Defects and Orbifolds of 2-dimensional Yang-Mills theory

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Defects in topological and conformal field theory
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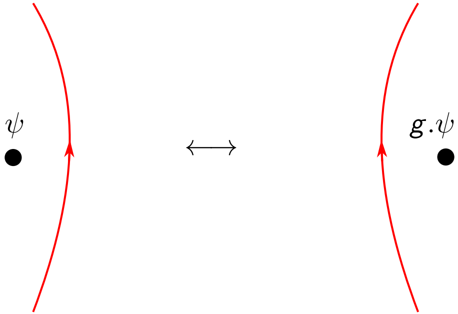
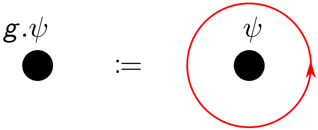
based on joint work with Richard J. Szabo and Lóránt Szegedy

Symmetries and Defects

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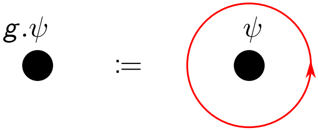


Goal for today

Study a simple class of symmetries and their corresponding defects in 2-dimensional Yang-Mills theory.

Yang-Mills theory

- Let G be a semi-simple compact Lie group with Lie algebra $\mathfrak{g}$ and $P \longrightarrow \Sigma$ a principal G-bundle over a Riemannian manifold Σ with connection A.
- Yang-Mills theory is defined by the action functional:

$$S_{\mathrm{YM}}(A) = \frac{1}{4e^2} \int_\Sigma \mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F})$$

- Partition function

$$Z_{YM}(\Sigma) = \int \mathcal{D}A \exp(-S_{\mathrm{YM}}(A))$$

- In 2-dimensions the partition function only depends on the area of Σ $\rightsquigarrow$ area dependent quantum field theory.

The symmetry

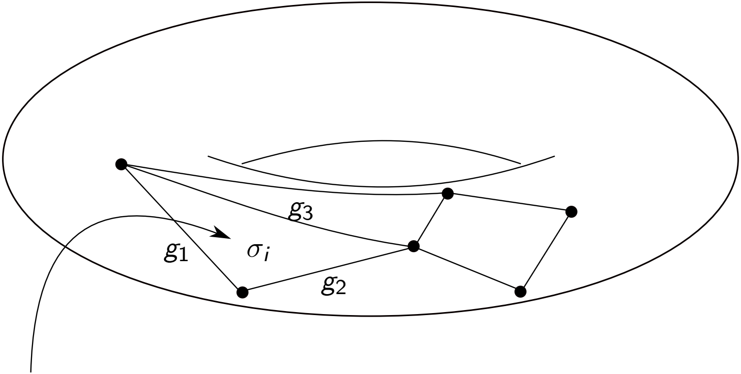
- We construct a symmetry from an outer automorphism $\varphi\colon G \longrightarrow G$, e.g. complex conjugation $SU(n) \longrightarrow SU(n)$.
- φ induces $\varphi_*\colon \mathfrak{g} \longrightarrow \mathfrak{g}$.
- The symmetry acts on a field configuration $(P, A \in \Omega^1(P;\mathfrak{g}))$ by sending it to

$$\varphi P \ : \ P \times G \xrightarrow{\mathrm{id}\times\varphi^{-1}} P \times G \longrightarrow P$$

  with connection $\varphi_* A$.
- The action transforms as

$$\frac{1}{4e^2}\int_\Sigma \mathrm{Tr}(\mathcal{F}\wedge *\mathcal{F}) \longmapsto \frac{1}{4e^2}\int_\Sigma \mathrm{Tr}(\varphi_*(\mathcal{F})\wedge *\varphi_*(\mathcal{F}))$$

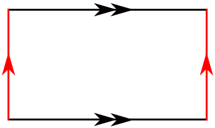
Lattice regularization and invariance of the quantum theory



$$\Gamma(\mathcal{U}_i := g_3 g_2 g_1, \sigma_i) := \sum_\alpha \dim(\alpha) \chi_\alpha(\mathcal{U}_i) \cdot \exp(-\sigma_i c_2(\alpha)/2)$$

$$Z(\Sigma, \sigma) := \int_{G^{|\Sigma_1|}} \prod_{\gamma_j \in \Sigma_1} dg_{\gamma_j} \prod_{w_i \in \Sigma_2} \Gamma(\mathcal{U}_i, \sigma_i)$$

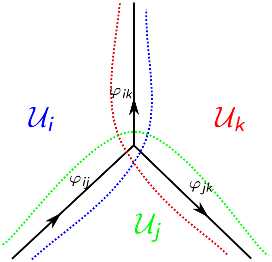
Defects and their partition function



$$Z(T^2,\sigma,\varphi) = \int_{G\times G} \sum_{\alpha} \dim(\alpha)\chi_\alpha(\varphi(g_2)^{-1}g_1^{-1}g_2g_1)\exp(-\sigma c_2(\alpha)/2)dg_1dg_2$$
$$= \sum_{\varphi^*\alpha\cong\alpha} \exp(-\sigma c_2(\alpha)/2),$$

Defects and twisted bundles

- A defect network defines an $\mathrm{Out}(G)$-bundle D as follows:



- There is a canonical map $r\colon \mathrm{Bun}_{G\rtimes \mathrm{Out}(G)}(\Sigma) \longrightarrow \mathrm{Bun}_{\mathrm{Out}(G)}(\Sigma)$

Definition

A D-twisted G bundle is a $G \rtimes \mathrm{Out}(G)$-bundle P together with a gauge transformation $r(P) \longrightarrow D$.

Defects and twisted bundles

- We can describe a D-twisted bundle with respect to the cover $\{U_i\}$ used to define D.
- The transition functions are of the from (g_{ij}, φ_{ij}) where the φ_{ij} are fixed by D.
- The 2-cocycle condition implies

$$g_{ki} = g_{kj}\varphi_{kj}(g_{ji})$$

- A connection can be described locally by 1-forms $A_i \in \Omega^1(\mathcal{U}_i, \mathfrak{g})$
- For g_{ij} trivial:

$$A_i = \mathrm{ad}_{(1,\varphi_{ij})}A_j = \varphi_{ij*}A_j.$$

$$\boxed{Z(\Sigma, D) = \int_{(P,A)\in \mathrm{Bun}^{\nabla}_{G\downarrow D}(\Sigma)} \mathcal{D}(P,A)\exp(-S_{YM}(P,A))}$$

A conjecture for the symplectic volume of flat twisted bundles

- Let $\mathcal{M}_G^D$ be the moduli space of flat D-twisted G-bundles.

Conjecture (LM, R.J. Szabo, L. Szegedy)

$Vol(\mathcal{M}_{SU(3)}^D) = \exp((2g-2)\Delta v)\zeta(6g-6)$

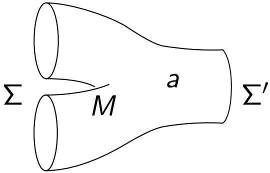
Area dependent quantum field theory

Definition

An area dependent 2-dimensional QFT is a symmetric monoidal functor $\text{a-Cob}_2 \longrightarrow \text{Hilb}$ which is continuous on hom-spaces.

Objects:

Morphisms:



Theorem (I. Runkel, L. Szegedy)

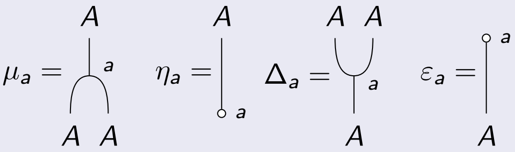
2-dimensional aQFTs are classified by regularized commutative Frobenius algebras.

Field theories with defects are defined using a version of a-Cob_2 containing labelled stratifications.

RFAs

Definition

A regularized Frobenius algebra (RFA) is a Hilbert space A equipped with



continuous in the parameter $a \in \mathbb{R}_{>0}$ with respect to the strong operator topology satisfying parametrized versions of the usual Frobenius relations.

Examples related to 2-dimensional Yang-Mills theory

Example

$L^2(G)$ is a RFA with structure maps:

$$\eta_a(1) = \sum_\alpha \dim(\alpha) \exp\left(-a\frac{c_2(\alpha)}{2}\right) \chi_\alpha(\cdot)$$

$$\mu(f \otimes g)(x) = \int_G f(xy^{-1})g(y)dy \ ,$$

$$\mu_a(f, g) = \mu(\eta_a(1), \mu(f, g))$$

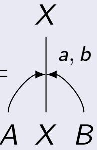
Δ_a and ϵ_a are the adjoint operators. $Z(L^2(G)) = \mathrm{Cl}(G)$ is the commutative RFA describing 2-dimensional Yang-Mills theory.

Bimodules

Definition

A bimodule over RFAs A and B is a Hilbert space X together with a family of maps $\rho_{a,b} : A \otimes X \otimes B \to X$ (the two-sided action) denoted by

$$\rho^X_{a,b} =$$



which satisfies a parametrised version of the usual bimodule conditions.

Remark

One can define the relative $X \otimes_A X'$ and cyclic $\circlearrowleft_A X$ tensor product of bimodules and dualizable bimodules using parametrized versions of the usual definitions.

Example

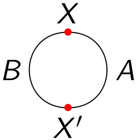
- Let $V \in \mathrm{Rep}(G)$. Wilson lines can be described by the bimodule $L^2(G) \otimes V$ with action

$$\varphi.(f \otimes v) = (\varphi * f) \otimes v \ ,$$
$$(f \otimes v).\psi = \left[x \mapsto \int_G y^{-1}.vf(xy^{-1})\psi(y) \right] \ .$$

- The twisted bimodules L_φ for $\varphi \in \mathrm{Out}(G)$ with action

$$\begin{aligned} \rho_{a,b} \colon L^2(G) \otimes L^2(G) \otimes L^2(G) &\longrightarrow L^2(G) \\ f \otimes h \otimes g &\longmapsto \mu_a(f, \mu_b(h, \varphi^* g)) \ . \end{aligned}$$

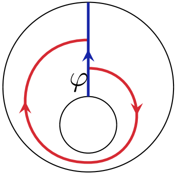
realizes the defects introduced at the beginning of the talk.



$$\mapsto \circlearrowleft_B X \otimes_A X'$$

- There is a way to define the value of the state sum construction on 2-dimensional bordisms with area.

Orbifolds via defects (See also Nils Carqueville's talk)



- Let Γ be a finite symmetry group of an aQFT Z with corresponding defects L_γ
- Set $M = \bigoplus_\Gamma L_\gamma$ and choose trivalent junction fields making M into a strongly separable symmetric Frobenius algebra in $\mathcal{B}$
- Add twisted sectors $H = \bigoplus_{\gamma \in \Gamma} Z(\mathbb{S}^1, \gamma)$. (In our case these are twisted class functions satisfying $f(gx\varphi(g^{-1})) = f(x)$)
- There is an action of Γ on H. The state space of the orbifold theory is the space of Γ invariants.
- We denote by P the projector onto this subspace.

The bicategory of topological defects

Definition

The topological defect bicategory of 2-dimensional Yang-Mills theories $\mathcal{B}$ has

- **Objects:** RFAs of the type $L^2(G)$
- **1-Morphisms:** Are the labels for topological defects, i.e. dualizable transmissive bimodules
- **2-Morphisms:** (Via operator state correspondence) A 2-morphism $X \to Y$ for $X, Y : A \to B$ is given by a family of maps $\{\phi_a : \mathbb{C} \to \circlearrowleft_A Y \otimes_B \bar{X}\}_{a \in \mathbb{R}_{>0}}$ which are invariant under the action of cylinders:

$$\mathcal{C}_b \circ \varphi_a = \varphi_{a+b}$$

Remark

It is possible to rewrite the 2-morphisms as families of bimodule maps.

Orbifold theory

Theorem (LM, R.J. Szabo, L. Szegedy)

The L-L-bimodule $M := \bigoplus_{\alpha \in \mathrm{Out}(G)} L_\alpha$ *is a is a separable symmetric Frobenius algebra in* $\mathcal{B}(L, L)$.

Proposition (LM, R.J. Szabo, L. Szegedy)

1. *The projector* P *is*

$$P = |\mathrm{Out}(G)|^{-1} \sum_{\beta \in \mathrm{Out}(G)} (\beta^{-1})^* : \mathcal{H} \to \mathcal{H} \ .$$

2. *The image of* P *is the subspace* $\mathcal{H}^{\mathrm{Out}(G)}$ *of* $\mathrm{Out}(G)$ *invariants under this action and we have*

$$\mathcal{H}^{\mathrm{Out}(G)} \simeq Cl^2(G \rtimes \mathrm{Out}(G)) \ .$$

Orbifold theory

Theorem (LM, R.J. Szabo, L. Szegedy)

The orbifold theory of 2dYM with gauge group G and orbifolding defect $\bigoplus_{\alpha \in \mathrm{Out}(G)} L_\alpha$ is 2dYM with gauge group $G \rtimes \mathrm{Out}(G)$.

Outlook

- Study of the homology of the moduli space of flat twisted bundles via 2-dimensional Yang-Mills theory.
- Quantization of the moduli space of flat twisted bundles via equivariant factorization homology.
- Generalization to q-deformed Yang-Mills and Chern-Simons theory.
- 2D-4D correspondence.

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Thank you for your attention!