A Defect Verlinde Formula [1901.08285]

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Topological Order

Brief history:

- Chiral spin state [Wen-Wilczek-Zee:89]
- Topological order [Wen:89, Wen:90]
- Topological quantum field theory [Atiyah:88, Witten:88]

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The mathematical structure of a $(2+1)D$ bulk topological order(TO) is described by unitary modular tensor category(UMTC) [Fredenhagen-Rehren-Schroer:89, Moore-Seiberg:89, Kitaev:05]

TO	UMTC
anyon	simple object
fusion	Grothendieck ring
braiding	S matrix ($S_{ab} \in \mathbb{Z}(\zeta_N)$ [Ng-Schauenburg:07, Ng-Schauenburg:10])
spin h_a	T matrix ($T_{aa} = e^{2\pi i h_a}$, $h_a \in \mathbb{Q}$ [Vafa:88])
...	...

Anyon Fusion

anyon fusion: $i \otimes j = \sum_k N_{ij}^k k, \qquad N_{ij}^k \in \mathbb{Z}^{\geq}$
fusion matrix: $(N_i)_{kj} := N_{ij}^k$
associativity of fusion = commutativity of fusion matrices

$$\sum_e N_{ab}^e N_{ec}^d = \sum_e N_{bc}^e N_{ea}^d$$

fusion matrices commute $\rightarrow$ simultaneous diagonalization (by S-matrix)

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Verlinde Formula (1988)

$$N_{ij}^k = \sum_r \frac{S_{ir} S_{jr} \overline{S_{kr}}}{S_{0r}}$$

Gapped Boundary

- Non-chiral TO $\rightarrow$ energy gap at the boundary.

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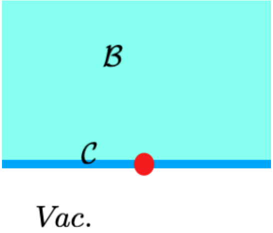
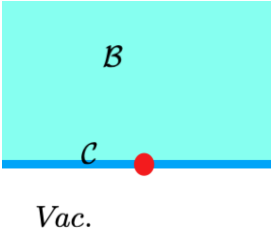
- Gapped boundary: an interface between TO and the vacuum.



Gapped boundary.

- Non-chiral TO $\to$ energy gap at the boundary.
- Gapped boundary: an interface between TO and the vacuum.
- Gapped boundary is described by unitary fusion category (UFC). [Kitaev-Kong:12, Kong:13]
- Bulk $=$ Center(Boundary): $\mathcal{B} = \mathcal{Z}(\mathcal{C})$ [Kong-Wen-Zheng:15, Kong-Wen-Zheng:2017]



Gapped boundary.

Physical: gapped boundary is described by anyon condensation

[Bais-Schroer-Slingerland:02, Kapustin-Saulina:11, Barkeshli-Jian-Qi:13, Levin:13, Hung-Wan:14, Wang-Wen:15, . . .]

Bulk-Boundary Relation

Physical: gapped boundary is described by anyon condensation
[Bais-Schroer-Slingerland:02, Kapustin-Saulina:11, Barkeshli-Jian-Qi:13, Levin:13, Hung-Wan:14, Wang-Wen:15, . . .]

Mathematical: (indecomposable module over) boundary UFC is equivalent to Lagrangian algebra $\mathcal{A} \in \mathcal{B}$. [Davydov-Müger-Nikshych-Ostrik:10, Fuchs-Schweigert-Valentino:12, Kitaev-Kong:12, . . .]
Bulk to boundary condensation is captured by

$$\begin{aligned} F_{\mathcal{A}}: \quad & \mathcal{B} \to \mathcal{C} \\ & a \mapsto a \otimes \mathcal{A} \end{aligned}$$

Informal definition

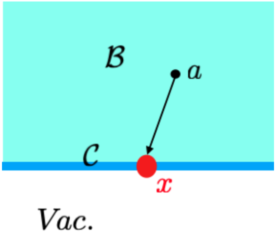
Condensed anyons form kernel of $F_{\mathcal{A}}$
Confined anyons label image of $F_{\mathcal{A}}$

Bulk-Boundary Relation

Bulk-to-boundary W matrix [Bais-Schroer-Slingerland:02, Bais-Slingerland-Haaker:09]

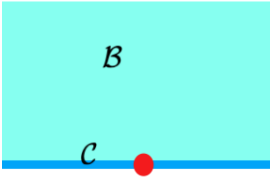
$$a = \oplus_x W_{ax} x, \qquad W_{ax} \in \mathbb{Z}^{\geq}$$

W_{ax} is the "multiplicity" of a decomposing into x



Condensate: $\mathsf{C} = \oplus_c W_{c0} c$.
Condensed anyon: $c \in \mathcal{B}$ s.t. $W_{c0} > 0$
Condensed anyons condense to trivial boundary excitation $x = 0$.

- Boundary excitations: live on one boundary, objects of boundary UFC

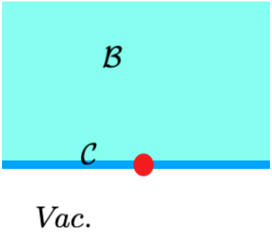


$Vac.$

Boundary excitation.
$\mathcal{Z}(\mathcal{C}) = \mathcal{B}$

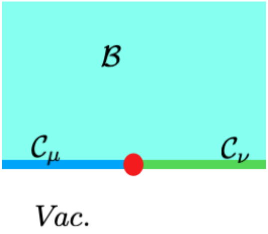
Boundary Excitation v.s. Boundary Defect

- Boundary excitations: live on one boundary, objects of boundary UFC
- Boundary defects: localized at the junction between two (different) boundaries, objects of multi fusion category. They're related to topological defect lines in CFT. [Petkova-Zuber:01, Fuchs-Runkel-Schweigert:08, Fuchs-Schweigert-Valentino:13, Cong-Cheng-Wang:17, . . .]



Boundary excitation.

$$\mathcal{Z}(\mathcal{C}) = \mathcal{B}$$



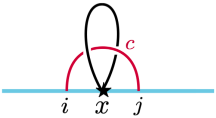
Boundary defect.

$$\mathcal{Z}(\mathcal{C}_\mu) = \mathcal{Z}(\mathcal{C}_\nu) = \mathcal{B}$$

Half-linking $\gamma_{x\, c_{(i,j)}}$

A natural topological number half-linking (non-abelian generalization of [Kapustin-Saulina:11])

1. create a boundary excitation x and its antiparticle
2. create a condensed anyon c from the boundary, move it around x in the bulk, and finally annihilate it at the boundary
3. annihilate the pair of x (close up x line)

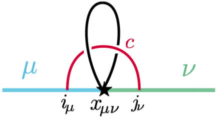


Half-linking $\gamma_{x\, c_{(i,j)}}$.

- x: boundary excitation
- c: condensed anyon
- i, j: condensation channel (when $W_{c0} > 1$)

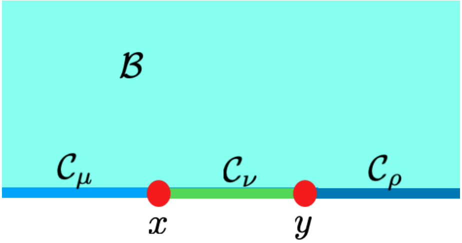
Half-linking $\gamma^{(\mu|\nu)}_{x\, c_{(i_\mu, j_\nu)}}$

For a boundary defect



Half-linking $\gamma^{(\mu|\nu)}_{x\, c_{(i_\mu, j_\nu)}}$

- $x_{\mu\nu}$: boundary defect
- c: $c \in \mathsf{C}_\mu \cap \mathsf{C}_\nu$ is shared condensed anyon
- i_μ: condensation channel (when $W^{(\mu)}_{c0} > 0$)
- j_ν: condensation channel (when $W^{(\nu)}_{c0} > 0$)

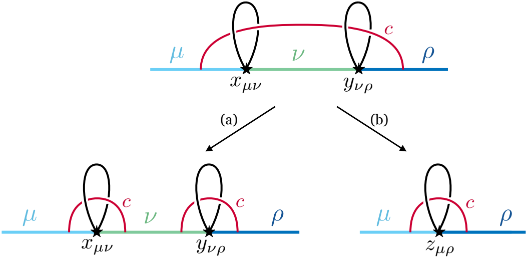


$$x \otimes y = \oplus_z n_{x\,y}^{z}\, z$$

Defect Verlinde Formula

$$n_{x\,y}^{z} = \sum_{c \in \mathsf{C}_\mu \cap \mathsf{C}_\nu \cap \mathsf{C}_\rho} \sum_{i_\mu, i_\nu, i'_\nu, i_\rho} \gamma^{(\mu|\nu)}_{x\, c_{(i_\mu, i_\nu)}} (M_c^\nu)^{-1}_{i_\nu i'_\nu} \gamma^{(\nu|\rho)}_{y\, c_{(i'_\nu, i_\rho)}} (\gamma^{(\mu|\rho)})^{-1}_{c_{(i_\rho, i_\mu)} z}$$

Defects Fusion



Normalization factor $(M_c^\nu)_{i_\nu i'_\nu} = \gamma^{(\nu|\nu)}_{0\, c_{(i_\nu, i'_\nu)}}$

Defect Verlinde Formula

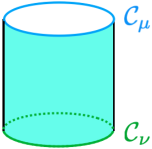
$$n_{x\,y}^{z} = \sum_{c \in \mathsf{C}_\mu \cap \mathsf{C}_\nu \cap \mathsf{C}_\rho} \sum_{i_\mu, i_\nu, i'_\nu, i_\rho} \gamma^{(\mu|\nu)}_{x\, c_{(i_\mu, i_\nu)}} (M_c^\nu)^{-1}_{i_\nu i'_\nu} \gamma^{(\nu|\rho)}_{y\, c_{(i'_\nu, i_\rho)}} (\gamma^{(\mu|\rho)})^{-1}_{c_{(i_\rho, i_\mu)} z}$$

Basis Transformation on a Cylinder

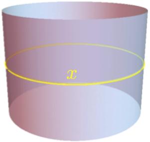
[Cong-Cheng-Wang:16]

$$\#(\text{defects}) = \sum_{c\in\mathcal{B}} W^{(\mu)}_{c0} W^{(\nu)}_{c0} = \text{GSD on a cylinder.}$$

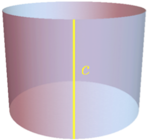
The ground state subspace of a TO on a cylinder has two sets of basis.
[Wang-Wen:15, Hung-Wan:15, Lou-Shen-Hung:19]



TO on a cylinder.

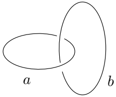
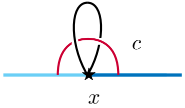
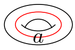
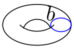


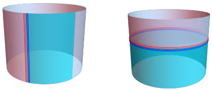
Wilson loop basis.



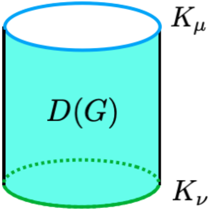
Wilson line basis.

$$\left| \underset{x}{\circlearrowleft} \right\rangle = \sum_{c\in \mathsf{C}_\mu\cap\mathsf{C}_\nu} \sum_{i_\mu, i_\nu} \gamma^{(\mu|\nu)}_{x\, c_{(i_\mu,i_\nu)}} \left| \begin{matrix} i_\mu \\ \uparrow c \\ i_\nu \end{matrix} \right\rangle .$$

Definition	$S_{ab} = \frac{1}{D}$ 	$\gamma_{xc}=$ 
Fusion	$N_{ab}^{c} = \sum_r \frac{S_{ar}S_{br}\overline{S_{cr}}}{S_{0r}}$	$n_{xy}^{z} = \sum_c \gamma_{xc} M_c^{-1} \gamma_{yc} \overline{\gamma_{cz}}$
(2+1)D TO	$\Big\vert$  $\Big\rangle = \sum_b S_{ab} \Big\vert$  $\Big\rangle$	$\Big\vert$  $\Big\rangle = \sum_c \gamma_{xc} \Big\vert$  $\Big\rangle$
(1+1)D CFT	$\chi_a(-\frac{1}{\tau}) = \sum_b S_{ab}\chi_b(\tau)$	$\chi_x^{\text{(open)}} = \sum_c \gamma_{xc}\chi_c^{\text{(closed)}}$



Example: Abelian Quantum Double $D(G)$



Quantum double with gapped boundaries.

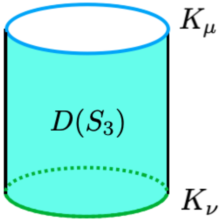
G is an abelian group, K_μ, K_ν are subgroups.

$$\#(\text{defects}) = |K_\mu \backslash G / K_\nu| \cdot |K_\mu \cap K_\nu|$$

$$\gamma_{xc}^{(\mu|\nu)} = \frac{1}{\sqrt{D}} \frac{\sqrt{|K_\mu| \cdot |K_\nu|}}{|K_\mu \cap K_\nu|} \cdot \tilde{S}_{xc}$$

$$n_{xy}^{z} = \sum_{c \in \mathsf{C}_\mu \cap \mathsf{C}_\nu \cap \mathsf{C}_\rho} \frac{\gamma_{xc}^{(\mu|\nu)} \gamma_{yc}^{(\nu|\rho)} \overline{\gamma_{zc}^{(\mu|\rho)}}}{\gamma_{0c}^{(\nu|\nu)}} = \delta_{x+y,z}$$

Example: Quantum Double $D(S_3)$



Quantum double $D(S_3)$ with gapped boundaries.

$\mathfrak{C}$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1 = A \oplus B \oplus 2C$ $K_1 = \{1\}$	Vec_{S_3}	$\{\sqrt{3}, \sqrt{3}\}$	$\{\sqrt{2}, \sqrt{2}, \sqrt{2}\}$	$\{\sqrt{6}\}$
$\mathcal{A}_2 = A \oplus B \oplus 2F$ $K_2 = \mathbb{Z}_3$	$\{\sqrt{3}, \sqrt{3}\}$	Vec_{S_3}	$\{\sqrt{6}\}$	$\{\sqrt{2}, \sqrt{2}, \sqrt{2}\}$
$\mathcal{A}_3 = A \oplus C \oplus D$ $K_3 = \mathbb{Z}_2$	$\{\sqrt{2}, \sqrt{2}, \sqrt{2}\}$	$\{\sqrt{6}\}$	$\mathrm{Rep}(S_3)$	$\{\sqrt{3}, \sqrt{3}\}$
$\mathcal{A}_4 = A \oplus F \oplus D$ $K_4 = S_3$	$\{\sqrt{6}\}$	$\{\sqrt{2}, \sqrt{2}, \sqrt{2}\}$	$\{\sqrt{3}, \sqrt{3}\}$	$\mathrm{Rep}(S_3)$

Table: Boundary defects of $D(S_3)$. [Cong-Cheng-Wang:17]

- Diagonal: boundary fusion category
- Off-diagonal: quantum dimension of defects

Example: Quantum Double $D(S_3)$

$$\gamma^{(3|2)} = 1 = \frac{1}{\sqrt{6}} \begin{array}{cc} A & \\ \begin{pmatrix} \sqrt{6} \end{pmatrix} & \{A\} \end{array}$$

$$\gamma^{(3|1)} = \frac{1}{\sqrt{6}} \begin{array}{cc} \begin{array}{ccc} A & C^1 & C^2 \end{array} & \\ \begin{pmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \end{pmatrix} & \begin{array}{c} \{A\} \\ \{F, p_1\} \\ \{F, p_2\} \end{array} \end{array}$$

$$\gamma^{(3|4)} = \frac{1}{\sqrt{6}} \begin{array}{cc} \begin{array}{cc} A & D \end{array} & \\ \begin{pmatrix} \sqrt{3} & \sqrt{3} \\ \sqrt{3} & -\sqrt{3} \end{pmatrix} & \begin{array}{c} \{A\} \\ \{B\} \end{array} \end{array}$$

$$\gamma^{(1|1)} = \frac{1}{\sqrt{6}} \begin{array}{cc} \begin{array}{cccccc} A & B & C^{1,1} & C^{1,2} & C^{2,1} & C^{2,2} \end{array} & \\ \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 0 & \sqrt{2} \\ 1 & -1 & \sqrt{2} & 0 & 0 & -\sqrt{2} \\ 1 & 1 & -\frac{1}{\sqrt{2}} & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \\ 1 & -1 & -\frac{1}{\sqrt{2}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \\ 1 & 1 & -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \\ 1 & -1 & -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} & \begin{array}{c} \{e\} \\ \{s\} \\ \{r\} \\ \{sr\} \\ \{r^2\} \\ \{sr^2\} \end{array} \end{array}$$

$$\gamma^{(3|3)} = \frac{1}{\sqrt{6}} \begin{array}{cc} \begin{array}{ccc} A & C & D \end{array} & \\ \begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 1 & \sqrt{2} & -\sqrt{3} \\ 2 & -\sqrt{2} & 0 \end{pmatrix} & \begin{array}{c} \{A\} \\ \{B\} \\ \{F\} \end{array} \end{array}$$

Fusion of defects between $\mathcal{A}_1 = A \oplus B \oplus 2C/\mathcal{A}_3 = A \oplus C \oplus D$, consider $\mathcal{A}_3|\mathcal{A}_1|\mathcal{A}_3$ defects fusion

$$n_{xy}^{z} = \frac{\gamma_{xA}^{(3|1)}\gamma_{yA}^{(1|3)}\overline{\gamma_{zA}^{(3|3)}}}{\gamma_{\{e\}A}^{(1|1)}} + \sum_{\mu,\nu=1,2} \gamma_{x\,C^{\mu}}^{(3|1)} \left(\gamma_{\{e\}}^{(1|1)}\right)^{-1}_{C^{\mu}C^{\nu}} \gamma_{y\,C^{\nu}}^{(1|3)}\overline{\gamma_{z\,C}^{(3|3)}}$$

the normalization M matrix is $\gamma_{\{e\}}^{(1|1)} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

$$\begin{aligned}
\{A\}^{(3|1)} \otimes \{A\}^{(1|3)} &= \{A\}^{(3|3)} \oplus \{B\}^{(3|3)} \\
\{F,p_1\}^{(3|1)} \otimes \{F,p_1\}^{(1|3)} &= \{A\}^{(3|3)} \oplus \{B\}^{(3|3)} \\
\{F,p_2\}^{(3|1)} \otimes \{F,p_2\}^{(1|3)} &= \{A\}^{(3|3)} \oplus \{B\}^{(3|3)} \\
\{A\}^{(3|1)} \otimes \{F,p_1\}^{(1|3)} &= \{F\}^{(3|3)} \\
\{A\}^{(3|1)} \otimes \{F,p_2\}^{(1|3)} &= \{F\}^{(3|3)} \\
\{F,p_1\}^{(3|1)} \otimes \{F,p_2\}^{(1|3)} &= \{F\}^{(3|3)}
\end{aligned}$$

Conclusion

Summary:

- Boundary excitations and defects on/between gapped boundaries of TO
- Fusion algebra of boundary excitations/defects (defect Verlinde formula)
- Abelian and non-Abelian $D(S_3)$ examples.

Outlook:

- Generalize to fermionic boundary
- More algebraic properties of γ matrix.

Thanks!