Boundary Conditions and the Swiss-Cheese Operad

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Workshop on *Defects in topological and conformal field theory*

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  - Homotopy theory has led to important constructions in topological field theory such as factorization homology, a kind of 'integration' of little disk algebras over manifolds (Lurie, Ayala-Francis, Calaque-Scheimbauer, . . .) and factorization algebras (Costello-Gwilliam)

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Principle

In a lot of the homotopy theoretic approaches to field theory, operads are used to encode the local structure of physical quantities.

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- The category $\mathcal{W}_a$ of Wilson lines at the boundary 'a' should be k-linear monoidal.
- Moving bulk Wilson lines to the boundary is modeled by a functor

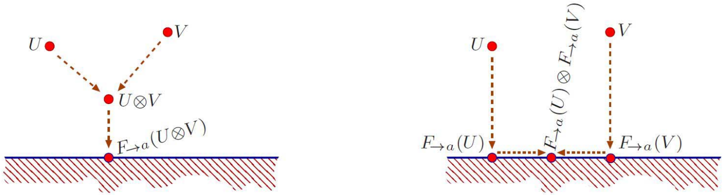
$$F_{\to a} : \mathcal{C} \longrightarrow \mathcal{W}_a \ .$$

Physical boundary conditions in 3D TFT after Fuchs-Schweigert-Valentino

- Fusing Wilson line in the bulk corresponds to fusing them in the boundary:

$$F_{\to a}(U \otimes V) \cong F_{\to a}(U) \otimes F_{\to b}(V) ;$$

i.e. $F_{\to a}$ is monoidal.



(Figure from Fuchs-Schweigert-Valentino)

Physical boundary conditions in 3D TFT after Fuchs-Schweigert-Valentino

- Boundary Wilson lines coming from the bulk commute with all boundary Wilson lines:

$$F_{\to a}(U) \otimes M \cong M \otimes F_{\to a}(U)$$

such that the boundary Yang-Baxter equations are satisfied.

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(Figure from Fuchs-Schweigert-Valentino)

- One concludes that $F_{\to a} : \mathcal{C} \longrightarrow \mathcal{W}_a$ lifts to a functor

$$\widetilde{F}_{\to a} : \mathcal{C} \longrightarrow Z(\mathcal{W}_a)$$

to the Drinfeld center of $\mathcal{W}_a$; 'minimal FSV conditions'.

Algebraic structures from open-closed field theories

The following type of algebraic structure is used to describe open-closed string field theory in a graded framework; [Kajiura-Stasheff 2006] based on ideas by Zwiebach, see [Hoefel 2009] for the precise relation:

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- graded vector spaces X and A which associative products $\bullet$ and $\star$ (of degree zero),
- $\bullet$ is graded symmetric,

$$x \bullet y = (-1)^{|x||y|} y \bullet x \ .$$

to be continued on the next slide

Algebraic structures from open-closed field theories

- a graded bracket $[-,-]$ on X of degree 1 which is graded anti-symmetric,

$$[x,y] = -(-1)^{(|x|-1)(|y|-1)}[y,x] \ ,$$

and satisfies the Jacobi identity,

$$[x,[y,z]] = [[x,y],z] + (-1)^{(|x|-1)(|y|-1)}[y,[x,z]] \ ,$$

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- a linear map $\phi : X \longrightarrow A$ respecting the grading satisfying

$$\phi(x \bullet y) = \phi(x) \star \phi(y) \ ,$$
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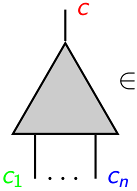
We will refer to this algebraic structure as a *HSC*-algebra.

Operads and their algebras

A (colored) operad encodes algebraic structures by giving objects (vector spaces, topological spaces, chain complexes) of operations with several inputs and one output.



$$\in \mathcal{O}\binom{c}{(c_1,\ldots,c_n)}$$

There is a composition of operations subject to associativity and unitality conditions.

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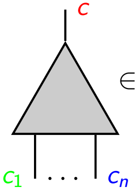

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Example: The associative operad

The n-ary operations of the associative operad As are $\mathrm{As}(n) = \Sigma_n$, the permutation group on n letters.

Operads and their algebras

An *algebra* A over a colored operad $\mathcal{O}$ is a concrete realization of the abstract operations in $\mathcal{O}$. It consists of colored objects A_c and morphisms

$$\alpha : \mathcal{O}\big({}^{\;\;\;c}_{(c_1,\dots,c_n)}\big) \otimes A_{c_1} \otimes \cdots \otimes A_{c_n} \longrightarrow A_c ,$$

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Example: Associative algebras

The algebras over the associative operad are unital associative algebras.

Little disks and braided monoidal categories

The little disk operad E_2 is a topological one-colored operad whose space of arity n operations $E_2(n)$ is given by the space of affine embeddings of n disks into one disk.

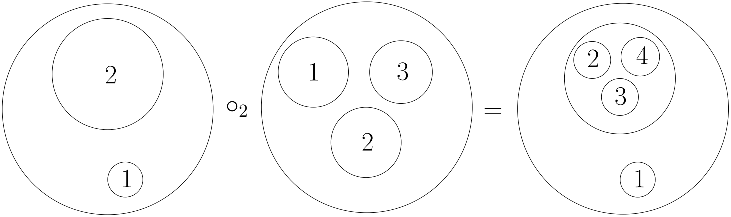

Figure: Example for the composition of little disks.

Little disks and braided monoidal categories

An E_2-algebra in categories has an underlying category $\mathcal{C}$. The embedding of two disks into one yields a 'multiplication' $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ which is homotopy commutative.

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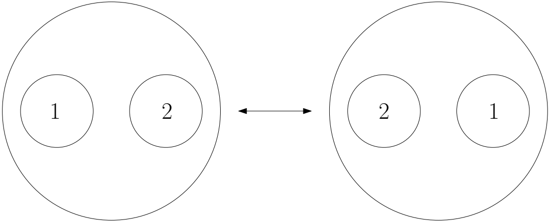

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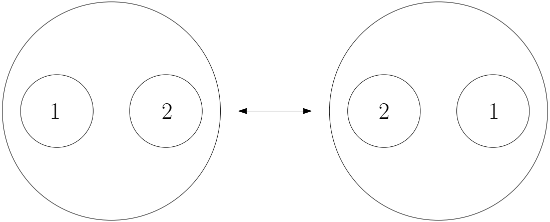

Theorem [Joyal-Street, Lurie, Wahl-Salvatore, . . . , see also Fresse]

Up to equivalence, little disk algebras in categories are braided monoidal categories.

Swiss-Cheese operad and its algebras

The Swiss-Cheese operad SC due to Voronov has two colors $\mathfrak{c}$ (closed) and $\mathfrak{o}$ (open).

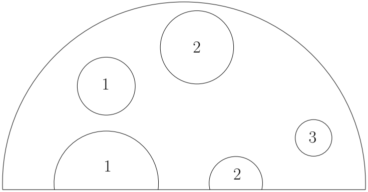

Figure: Element in $SC(2, 3)$.

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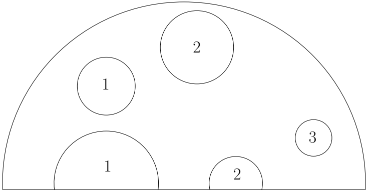

Figure: Element in $SC(2,3)$.

Theorem [Idrissi 2017]

Up to equivalence, Swiss-Cheese algebras in categories are triples $(\mathcal{M}, \mathcal{N}, F)$ of a braided monoidal category $\mathcal{M}$, a monoidal category $\mathcal{N}$ and a braided monoidal functor $F : \mathcal{M} \longrightarrow Z(\mathcal{N})$.

Boundary conditions via Swiss-Cheese algebras

We can now conclude: The minimal FSV boundary condition give a full description of the topological situation present at a boundary.

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More precisely:

- All the structure and relations found by Fuchs-Schweigert-Valentino correspond to Swiss Cheese operations and paths in the operation spaces of the Swiss Cheese operad (not too hard to see).

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More precisely:

- All the structure and relations found by Fuchs-Schweigert-Valentino correspond to Swiss Cheese operations and paths in the operation spaces of the Swiss Cheese operad (not too hard to see).
- . . . and there is nothing more (non-trivial).

Application: Boundary conditions and *HSC*-algebras

For a k-linear category $\mathcal{C}$ we can consider the derived coend $\int_{\mathbb{L}}^{X \in \operatorname{Proj} \mathcal{C}} \mathcal{C}(X, X)$ which is the differential graded vector space which in degree $n \geq 0$ is given by

$$\bigoplus_{X_0, \ldots, X_n \in \operatorname{Proj} \mathcal{C}} \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_n, X_0) ,$$

i.e. by loops of morphisms in projective objects in $\mathcal{C}$ (this is a form a Hochschild chains).

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Theorem [Schweigert-W.]

Let $\mathcal{O}$ be an operad in groupoids. If a k-linear category $\mathcal{C}$ is a $k[\mathcal{O}]$-algebra, then $\int_{\mathbb{L}}^{X \in \operatorname{Proj} \mathcal{C}} \mathcal{C}(X,X)$ is a differential graded $N_* B\mathcal{O}$-algebra.

Application: Boundary conditions and HSC-algebras

Corollary

If a braided category $\mathcal{C}$ and a monoidal category $\mathcal{W}_a$ satisfy the minimal FSV boundary conditions, then we have on the Hochschild homologies $H_* \int_{\mathbb{L}}^{X \in \operatorname{Proj} \mathcal{C}} \mathcal{C}(X, X)$ and $H_* \int_{\mathbb{L}}^{Y \in \operatorname{Proj} \mathcal{W}_a} \mathcal{C}(Y, Y)$ of $\mathcal{C}$ and $\mathcal{W}_a$ the structure of a HSC-algebra.

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Proof. As noted above the pair $(\mathcal{C}, \mathcal{W}_a)$ forms a categorical ΠSC-algebra. By the Theorem on the last slide, this implies that the Hochschild chains carry an action of $N_* B\Pi SC$. The latter operad is equivalent to $N_* SC$ because SC is aspherical. As a consequence, the Hochschild homologies of $\mathcal{C}$ and $\mathcal{W}_a$ form a the homology of the Swiss-Cheese operad. By [Hoefel 2009] the resulting algebraic structure actually coincides with what has been (suggestively) called HSC-algebra above. $\square$

The Gerstenhaber bracket reflects the 'quantum part' of the braiding.

$$[\underline{f}, \underline{g}] = (-1)^p \sum_{\substack{(p,q)\text{-shuffles } (\mu,\nu) \\ \text{of } p+q}} \sum_{j=0}^{p+q} (-1)^j \operatorname{sign}(\mu,\nu) \quad \Big(s_\nu(\underline{f}) \bullet_j s_\mu(\underline{g}) + (-1)^{pq} s_\mu(\underline{g}) \bullet_j s_\nu(\underline{f}) \Big) \ ,$$

where

- $\underline{f}$ and $\underline{g}$ are loops of morphisms chaising through projective objects,
- s_ν and s_ν are degeneracy operators associated to shuffles,
- $\bullet_j$ are insertion operators for the braiding.

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- Operads allow us to discover interesting operations in Hochschild homology (without re-inventing the wheel).
- Relation of the Gerstenhaber bracket and non-degeneracy?
- Generalization to the equivariant case (joint with Lukas Müller).