Fusion quotients and topological defects in Temperley-Lieb Lattice models

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based on joint work with:
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Defects in topological and conformal field theory
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The plan

- Introduction
- Example: the unitary case
- The TL algebras
- The TL lattice models
- More examples

Why TL models?

- Regularization of a CFT by a finite lattice model
- Type A: Chiral CFT on a cylinder
  - RSOS $\rightarrow$ Minimal models $M(p+1,p)$ - $\phi_{1,s}$ only
  - Others $\rightarrow$ Some are log others aren't.
- Type B: Bulk CFT on a torus
  - Twisted RSOS $\rightarrow$ Minimal models $M(p+1,p)$ (ADE and some others, but not all)
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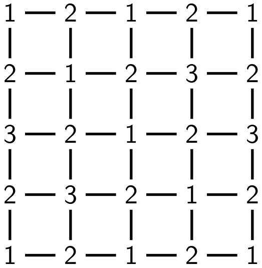
Can we understand defects in these lattice models?

Example I: The A_p RSOS models

Includes Ising ($p=3$), tri-critical Ising ($p=4$), 3-states Potts ($p=5$), etc.



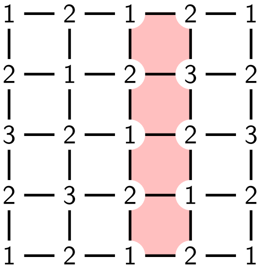
Heights $1, 2, \dots, p$, $\mathfrak{q} \equiv e^{i\frac{\pi}{p+1}}$

$$\begin{array}{ccc} d & — & c \\ | & & | \\ a & — & b \end{array} = \frac{\mathfrak{q}x^{-1} - \mathfrak{q}^{-1}x}{\mathfrak{q} - \mathfrak{q}^{-1}}\delta_{a,c} + \frac{x - x^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}}\delta_{b,d}$$

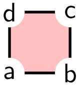
$$Z_{n,m} = \mathrm{Tr}(T_n)^m$$

$$\langle a_1, \dots, a_6 | T_n | b_1, b_2, \dots, b_6 \rangle = \begin{array}{ccccccccccc} b_1 & — & b_2 & — & b_3 & — & b_4 & — & b_5 & — & b_6 \\ | & & | & & | & & | & & | & & | \\ a_1 & — & a_2 & — & a_3 & — & a_4 & — & a_5 & — & a_6 \end{array}$$

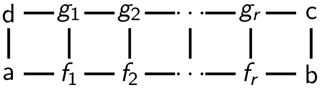
Example I: The A_p RSOS models



The width $k = 0, 1, 2, \ldots, p-2$



$$= \lim_{x \to 0} \sum_{\substack{f_1, \ldots, f_k \\ g_1, \ldots, g_k}} \lambda_{\substack{f_1, \ldots, f_k \\ g_1, \ldots, g_k}}(a, b, c, d)$$



$$\tilde{Z}_{n,m}(k) = \mathrm{Tr}(\tilde{T}_n(k))^m$$

The functions $\lambda_{\substack{f_1, \ldots, f_k \\ g_1, \ldots, g_k}}(a, b, c, d)$ are implicitly given as products of components of the eigenvectors of some matrix.

Example I: The A_p RSOS models

Continuum limit: Virasoro minimal model $M(p+1,p)$

$$H \sim \bigoplus_{s=1}^{p} \bigoplus_{r=1}^{p-1} (\phi_{r,s} \otimes \bar{\phi}_{r,s}) \tag{1}$$

Example I: The A_p RSOS models

Continuum limit: Virasoro minimal model $M(p+1,p)$

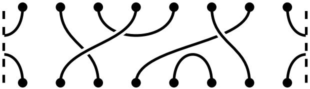
$$H \sim \bigoplus_{s=1}^{p} \bigoplus_{r=1}^{p-1} (\phi_{r,s} \otimes \bar{\phi}_{r,s}) \tag{1}$$

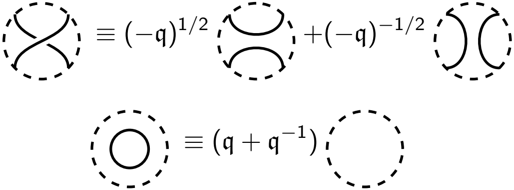
$$\tilde{H}(k) \sim \bigoplus_{s=1}^{p} \bigoplus_{r=1}^{p-1} ((\overbrace{\phi_{r,s} \times_f \phi_{1,k+1}}^{\text{Virasoro fusion!}}) \otimes \bar{\phi}_{r,s}).$$

- Defects give the twisted-boundary RSOS models.
- Computing products require extensive numerics.
- What's the meaning of the λs? How to generalize to other models?

The Temperley-Lieb algebras

Affine n-diagrams:

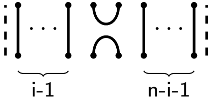




The Temperley-Lieb algebras

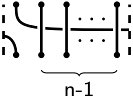
Type A or *regular*

$i = 1, 2, \dots, n-1$

$e_i =$ 

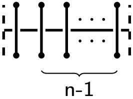
$$e_i e_i = (\mathfrak{q} + \mathfrak{q}^{-1}) e_i,$$
$$e_i e_{i\pm1} e_i = e_i,$$
$$e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2$$

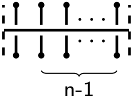
Type B or *affine*

$b = (-\mathfrak{q})^{-3/2}$ 

$$e_1 b e_1 = \overbrace{(\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1})}^{-Y} e_1,$$
$$e_i b = b e_i \quad i \neq 1$$

Two *Hoop* operators:

$Y =$ 

$\bar{Y} =$ 

The TL Tower structure

$$\phi^o : \mathsf{aTL}_n \otimes_{\mathbb{C}} \mathsf{TL}_m \to \mathsf{aTL}_{n+m}$$

$$\phi^u : \mathsf{aTL}_n \otimes_{\mathbb{C}} \mathsf{TL}_m \to \mathsf{aTL}_{n+m}$$

The TL Tower structure

$$\phi^o : \mathrm{aTL}_n \otimes_{\mathbb{C}} \mathrm{TL}_m \to \mathrm{aTL}_{n+m}$$

$$\phi^u : \mathrm{aTL}_n \otimes_{\mathbb{C}} \mathrm{TL}_m \to \mathrm{aTL}_{n+m}$$

For any $V \in \mathrm{mod}(\mathrm{TL}_m)$, $\mathrm{aTL}_{n+m} \otimes_{\mathrm{TL}_m} V$ is a $(\mathrm{aTL}_{n+m}, \mathrm{aTL}_n)$-bimodule

TL fusions

$$M_n \equiv \mathrm{mod}(\mathrm{aTL}_n),\ V \in \mathrm{mod}(\mathrm{TL}_m)$$

$$- \times_f^{u/o} V$$

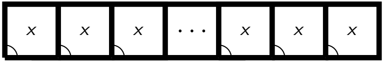
$$M_{n-1} \subset M_n \subset M_{n+1} \subset \cdots \subset M_{n+m-1} \subset M_{n+m} \subset M_{n+m+1} \subset \cdots$$

$$- \div_f^{u/o} V$$

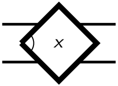
$$M \times_f^{u/o} V \equiv (\mathrm{aTL}_{n+m} \otimes_{\mathrm{TL}_m} V) \otimes_{\mathrm{aTL}_n} M, \qquad \text{Fusion product}$$

$$\bar{M} \div_f^{u/o} V \equiv \mathrm{Hom}_{\mathrm{aTL}_{n+m}} \left(\mathrm{aTL}_{n+m} \otimes_{\mathrm{TL}_m} V, \bar{M}\right), \qquad \text{Fusion quotient}$$

The TL Transfer matrix

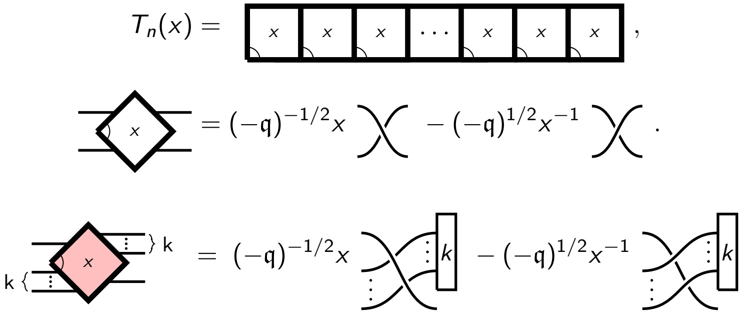
$T_n(x) =$ 

,



$= (-\mathfrak{q})^{-1/2} x$  $- (-\mathfrak{q})^{1/2} x^{-1}$  .

The TL Transfer matrix

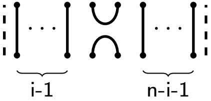


The box is the Jones-Wenzl projector: unique $\rho_k \in \mathrm{TL}_k$ s.t. $\rho_k^2 = \rho_k$ and $e_i \rho_k = 0$ for all $i = 1, \dots, k-1$. Only exists for $k = 1, \dots, p-1$ ($\mathfrak{q} = e^{i\pi/(p+1)}$).

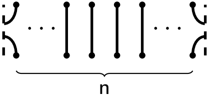
The Hamiltonian

Classical: $H_n = \sum_{i=1}^{n} e_i$

$e_i =$

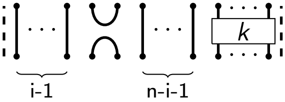


$e_n =$

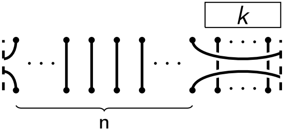


With a defect $\tilde{H}_n^k = \sum_{i=1}^{n-1} \tilde{e}_i$,

$e_i =$



$e_n =$



The theorem

For $M \in \text{mod}(\text{aTL}_{n+m})$, let $\tilde{M} \equiv \phi^u(1_{\text{aTL}_n} \otimes \rho_k)M$, then

$$\tilde{H}_n^k|_{\tilde{M}} \sim H_n|_{M \div_f^u \text{TL}_k \rho_k} \quad \text{or} \quad H_n|_{M \div_f^o \text{TL}_k \rho_k}$$
$$\tilde{T}_n^k|_{\tilde{M}} \sim T_n|_{M \div_f^u \text{TL}_k \rho_k} \quad \text{or} \quad T_n|_{M \div_f^o \text{TL}_k \rho_k}.$$

In particular,

$$\tilde{Z}_n^k|_M = Z_n|_{M \div_f^u \text{TL}_k \rho_k} \text{or} \quad Z_n|_{M \div_f^o \text{TL}_k \rho_k},$$

i.e. if M describe some model, $M \div_f^{u/o} V$ describes the same model with a V-defect.

Pros and cons

Pros

- Reps can be arranged in families $A[n]$, $B[n]$, etc. which describes the regulariz. of the same sector of the CFT.

$$A[n+m] \div_f^{u/o} B[m] \simeq A[n+k] \div_f^{u/o} B[k],$$

- Products of defects are easy

$$(A \div_f^u B) \div_f^u C \simeq A \div_f^u \underbrace{(B \otimes C)}_{\text{Tensor prod. of the TL cat.}} .$$

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Cons

- Any info. that is basis dependent is lost.
- Computing the iso-class of a fusion quotient is (relatively) easy, finding what it is is difficult.

The tensor product in open TL

Simple modules I_k, $k = 0, ..., n$ $\quad 0 \leq s_1, s_2 \leq p-1 \qquad \mathfrak{q} = e^{\frac{i\pi}{p+1}}$,

$$\mathrm{I}_{s_1} \otimes \mathrm{I}_{s_2} \simeq \bigoplus_{\substack{k=|s_1-s_2| \\ \text{step } =2}}^{\min(s_1+s_2, 2p-(r+s)-2)} \mathrm{I}_k, \tag{2}$$

Continuum limit: $\mathrm{I}_s \to \phi_{1,1+s}$ in $M(p+1, p)$.

The tensor product in open TL

Simple modules $\mathrm{I}_k,\ k = 0, \ldots, n \quad 0 \leq s_1, s_2 \leq p - 1 \qquad \mathfrak{q} = e^{\frac{i\pi}{p+1}},$

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Continuum limit: $\mathrm{I}_s \to \phi_{1,1+s}$ in $M(p+1, p)$. $\quad 0 \leq s_2 \leq p - 1 < s_2$

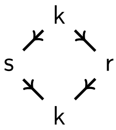
$$\mathrm{I}_{s_1} \otimes \mathrm{I}_{s_2} \simeq 0 \tag{3}$$

The tensor product in open TL

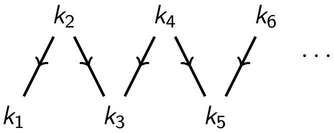
Simple modules $\mathrm{I}_k,\ k = 0, ..., n \quad p \leq s_1, s_2$

$$\mathrm{I}_{s_1} \otimes \mathrm{I}_{s_2} \simeq \text{ Complicated formula} \sim B \oplus P \oplus I \qquad (2)$$

$P \sim$



$B \sim$



Fusion quotient in aTL_n

Twisted RSOS $s_1, s_2, s_3 \in 1, \ldots, p$

$$\chi_{s_1,s_2} \div_f^u \mathsf{I}_{s_3-1} \simeq \sum_{\substack{k=|s_2-s_3|+1 \\ \text{step }=2}}^{\min(s_2+s_3-1, 2(p+1)-s_2+s_3-1)} \chi_{s_1,k} \ ,$$

Continuum limit:

$$\chi_{s_1,s_2} \sim \bigoplus_{r=1}^{p} \phi_{r,s_1} \otimes \bar{\phi}_{r,s_2}$$

$$\chi_{s_1,s_2} \div_f^u \mathsf{I}_{s_3-1} \sim \bigoplus_{r=1}^{p} \phi_{r,s_1} \otimes (\bar{\phi}_{r,s_2} \times_f \bar{\phi}_{1,s_3})$$

Fusion quotient in aTL_n

Twisted RSOS $s_1, s_2, s_3 \in 1, \dots, p$

$$\chi_{s_1,s_2} \div^o_f \mathsf{I}_{s_3-1} \simeq \sum_{\substack{k=|s_2-s_3|+1 \\ \text{step } = 2}}^{\min(s_1+s_3-1,2(p+1)-s_1+s_3-1)} \chi_{k,s_2} \ ,$$

Continuum limit:

$$\chi_{s_1,s_2} \sim \bigoplus_{r=1}^{p} \phi_{r,s_1} \otimes \bar{\phi}_{r,s_2}$$

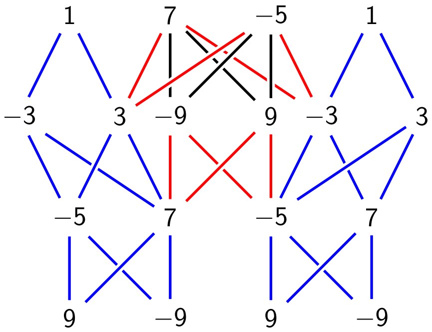
$$\chi_{s_1,s_2} \div^o_f \mathsf{I}_{s_3-1} \sim \bigoplus_{r=1}^{p} (\phi_{r,s_1} \times_f \phi_{1,s_3}) \otimes \bar{\phi}_{r,s_2}$$

Fusion quotient in aTL_n

Unknown in general: for $Y = -1$, $p = 2$, simples $\sim \mathbb{Z}$

$$\mathrm{W}_0 \div^{o}_{f} S_3[3] \simeq \mathrm{W}_{-1} \oplus Q^1_1$$

$Q^1_1 \sim$



So what now?

- Modular transformations?
- Massive gap in the rep. theory of aTL at root of unity.
- Rep. theory when hoops are non-diagonal is (mostly) in-existent.
- Physical meaning of those new boundary conditions?

The end

Thank you!

- Defects in minimal models
  - V.B. Petkova and J.-B. Zuber, *Generalized twisted partition functions*, Phys. Lett. B **504**, 157 (2001).
- Defects in A_p RSOS models
  - C. Chui, C. Merkat, P. Orrick and P.A. Pearce, *Integrable lattice realizations of conformal twisted boundary conditions*, Phys. Lett. B **517**, 429–435 (2001).
- Defects in aTL models, both cross and direct channel
  - J. Belletête, A.M. Gainutdinov, J.L. Jacobsen, H. Saleur, T.S. Tavares, *Topological defects in lattice models and affine Temperley-Lieb algebra* 1811.02551 (2018)
- Fusion products and quotients for the open case:
  - J. Belletête, *The fusion rules for the Temperley-Lieb algebra and its dilute generalization*, J. Phys. A: Math. Theor. **48**, 395205 (2015).

Example II: The XXZ spin-chain

The Hilbert space is $\mathbb{C}_2^{\otimes n}$ and

$$H_n(Q) = \sum_{j=1}^{n} \left(\sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ + \frac{\mathfrak{q} + \mathfrak{q}^{-1}}{4} \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right),$$

$$\sigma_{n+1}^z \equiv \sigma_1^z, \qquad \sigma_{n+1}^\pm \equiv Q^{\mp 2} \sigma_1^\pm .$$

$$Y = (-1)^n \left(\mathfrak{q}^{S_z} Q^{-1} + \mathfrak{q}^{-S_z} Q \right), \qquad \bar{Y} = \mathfrak{q}^{S_z} Q + \mathfrak{q}^{-S_z} Q^{-1},$$

Example II: The XXZ spin-chain

$$H^u_{n-1}(Q) = \sum_j^{n-1}(a_j^- a_{j+1}^+ + a_{j+1}^- a_j^+ + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{4}\left(a_j^z a_{j+1}^z - 1\right)) + \left((1-\mathfrak{q}^{2a_1^z})a_{n-1}^- + Q^2(1-\mathfrak{q}^{-2a_{n-1}^z})a_1^-\right)\sigma_n^+,$$

$$\begin{array}{cc} (\dots)\otimes|\uparrow\rangle & (\dots)\otimes|\downarrow\rangle \end{array}$$
$$\sim \begin{pmatrix} H_{n-1}(-Q\mathfrak{q}^{-1/2}) & \Delta \\ 0 & H_{n-1}(-Q\mathfrak{q}^{1/2}) \end{pmatrix},$$

$$\Delta = (1-\mathfrak{q}^{2a_1^z})a_{n-1}^- + Q^2(1-\mathfrak{q}^{-2a_{n-1}^z})a_1^-$$

where $a_j^k = \sigma_j^k$, $k = z, \pm$, $j = 1, 2, \dots n-1$, and

$$a_n^z \equiv a_1^z, \qquad a_n^\pm \equiv (Q^2\mathfrak{q}^{-\sigma_n^z})^{\mp 1}a_1^\pm. \tag{3}$$

Example II: The XXZ spin-chain

$$Y = (-1)^n(\mathfrak{q}^{S_z}Q^{-1} + \mathfrak{q}^{-S_z}Q),$$

$$\bar{Y} \sim \begin{array}{c} \\ \end{array}\overset{\qquad(\dots)\otimes|\uparrow\rangle \qquad\qquad\qquad (\dots)\otimes|\downarrow\rangle}{\begin{pmatrix} Q_-\mathfrak{q}^{\tilde{S}_z} + Q_-^{-1}\mathfrak{q}^{-\tilde{S}_z} & Q(\mathfrak{q}-\mathfrak{q}^{-1})^2\tilde{S}_- \\ 0 & Q_+\mathfrak{q}^{\tilde{S}_z} + Q_+^{-1}\mathfrak{q}^{-\tilde{S}_z} \end{pmatrix}},$$

where $Q_\pm \equiv -Q\mathfrak{q}^{\pm 1/2}$, and $\tilde{S}_-$, $\mathfrak{q}^{\pm\tilde{S}_z}$ are the standard $U_\mathfrak{q}(\mathfrak{sl}_2)$ generators on $n-1$ spins.