.$$  

This establishes that the basic algebraic similarity between the properties of distance and the properties of standard deviation, but we can go further. Let

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1  Markowitz’s paper *Portfolio Selection* (1952) was a revolutionary attempt to apply mathematical techniques to the question of choosing optimal investments and lead to the theory known as modern portfolio theory or mean-variance analysis [3]. He won the 1990 Nobel Memorial Prize in Economics for this work.
us assume all non-trivial risk-free investments in the market have a non-zero standard deviation (and hence non-zero risk). With this assumption, it will be possible to prove that distance in Euclidean geometry and standard deviation in markets are mathematically identical. We will also see how to use this to obtain some interesting investment advice.

1 Diversification in a market of identical stocks

Before tackling the full theory, let’s examine an important special case which we will call a market of identical stocks. In a market of identical stocks we assume that \( n \) stocks are traded. We assume that these stocks are independent, and all the stocks financially equivalent to each one another in the sense that:

(i) they all have the same cost which we can suppose to be $1;

(ii) they all have the same probability distribution for their payoff.

We will suppose that this probability distribution has mean \( \mu \) and standard deviation \( \sigma > 0 \).

If one has an amount \( C \) to invest in a market of identical stocks, how should one invest? You might guess that it wouldn’t matter how you invested since the stocks are identical, but if you were to guess that, you would be wrong!

The trick is to realise it is possible to invest in a portfolio of stocks. We can invest an amount \( q_1 \) in stock 1, \( q_2 \) in stock 2, and so on. We represent this portfolio of stocks as a vector \( q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n \). We will see that investing in a portfolio of stocks rather than in just one specific stock allows you to reduce risk without reducing the expected payoff.

Let us calculated the expected payoff and its standard deviation. Write \( X_i \) for the random variable representing the payoff of stock \( i \). The value of our portfolio at time \( T \) is given by a random random variable, \( Q \), given by

\[
Q = q_1 X_1 + q_2 X_2 + \ldots + q_n X_n. \tag{3}
\]

Since each stock has cost 1 and expected payoff \( \mu \), the cost, \( C \), and expected payoff, \( P \), of the portfolio are given by

\[
C = q_1 + q_2 + \ldots + q_n. \tag{4}
\]
\[
P = (q_1 + q_2 + \ldots + q_n)\mu = C\mu.
\]

We conclude that the expected payoff for any portfolio is determined by \( C \).

The standard deviation is more interesting. Note that the standard deviation of \( q_i X_i \) is equal to \( q_i \sigma(X_i) = q_i \sigma \). We find from (3) and repeated applications of our analogue of Pythagoras’s Theorem, equation (??), that

\[
\sigma(Q)^2 = q_1^2\sigma^2 + q_2^2\sigma^2 + \ldots + q_n^2\sigma^2.
\]

\[
= |q|^2\sigma^2.
\]

We have shown:
Lemma 1. The standard deviation of payoff of the portfolio corresponding to \( q \) in a market of identical stocks is proportional to the length of \( q \). The expected payoff is determined by the cost.

Now consider a two dimensional market as illustrated in Figure 1. Each point in Figure 1 represents a portfolio with quantities \( (q_1, q_2) \). We have drawn lines indicating the cost of each portfolio. These lines are at 45 degrees because each stock in the market has the same cost: so the cost of the portfolio \( (1, 0) \) is equal to the cost of the portfolio \( (0, 1) \).

![Figure 1: Contour lines of constant cost, \( C \), when each stock costs $1. The optimal portfolio of cost 1 is labelled \( C^* \)](image)

Among all the portfolios of cost $1, which has the lowest risk? It is the portfolio on the line \( C = 1 \) closest to the origin, namely \( C^* = (\frac{1}{2}, \frac{1}{2}) \), as shown in Figure 1. By Pythagoras’ theorem, it has standard deviation

\[
\left( \sqrt{\frac{1}{2}^2 + \frac{1}{2}^2} \right) \sigma = \frac{1}{\sqrt{2}} \sigma
\]

We conclude that investing equal amounts in each stock allows us less risky than investing in just one stock even though both portfolios have identical expected return.

In a general when investing $1 in an \( n \)-dimensional market of identical stocks, one should divide your investment equally between all \( n \) stocks to give a portfolio

\[
C_* = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right).
\]

This will have payoff with standard deviation

\[
\frac{1}{\sqrt{n}} \sigma.
\]

Dividing your investment across multiple assets in this way is called diversification.

The three dimensional case is illustrated in Figure 2.
To see why this will be optimal, notice that the vector $C_*$ given by (??) connects the origin perpendicularly to the hyperplane $C = 1$. So if $P$ is another point on the hyperplane, we have by Pythagoras’s theorem that:

$$|P|^2 = |C_*|^2 + |P - C_*|^2 \geq |C_*|^2,$$

as illustrated in Figure 3.

We summarize our results.

**Theorem 1.** When investing an amount $C$ in a market of identical stocks, if you wish to minimize the risk of your investments (as measured by standard deviation) you should diversify your investment equally among all stocks. The expected payoff is determined entirely by the cost.

This gives a mathematical explanation of the standard investment advice “don’t put all your eggs in one basket”. This simple, but important, result is one of the major achievements of Markowitz’s theory.
General markets and Euclid’s axioms

In the general case, we will assume that we are given an $n$-dimensional set of random variables, $\mathcal{V}$, which we call the space of assets or the market. Each random variable in $\mathcal{V}$ represents the payoff distribution of an asset or a portfolio of assets. We will assume that $\mathcal{V}$ contains all the assets in our market and all possible portfolios.

We will allow portfolios to include either positive or negative quantities of an asset. Holding negative quantities of an asset can be interpreted as promising to give someone that asset in the future. This generalizes the familiar idea of using negative quantities in your accounts to represent owing someone money. In finance taking a negative position in a stock is called going short on the stock. A positive position is called a long position.

Our assumptions that $\mathcal{V}$ contains all portfolios including both long and short positions can be expressed mathematically by saying that we assume $\mathcal{V}$ is a vector space.

We would like to interpret the standard deviation geometrically as the distance from the origin in this more general set up. To do this we will need to assume that all non-zero assets have a finite, non-zero standard deviation. If we make this assumption, it turns out that the standard deviation has all the same properties as distance in Euclidean geometry.

One way to understand why this might be true, is to recall that Euclid developed his theory of geometry by assuming only a very few basic facts about points, lines and distances. These assumptions are called the axioms of Euclidean geometry. If we can show that the distance in our market shares these same basic properties with Euclidean distance, it will follow that every theorem in Euclidean geometry will be equally true of the standard deviation in our market.

At this point in our argument you could go off and brush up your ancient Greek to find out what Euclid’s axioms actually were. It’s easier, however, to see Euclid’s axioms as motivation and use instead a modern axiomatic treatment of geometry. In modern treatments, the standard approach is to notice that Euclidean space is not just a vector space, there is also a dot product that associates a scalar to the product $\mathbf{u} \cdot \mathbf{v}$. An inner product space is defined to be a real vector space equipped with a product that satisfies the same algebraic rules as the familiar dot product.\(^2\)

Our space $\mathcal{V}$ is, by assumption, a vector space. Our assumption that the standard deviation of any non-zero asset exists and is non-zero is enough to make $\mathcal{V}$ into an inner product space if we define the product of two assets using the covariance. It now follows from the axiomatic development of Euclidean

\(^2\)These algebraic properties are:

(i) (Symmetry) $\mathbf{u} \cdot \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$;

(ii) (Linearity) $(\mathbf{u} + \lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \lambda \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and $\lambda \in \mathbb{R}$;

(iii) (Positive Definiteness) $|\mathbf{v}|^2 := \mathbf{v} \cdot \mathbf{v} \geq 0$ with equality if and only if $\mathbf{X} = 0$.

The last property shows how we can define distance in terms of an inner product.
geometry that we can use theorems about distance in Euclidean geometry to prove results about the standard deviation, and hence the risk, of assets.

One result from Euclidean geometry that is particularly useful is that in an $n$-dimensional Euclidean space, we can find $n$ orthogonal vectors of length 1. If we translate this into the language of risk, we find that we can find $n$ uncorrelated portfolios of standard deviation 1. Let’s call these portfolios $e_1, e_2, \ldots, e_n$. If we treat these $e_i$ as the “basic” assets in our market, then any portfolio can be written as a combination $q_1 e_1 + q_2 e_2 + \ldots + q_n e_n$ and the risk will be given by the length of $|q|$. In other words: our market is exactly the same as standard Euclidean space, you just need to choose the right coordinates.

3 The two mutual fund theorem

In the last section, we focussed entirely on risk. We should also consider how much a portfolio costs and the mean payoff of a portfolio. Doing this will allow us to prove the celebrated two mutual fund theorem.

We’ll assume that the cost of a portfolio is just given by the total cost of the component assets scaled by the quantities. This means that the cost, $C$, is a linear function on the space of portfolios, so we can divide the space of portfolios into slices of equal cost as shown in Figure 4. Each slice is a $(n-1)$-dimensional hyperplane in the space of all portfolios.

![Figure 4: Hyperplanes of constant cost](image)

In Euclidean geometry, given a hyperplane, we can draw a unique line through the origin that intersects the plane at right angles. The line and plane intersect at the closest point on the hyperplane to the origin. This can be proved using Pythagoras’s theorem just as we did in the two dimensional case.

Translating this into the language of our market, this means that we can
find a portfolio that minimizes the standard deviation among all portfolios of cost \( C = 1 \). We will call this \( C_\ast \). This portfolio is shown in Figure 4.

Similarly, the mean payoff of a portfolio will be a linear function, \( P \), on the space of portfolios. So we can find a portfolio that minimizes the standard deviation among all portfolios of mean payoff \( P = 1 \). We call this \( P_\ast \).

Portfolios that have both cost \( C = 1 \) and mean payoff \( P = 1 \) will lie in the intersection of the hyperplanes \( C = 1 \) and \( P = 1 \). These portfolios form a \((n - 2)\)-dimensional subspace of the space of all portfolios which we will call \( L \). The portfolios matching any specified combination of cost and mean payoff will lie in a parallel \((n - 2)\)-dimensional subspace, \( L' \). Both \( L \) and \( L' \) subspaces are perpendicular to the plane \( OC_\ast P_\ast \) containing the origin , \( C_\ast \) and \( P_\ast \). The situation is illustrated in the three dimensional case in Figure 5.

![Figure 5: The geometry of the two mutual fund theorem](image)

It follows that if \( Q \) is closest point to the origin in the subspace \( L' \), then \( Q \) lies in the plane \( OC_\ast P_\ast \).

Translating this into the language of portfolios we have: The portfolio, \( Q \), of lowest risk among all portfolios, \( L \), with a given cost and given mean payoff lies on the plane \( OC_\ast P_\ast \) and can be written as a combination of the portfolios \( C_\ast \) and \( P_\ast \).

**Theorem 2** (Two mutual fund theorem). *Given a portfolio \( A \), we can always find a portfolio, \( Q \) with the same cost and mean payoff in the plane \( OC_\ast P_\ast \) that is no riskier. This portfolio can be written as a linear combination of the two portfolios \( C_\ast \) and \( P_\ast \) (or indeed any two linearly independent portfolios in the plane \( OC_\ast P_\ast \)).*
A mutual fund is a financial investment where all the money invested in the fund is pooled and invested in a specified way. For example, a mutual fund might be invested 75% in Apple and 25% in Bitcoin. The theorem above is called the Two Mutual Fund Theorem because we can design two funds that invests in stocks in fixed proportions that match the proportions in $C_*$ and $P_*$. The optimal investment strategy $Q$ in the theorem above then corresponds to investing in a combination of these two mutual funds.

One of the jobs of a fund manager is to design funds that will attract investors. Our theorem says that a fund manager need only create two mutual funds in order to attract all customers who believe our model is correct, irrespective of their risk preferences.

The two mutual fund theorem is closely related to the symmetry of our market. Notice that given a portfolio $A$, we can always find another portfolio $A'$ with the same mean payoff and the same mean cost by a reflection in the plane $OC_*P_*$. See Figure 6. The portfolio $A$ will be equal to $A'$ only if $A$ actually lies in the plane $OC_*P_*$. From the point of view of risk, cost and mean payoff, the portfolios $A$ and $A'$ are identical. So any financial question involving only standard deviation, cost and mean payoff that yields $A$ as a possible answer will also yield $A'$ as an answer. It follows that any question of this type that also has a unique answer must have an answer that lies in $OC_*P_*$. This includes the question of which portfolio minimizes standard deviation given a fixed cost and mean payoff, so this symmetry argument can be seen as a generalization of the two mutual fund theorem. An ambitious fund manager might try to think of other financial problems that their customers might want to solve in order to devise new, attractive financial products. Our result shows that they will need to consider market features beyond standard deviation, cost and mean payoff in order to this.\(^3\)

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\(^3\)The two mutual fund theorem was first proved by Merton in [4]. Our geometric proof is based on [2], as are our symmetry arguments. The symmetry argument was generalized to other markets in [1].

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Figure 6: A symmetry of the market
4 The efficient frontier

Because of their risk-minimizing properties, portfolios in the plane $OP^* R^*$ are called efficient portfolios. If we choose a fixed cost $C = 1$, then we obtain a line of $\ell$ of efficient portfolios of cost 1. This is illustrated in Figure 7. The mean payoff will vary linearly as one moves along $\ell$. The risk at each point in $\ell$ can be computed by intersecting the vertical plane through $\ell$ with the cone. This slice of the cone gives us a graph showing the relationship between mean payoff and risk for efficient portfolios, as illustrated in Figure ??.

Figure 7: A graph of the risk of each efficient portfolio gives a cone

This graph gives the minimum achievable risk for any given payoff in the market, the curve is called the efficient frontier. The efficient frontier is the iconic image of Markowitz’s theory of portfolio optimization. We have just proved using simple geometry that the shape of this curve is a conic section, specifically a hyperbola.

Figure 8: The efficient frontier

So it turns out that remarkably the ancient Greek theory of conic sections allows us to understand the relationship between risk and return in financial markets. Had Euclid known this, he could have been a millionaire.
References


