Orbifolds via defects

Nils Carqueville

Universität Wien

$$\begin{tikzcd}\textbf{spacetime} \arrow[r, "\textbf{QFT}"] & \textbf{algebra}\end{tikzcd}$$

Motivation: basic features of quantum physics

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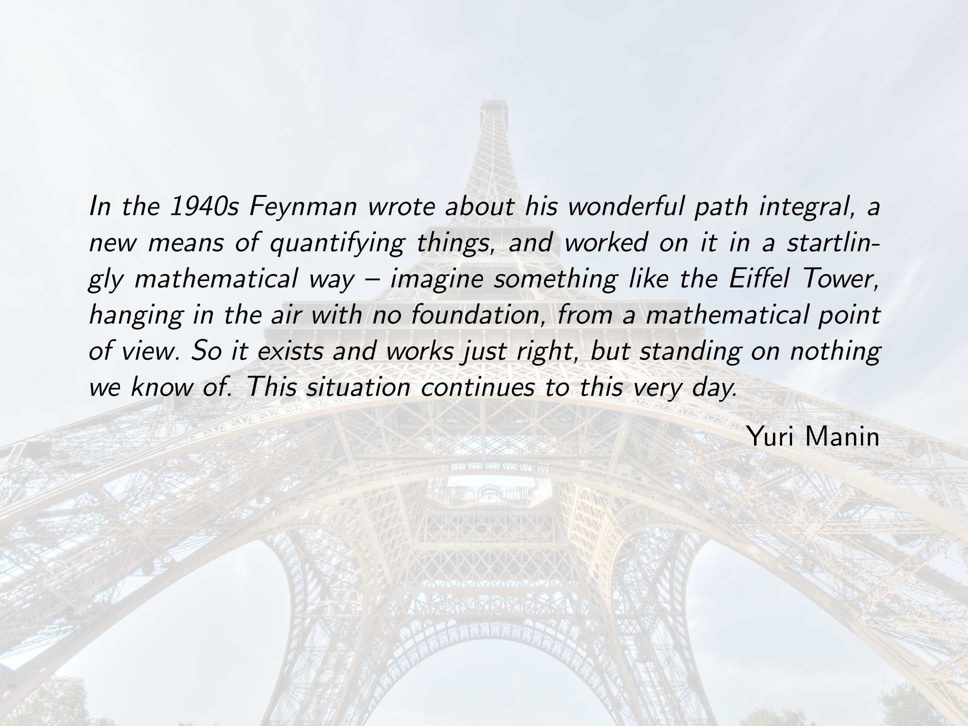
$$\overset{??}{=} \int \mathrm{d}y \int_{x(0)=y}^{x(t)=z} \mathcal{D}x\ \mathrm{e}^{-S[x]}\ \Psi(y)$$



In the 1940s Feynman wrote about his wonderful path integral, a new means of quantifying things, and worked on it in a startlingly mathematical way – imagine something like the Eiffel Tower, hanging in the air with no foundation, from a mathematical point of view. So it exists and works just right, but standing on nothing we know of. This situation continues to this very day.

Yuri Manin

PARIS


PARIS
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TQFT
??
??

$$\begin{tikzcd}\textbf{spacetime} \arrow[r, "\textbf{QFT}"] & \textbf{algebra}\end{tikzcd}$$

$$\textbf{spacetime} \supset \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \xrightarrow{\hspace{5cm}} \mathrm{Vect}_{\mathbb{k}} \subset \textbf{algebra}$$

$$\textbf{spacetime} \supset \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \xrightarrow{\quad\textbf{defect TQFT}\quad} \mathrm{Vect}_{\Bbbk} \subset \textbf{algebra}$$

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Let G be a group. A ***G*-representation** is a functor

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single object $*$ and $\mathrm{End}(*) = G$

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(Functoriality means $\rho(e) = \mathrm{id}_V$ and $\rho(gh) = \rho(g) \circ \rho(h)$ for all $g, h \in G$.)

Topological quantum field theory

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Atiyah 1988

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↗ orient. circles S^1 and surfaces with bdry./diffeom.

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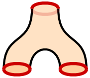

$$\longmapsto \left(C \otimes C \xrightarrow{\mu} C \right)$$

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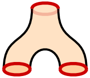

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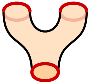

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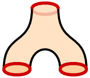

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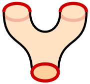

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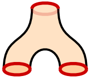

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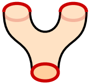

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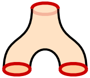
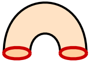


$=$



$\longmapsto \left(\langle -,- \rangle \colon C \otimes C \xrightarrow{\mathrm{tr}\circ\mu} \Bbbk\right)$

Atiyah 1988

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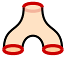

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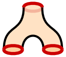

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Theorem. $\big\{$2d TQFTs$\big\} \cong \big\{$**commutative Frobenius algebras**$\big\}$

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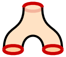

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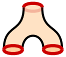

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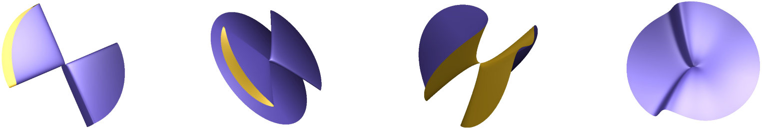
- $C = \mathbb{C}[x_1, \dots, x_n]/(\partial_{x_1} W, \dots, \partial_{x_n} W)$ ($\langle -,- \rangle$ from residue theory)



Defect TQFT

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$$\mathcal{Z}\colon\ \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\Bbbk}$$

Davydov/Kong/Runkel 2011

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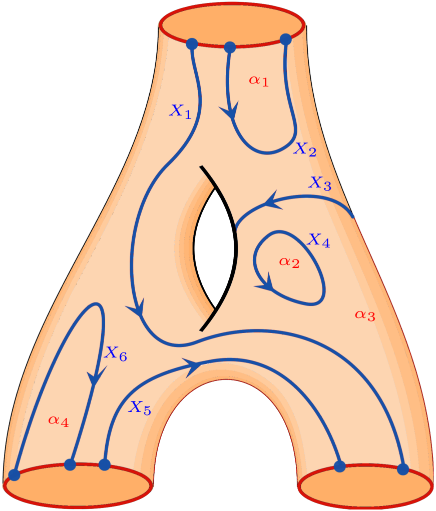

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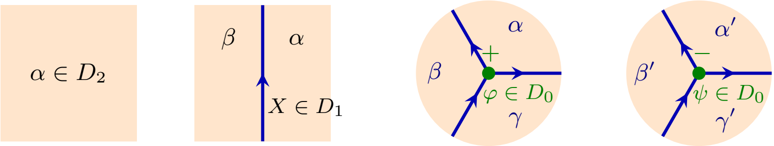
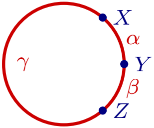
depending on **defect data** $\mathbb{D}$ consisting of:

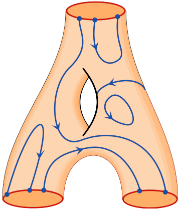
- sets D_j, whose elements decorate j-strata of bordisms
- rules how strata are allowed to meet:



objects:



morphisms:



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Examples of 2d defect TQFTs

Trivial defect TQFT $\mathcal{Z}^{\mathrm{triv}}$:

$D_2 := \{\mathbb{k}\}$

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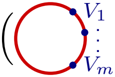

$$\mathcal{Z}^{\text{triv}}\Big(\quad \Big) \overset{\text{def}}{=} V_1 \otimes \cdots \otimes V_m$$

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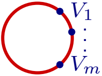
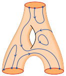
$\mathcal{Z}^{\text{triv}}\Bigg($  $\Bigg) \overset{\text{def}}{=} V_1 \otimes \cdots \otimes V_m$

$\mathcal{Z}^{\text{triv}}\Bigg($  $\Bigg) \overset{\text{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in $\mathrm{vect}_{\Bbbk}$)

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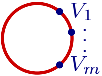
$\mathcal{Z}^{\mathrm{triv}}\Big($ 

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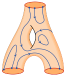
$\mathcal{Z}^{\mathrm{triv}}\Big($  $\Big) \overset{\mathrm{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in $\mathrm{vect}_{\Bbbk}$)

B-twisted sigma models:

complex manifolds and holomorphic vector bundles

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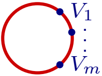
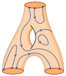
$\mathcal{Z}^{\mathrm{triv}}\Big($  $\Big) \overset{\mathrm{def}}{=} V_1 \otimes \cdots \otimes V_m$

$\mathcal{Z}^{\mathrm{triv}}\Big($  $\Big) \overset{\mathrm{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in $\mathrm{vect}_{\Bbbk}$)

B-twisted sigma models:
complex manifolds and holomorphic vector bundles

Landau-Ginzburg models:
isolated singularities and homological algebra

(more soon. . .)

State sum models

Input: Δ-separable symmetric Frobenius $\Bbbk$-algebra (A, μ, Δ)

Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

State sum models

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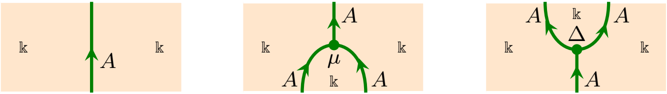

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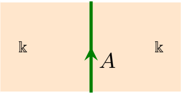

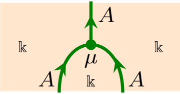

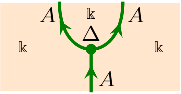

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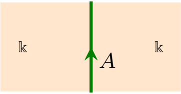

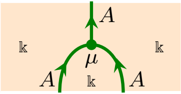

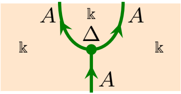

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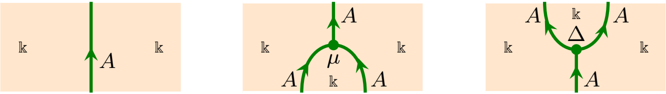

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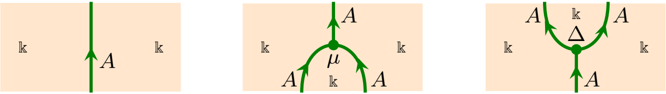

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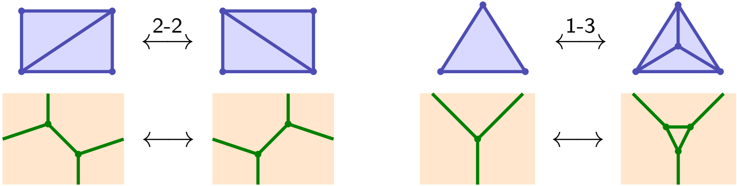

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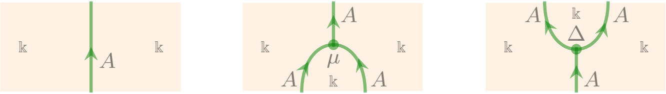

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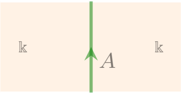

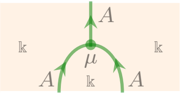

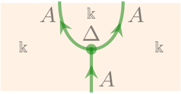

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No need to consider only algebras over $\mathbb{k}$!

Orbifolds

Definition. Let $\mathcal{Z}\colon \operatorname{Bord}_2^{\text{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}_{\Bbbk}$ be defect TQFT.

Carqueville/Runkel 2012

Orbifolds

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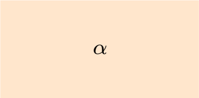
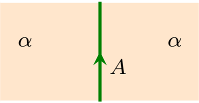
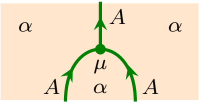
An **orbifold datum** for $\mathcal{Z}$ is $\mathcal{A} \equiv (\alpha, A, \mu, \Delta)$:



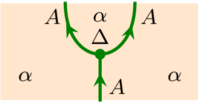
$\alpha \in D_2$



$A \in D_1$

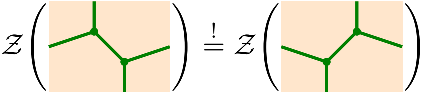


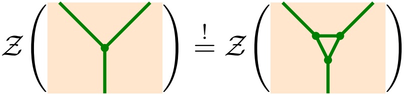
$\mu \in D_0$



$\Delta \in D_0$

such that *Pachner moves become identities* under $\mathcal{Z}$:





Carqueville/Runkel 2012

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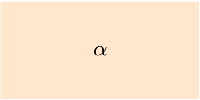
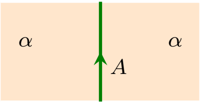


$\alpha \in D_2$



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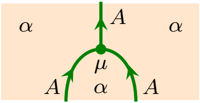
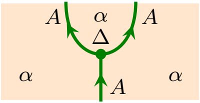


$\mu \in D_0$



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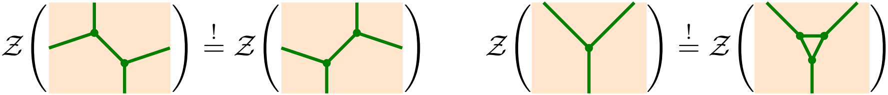

Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$

Carqueville/Runkel 2012

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Definition. Let $\mathcal{Z}\colon \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\Bbbk}$ be defect TQFT.

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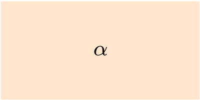

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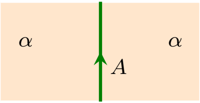

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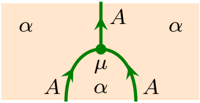
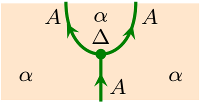


$\mu \in D_0$



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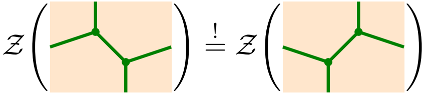

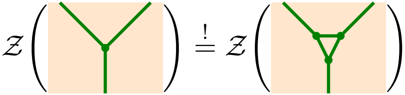

Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$ = **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}_{\Bbbk}$$

Carqueville/Runkel 2012

Algebraic characterisation

Theorem.

2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{B}_{\mathcal{Z}}$

Davydov/Kong/Runkel 2011

Algebraic characterisation

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Proof sketch:

- objects = elements of D_2 = 'theories'

Davydov/Kong/Runkel 2011

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Proof sketch:

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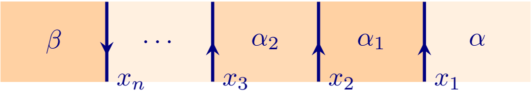

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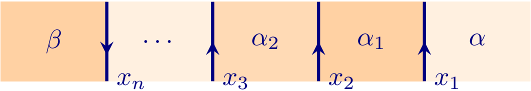
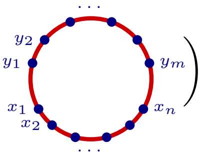


- 2-morphisms = 'junction fields':

$\mathrm{Hom}(X,Y) = \mathcal{Z}\Bigg($



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Davydov/Kong/Runkel 2011

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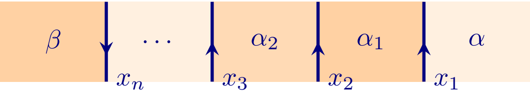

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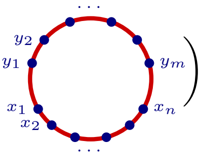

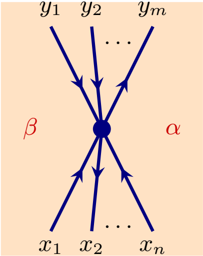

Davydov/Kong/Runkel 2011

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Davydov/Kong/Runkel 2011

Algebraic characterisation

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Examples.

- **vector spaces**: $\mathrm{Bvect}_{\Bbbk}$
  $*$, finite-dimensional $\Bbbk$-vector spaces, linear maps

Davydov/Kong/Runkel 2011

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Davydov/Kong/Runkel 2011

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- **B-twisted sigma models**
  Calabi-Yau varieties, Fourier-Mukai kernels, RHom
- **A-twisted sigma models**
  symplectic manifolds, Lagrangian correspondences, Floer homology
- **Landau-Ginzburg models**
  isolated singularities, matrix factorisations
- **differential graded categories**
  smooth and proper dg categories, dg bimodules, intertwiners
- **categorified quantum groups**
  weights, functors $\mathcal{E}_i, \mathcal{F}_j \ldots$, string diagrams. . .

Davydov/Kong/Runkel 2011, Carqueville 2016

Algebraic characterisation of orbifolds

Theorem.

2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{B}_{\mathcal{Z}}$

Davydov/Kong/Runkel 2011

Algebraic characterisation of orbifolds

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Lemma.

$\big\{$orbifold data for $\mathcal{Z}\big\} \cong \big\{\Delta$-separable symmetric Frobenius algebras in $\mathcal{B}_{\mathcal{Z}}\big\}$

Davydov/Kong/Runkel 2011

Algebraic characterisation of orbifolds

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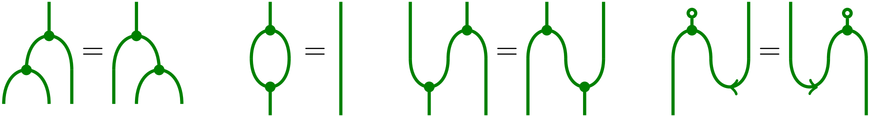

Davydov/Kong/Runkel 2011

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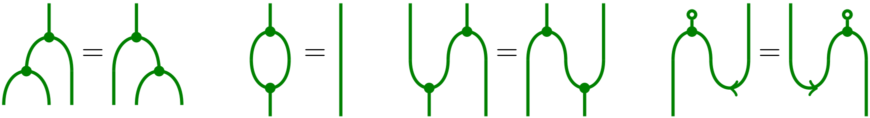
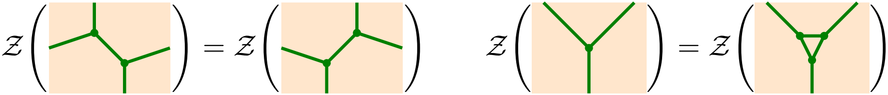


$$\iff$$



Davydov/Kong/Runkel 2011

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Davydov/Kong/Runkel 2011

Algebraic characterisation of orbifolds

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Orbifolds unify G-equivariantisations and state sum models.

Davydov/Kong/Runkel 2011, Fröhlich/Fuchs/Runkel/Schweigert 2009, Brunner/Carqueville/Plencner 2014

Orbifold equivalence

The **orbifold completion** of a pivotal 2-category $\mathcal{B}$ is a pivotal 2-category $\mathcal{B}_{\mathrm{orb}}$ with:

- *objects* = **Δ-separable symmetric Frobenius algebras** $A \in \mathcal{B}(\alpha, \alpha)$
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Carqueville/Runkel 2012

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Carqueville/Runkel 2012

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Corollary.

$$\alpha \sim \beta \implies \mathcal{B}\big(\gamma, \beta\big) \cong \mathcal{B}_{\mathrm{orb}}\big((\gamma, 1_\gamma), (\alpha, A)\big) \cong \mathrm{mod}(A) \quad \text{for all } \gamma \in \mathcal{B}.$$

$\implies$ Theory β determined by theory α and defect A.

Carqueville/Runkel 2012

Orbifolds of Landau-Ginzburg models

Theorem. There is a pivotal 2-category $\mathcal{LG}$ with:

– objects = isolated singularities $W \in \mathbb{C}[x_1, \dots, x_n]$ $(\dim \mathbb{C}[x]/(\partial_x W) < \infty)$

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Examples. $W_{\mathrm{A}_{n-1}} = x_1^n + x_2^2, \quad W_{\mathrm{D}_{n+1}} = x_1^n + x_1 x_2^2, \quad W_{\mathrm{E}_7} = x_1^3 + x_1 x_2^3$

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Examples. $\mathcal{D} = \left(\begin{smallmatrix} 0 & u^{n-i} \\ u^i & 0 \end{smallmatrix}\right)$ for u^n, $\mathcal{D} = \begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & y^2 & -x \\ x^2 & xy & 0 & 0 \\ xy^2 & -x^2 & 0 & 0 \end{pmatrix}$ for $x^3 + xy^3$

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Theorem. (**Orbifolds** in $\mathcal{LG}$)

$$\begin{aligned} x^k + xy^2 &\sim u^{2k} + v^2 && \big(\mathrm{D}_{k+1} \sim \mathrm{A}_{2k-1}\big) \\ x^3 + y^4 &\sim u^{12} + v^2 && \big(\mathrm{E}_6 \sim \mathrm{A}_{11}\big) \\ x^3 + xy^3 &\sim u^{18} + v^2 && \big(\mathrm{E}_7 \sim \mathrm{A}_{17}\big) \\ x^3 + y^5 &\sim u^{30} + v^2 && \big(\mathrm{E}_8 \sim \mathrm{A}_{29}\big) \end{aligned}$$

Carqueville/Murfet 2012, Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013

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x^5 y + y^3 & \sim & u^3 v + v^5 & \qquad \big(\mathrm{E}_{13} \sim \mathrm{Z}_{11}\big) \\
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Carqueville/Murfet 2012, Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

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Corollary. $\operatorname{hmf}\big(W^{(\mathrm{E}_6)}\big) \cong \operatorname{mod}\Big(\operatorname{hmf}\big(W^{(\mathrm{A}_{11})}\big) \xrightarrow{A\otimes(-)} \operatorname{hmf}\big(W^{(\mathrm{A}_{11})}\big)\Big)$ etc.

Carqueville/Murfet 2012, Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

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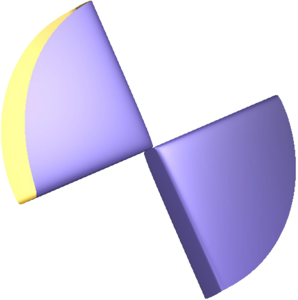
$\mathrm{A}_{11} : \Big\{u^{12} + v^2 = 0\Big\}$



$\neq$

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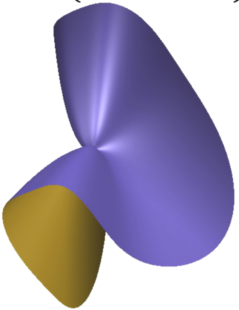

Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

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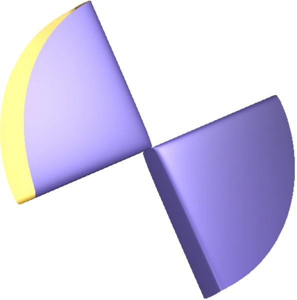

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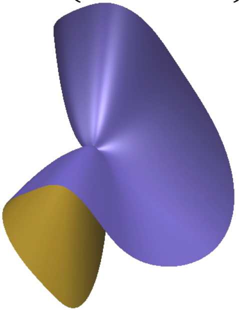

Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

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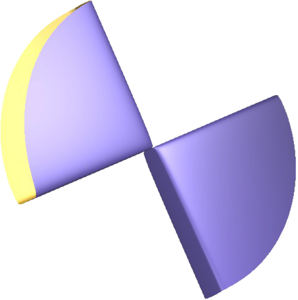

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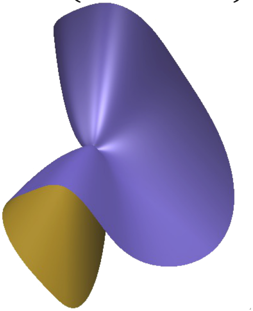
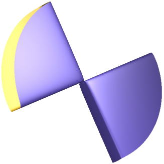


Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

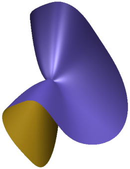
Summary so far

A_{11}



$\sim$

E_{6}



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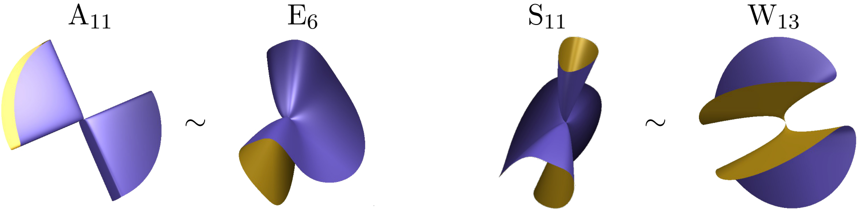

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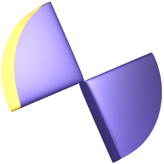
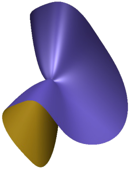
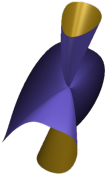


$\sim$

E_6

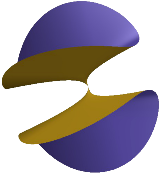


S_{11}



$\sim$

W_{13}



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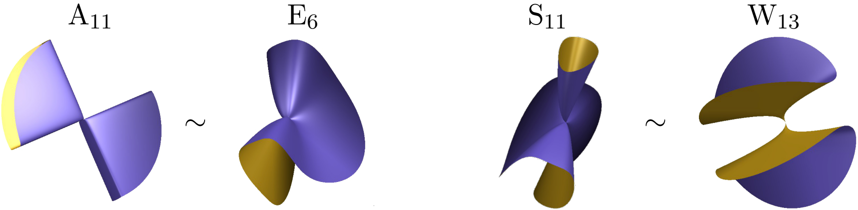

$$\mathcal{C}^G \;\cong\; \operatorname{mod}\left(\mathcal{C} \xrightarrow{A_G \otimes (-)} \mathcal{C}\right) \qquad\qquad \mathcal{Z}_A^{\mathrm{ss}} = \left(\mathcal{Z}^{\mathrm{triv}}\right)_A$$

2d orbifolds

- **encode triangulation invariance in algebraic structure**
- representation theory of algebras in 2-categories
- unify G-equivariantisation and state sum models
- give new relations in algebra and geometry

The **orbifold construction** *can be generalised to* n*-dimensional defect TQFTs*

$$\mathcal{Z}\colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}_{\Bbbk}$$

in any dimension $n \geqslant 1$.

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018

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- **triangulation invariance $\Longrightarrow$ algebraic structures**
  - $n = 2$: Frobenius algebras in 2-categories
  - $n = 3$: spherical fusion categories in 3-categories
- rich representation theory
- unify G-equivariantisation and state sum models
- ...

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018

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An **n-dimensional defect TQFT** is a symmetric monoidal functor

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Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–18

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that depends on **defect data** $\mathbb{D}$, consisting of:

- sets D_j, whose elements decorate j-strata of bordisms
- rules how strata are allowed to meet (defined recursively via cones and cylinders)

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–18

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Examples of 3d defect TQFTs.

- quantum **Chern-Simons theory** ($=$ Reshetikhin-Turaev theory $\mathcal{Z}^{\mathcal{C}}$)
  - $D_3 = \{$gauge group$\}$ (more generally: modular tensor category $\mathcal{C}$)
  - $D_2 = \{\Delta$-separable symmetric Frobenius algebras in $\mathcal{C}\}$
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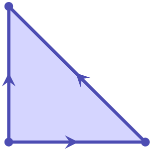
- **Rozansky-Witten theory** (conjecturally)
  - $D_3 = \{\text{holomorphic symplectic manifolds}\}$
  - $D_2 = \{\text{“generalised Landau-Ginzburg models”}\}$
  - $D_1 = \{\text{“fibred matrix factorisations”}\}$

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–18, Kapustin/Rozansky/Saulina 2009 + wip

Triangulations

standard n-simplex $\Delta^n := \left\{ \sum_{i=1}^{n+1} t_i e_i \;\middle|\; t_i \geqslant 0\,,\ \sum_{i=1}^{n+1} t_i = 1 \right\} \subset \mathbb{R}^{n+1}$

$\Delta^2 =$



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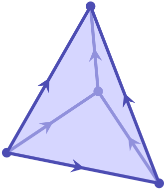

simplicial complex C is collection of simplices such that

- all faces of all $\sigma \in C$ are also in C
- $\sigma, \sigma' \in C \quad \Longrightarrow \quad \sigma \cap \sigma' = \emptyset$ or $\sigma \cap \sigma' =$ face

triangulation of manifold M is simplicial complex C with homeomorphism $\varphi\colon |C| \xrightarrow{\cong} M$

(details for smooth, oriented, ...)

Pachner moves

Let $\varphi\colon |C| \xrightarrow{\cong} M$ be triangulated n-manifold.

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A **Pachner move** “glues the other side of $\partial\Delta^{n+1}$ into M”:

$$M \longmapsto \left|\partial\Delta^{n+1} \setminus \mathring{F}\right| \cup_{\varphi|_{|\partial F|}} \big(M \setminus \varphi(|F|)\big)$$

Pachner 1991

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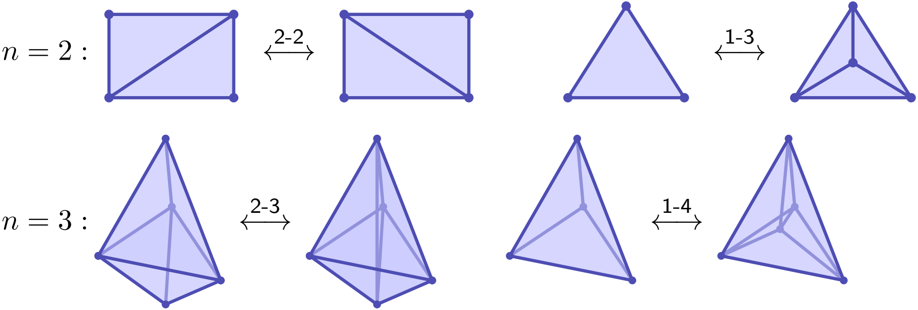

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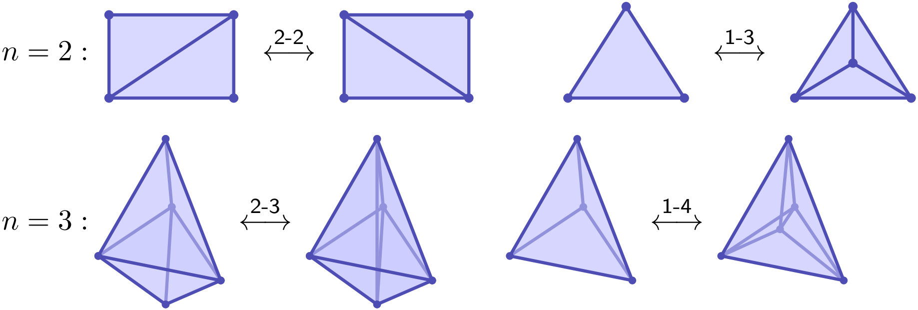

Theorem. If triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves between them.

Pachner 1991

Orbifolds in any dimension n

An **orbifold datum** $\mathcal{A}$ for $\mathcal{Z}\colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}_{\Bbbk}$

Carqueville/Runkel/Schaumann 2017

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Carqueville/Runkel/Schaumann 2017

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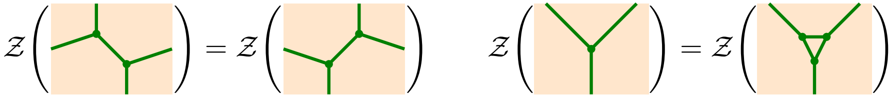

Carqueville/Runkel/Schaumann 2017

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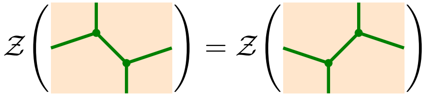

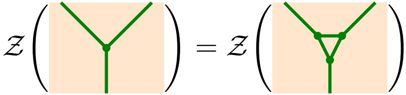

Definition & Theorem.

Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$

Carqueville/Runkel/Schaumann 2017

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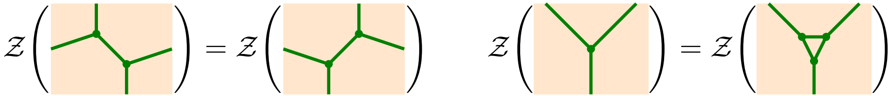
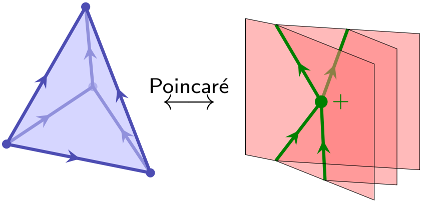


Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$ = **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_n \longrightarrow \mathrm{Vect}_{\Bbbk}$$

Carqueville/Runkel/Schaumann 2017

Orbifold datum $\mathcal{A}$ for $n = 3$



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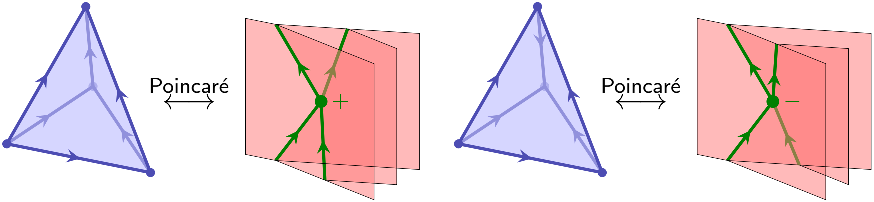

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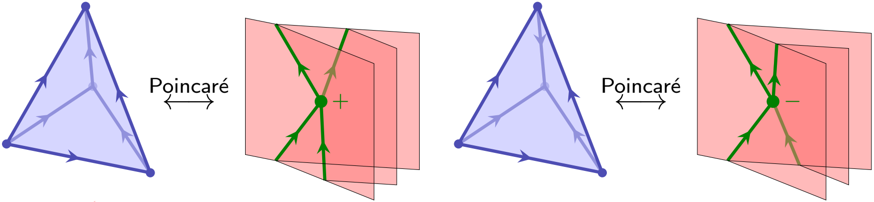

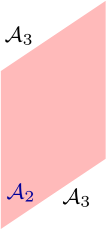

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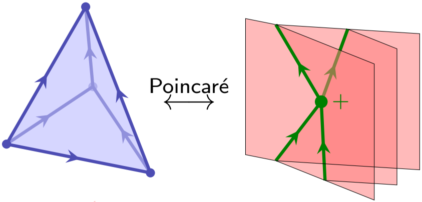

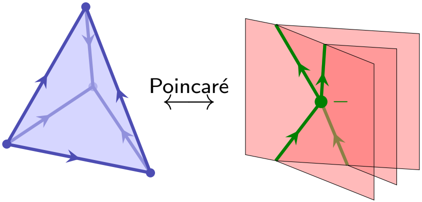

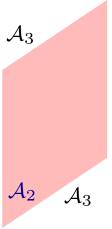

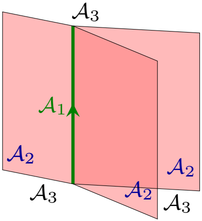

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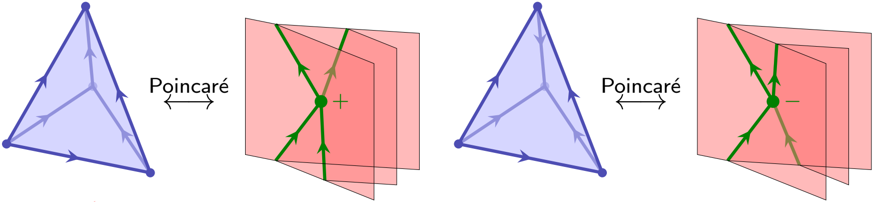

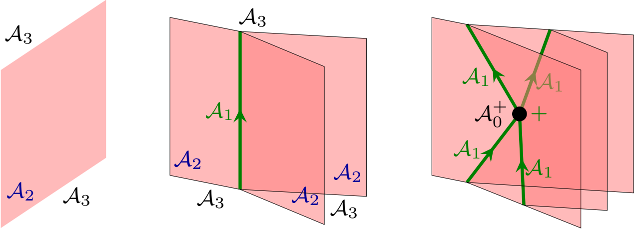

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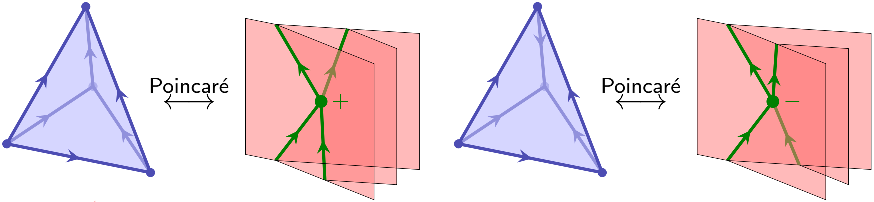

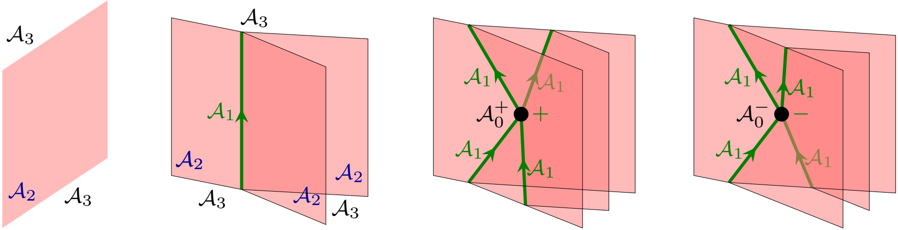

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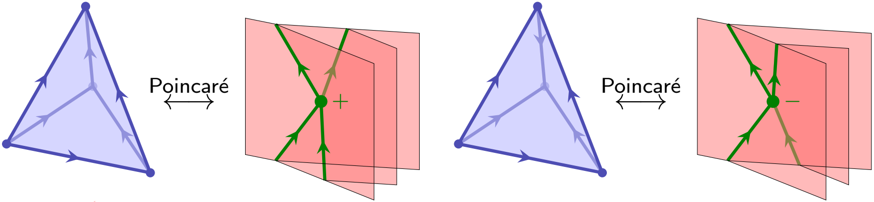

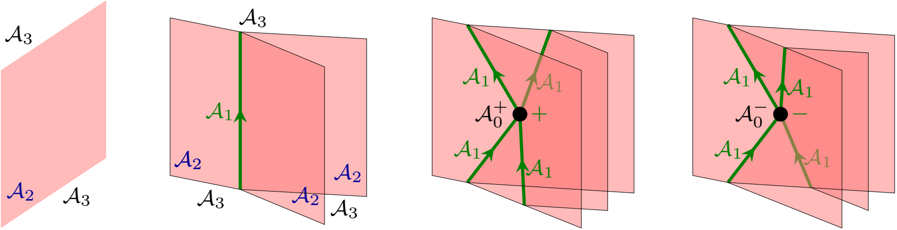

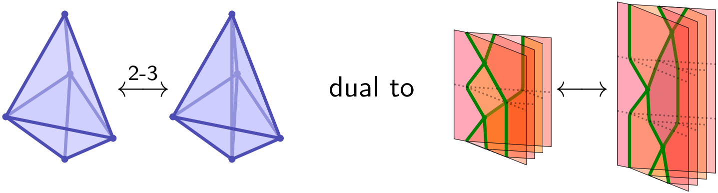

3d orbifolds

Theorem.

3d defect TQFT $\mathcal{Z}$ $\implies$ 3-category $\mathcal{T}_{\mathcal{Z}}$

Carqueville/Meusburger/Schaumann 2016

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Theorem.
Spherical fusion categories in $\mathcal{T}_{\mathcal{Z}}$ are orbifold data for $\mathcal{Z}$.

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018

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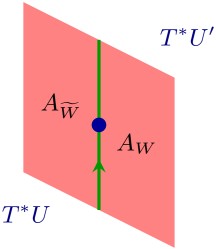
Upshot: theory well-developed and ready for applications

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018

Application 1: 3d TQFT $\supsetneq$ semisimplicity

Rozansky-Witten theory as 3-category $\mathcal{C}^{\mathrm{RW}}$ with duals:



objects: T^*U for complex manifolds U

1-morphisms $T^*U \longrightarrow T^*U'$ are *curved differential graded algebras* $A_W := \Big(\Omega^{0,\bullet}(U \times U'),\ \bar{\partial},\ W\Big)$

2- and 3-morphisms: $\mathbb{D}\big(A_W^{\mathrm{op}} \otimes A_{\widetilde{W}}\big)$

Theorem. $\mathrm{B}\mathcal{LG} \overset{U\,=\,\mathrm{pt}}{\subset} \mathcal{C}^{\mathrm{RW}}$ ("LG models are surface defects")

Rozansky-Witten 1996, Kapranov 1999, Kapustin-Rozansky 2009, Carqueville/Montiel Montoya 2018

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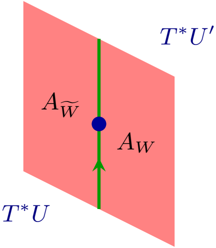

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Future:

- 3d TQFTs from tensor categories $\mathbb{D}\left(A_W^{\mathrm{op}} \otimes A_W\right)$
- representation theory of **non-semisimple tensor categories**
- new invariants of knots and surface embeddings

Rozansky-Witten 1996, Kapranov 1999, Kapustin-Rozansky 2009, Carqueville/Montiel Montoya 2018

Application 2: topological quantum computation

Interpretation of Reshetikhin-Turaev theory $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$:

- objects u_i in $\mathcal{C}$: anyonic quasiparticles in 2+1 dimensions
- $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}(\Sigma_{u_1,\dots,u_m})$: qubit storage on surface Σ with m anyons
- braiding matrices β_{u_i,u_j}: quantum gates
- $\big\langle \beta_{u_i,u_j} \big\rangle$ dense in $\mathrm{U}(N)$ for $N \gg 1$: *universal* quantum computations

Kitaev 1997, Freedman/Kitaev/Larsen/Wang 2001

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Fact. $\mathcal{C} =$ Ising category not universal.

Kitaev 1997, Freedman/Kitaev/Larsen/Wang 2001, Barkeshli/Jian/Qi 2012, Fuchs/Schweigert 2013

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- objects u_i in $\mathcal{C}$: anyonic quasiparticles in 2+1 dimensions
- $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}(\Sigma_{u_1,\dots,u_m})$: qubit storage on surface Σ with m anyons
- braiding matrices β_{u_i,u_j}: quantum gates
- $\langle \beta_{u_i,u_j} \rangle$ dense in $\mathrm{U}(N)$ for $N \gg 1$: *universal* quantum computations

Fact. $\mathcal{C} =$ Ising category not universal.
"Gauging" of S_2-symmetry of $\mathcal{C} \boxtimes \mathcal{C}$ is universal!

Kitaev 1997, Freedman/Kitaev/Larsen/Wang 2001, Barkeshli/Jian/Qi 2012, Fuchs/Schweigert 2013

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Conjecture. **Orbifolds of $\mathcal{Z}^{\mathcal{C}}$** construct universal quantum computers with larger qubit storages $\mathcal{Z}^{\mathcal{C}}(\Sigma_{u_1,\dots,u_m})$;
in particular

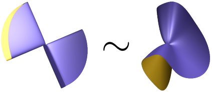
- $\rho \colon \mathrm{B}S_N \longrightarrow \mathrm{Bimod}_{\Bbbk}$ with $\rho(*) = \mathcal{C}^{\boxtimes N}$
- $\mathcal{C}$-$\mathcal{C}'$-bimodules with "invertible quantum bubble"

Kitaev 1997, Freedman/Kitaev/Larsen/Wang 2001, Barkeshli/Jian/Qi 2012, Fuchs/Schweigert 2013

The **orbifold construction** *can be generalised to n-dimensional defect TQFTs*

$$\mathcal{Z} \colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}_{\Bbbk}$$

in any dimension $n \geqslant 1$.



- **triangulation invariance $\Longrightarrow$ algebraic structures**
  - $n = 2$: Frobenius algebras in 2-categories
  - $n = 3$: spherical fusion categories in 3-categories
- rich representation theory
- unify G-equivariantisation and state sum models
- **Applications**:
  - 3-manifold invariants from non-semisimple tensor categories
  - embedding invariants from modular tensor categories
  - models for topological quantum computation

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018, Carqueville/Montiel Montoya 2018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