Boundary renormalisation group interfaces

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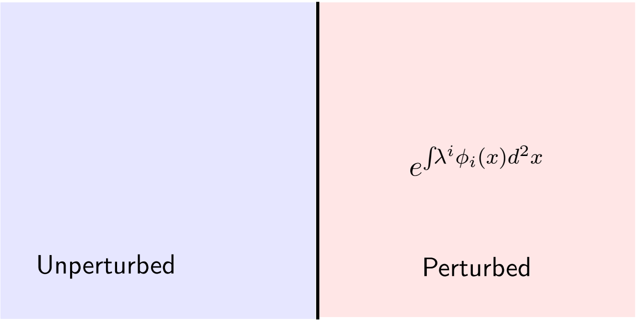
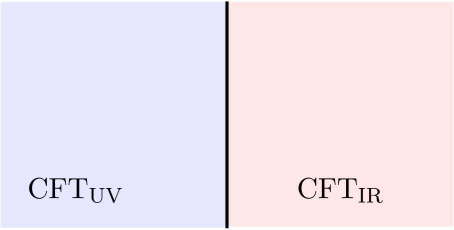
We are interested in RG flows of 2D Euclidean QFTs. We would like to have a tool for establishing a global picture of flows that start from a given UV fixed point. An interesting object called RG interface was proposed by I. Brunner and D. Roggenkamp (2007). If the flow is triggered by a perturbation $\Delta S = \int d^2x \, \lambda^i \phi_i(x)$ of the UV fixed point one can consider putting this perturbation on a half plane and letting it flow with the renormalization group flow.



In the far infrared, if the RG flow end up in a non-trivial fixed point, we obtain a conformal interface that is a 1-dimensional object separating two conformal field theories and respecting the conformal symmetry.



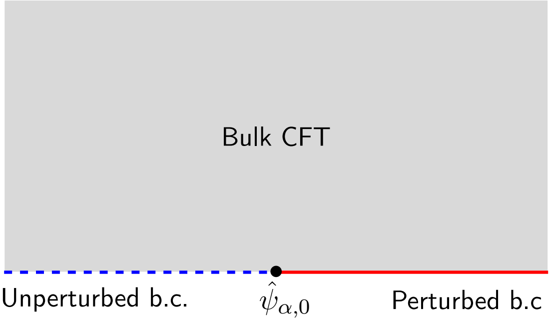
This object is conformal that imposes lots of restrictions on it and at the same time it must carry some information about RG flow between $\mathrm{CFT}_{\mathrm{UV}}$ and $\mathrm{CFT}_{\mathrm{IR}}$.

In general it is hard to construct conformal interfaces between two given CFTs. For bulk RG flows between two non-trivial CFTs the RG interfaces were constructed by D. Gaiotto, 2012 for the RG flows between neighbouring Virasoro minimal models triggered by $\psi_{1,3}$ perturbation. The construction generalises to similar flows between other series of CFTs also considered in Ahn, Stanishkov, 2014; Stanishkov, 2016. Gaiotto's construction is rather involved algebraically and does not allow to calculate many quantities related to the interface.

Most RG flows end up in a trivial CFT with only the vacuum state remaining. The corresponding RG interface is given by a conformal boundary condition in the UV theory. This was studied by A.K., 2016; J. Cardy, 2017.

Boundary RG interfaces

The situation is more tenable to analytic control for boundary RG flows. In this case we consider a UV CFT on a half-plane with a conformal boundary condition. We perturb this theory by a *boundary* relevant operator with a coupling α on a half line belonging to the boundary.



This triggers an RG flow between two conformal boundary conditions.

This set up has two interesting features:

- Boundary RG flows always end in a non-trivial fixed point that at least contains the Virasoro identity tower.
- The boundary RG interface is 0-dimensional and is thus described by a local boundary condition changing operator. We will call it an RG operator.

For Virasoro minimal models we know very little about conformal interfaces between the bulk theories but we do know all conformal boundary conditions (given by Cardy's construction in the unitary series) and we do know all boundary-condition changing operators between them.

If we could somehow describe what special properties distinguish RG interfaces amongst all conformal interfaces we could potentially find some selection rules that would restrict possible end points for given flows and thus help to chart the global phase diagram of the flows originating from a given conformal boundary condition.

Plan for the rest if the talk

- discuss the simplest model for a boundary flow in which the RG operator can be explicitly constructed
- discuss a variational method based on RG operators
- discuss applications to phase diagram of boundary flows in the tricritical Ising model
- formulate general conjectures about the RG operators

Boundary magnetic field model

There is a boundary RG flow for which we can construct the RG operator $\hat{\psi}_{\alpha,0}$ for arbitrary values of the coupling α. Consider the two-dimensional critical Ising model on an upper half-plane with complex coordinates $z = x + iy$, $\bar{z} = x - iy$ and the boundary at $z = \bar{z}$. The model admits two elementary conformal boundary conditions corresponding to keeping the boundary spin fixed (up or down) or allowing it to fluctuate freely. In terms of free massless fermions $\psi(z)$, $\bar{\psi}(\bar{z})$:

$$\text{free b.c.} \; : \; \psi(z) = \bar{\psi}(\bar{z})\,, \quad z = \bar{z}$$

$$\text{fixed b.c.} \; : \; \psi(z) = -\bar{\psi}(\bar{z})\,, \quad z = \bar{z}\,.$$

For the free boundary condition the vacuum is doubly degenerate. In the quantisation with time along the boundary we can choose a basis of spin up and spin down vacua: $|0, \pm\rangle$.

To describe the boundary spin operator we introduce following Goshal, Zamolodchikov, 1993 , Chatterjee, Zamolodchikov, 1993 a boundary fermion field $a(x)$ with a two-point function

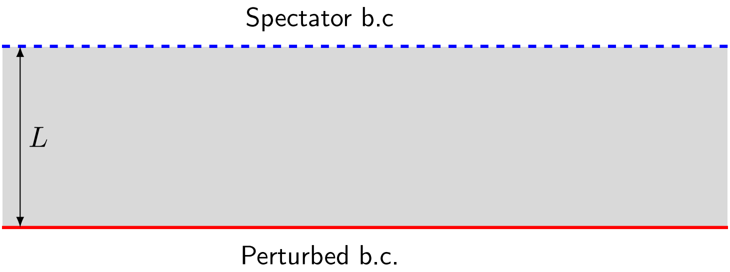
$$\langle a(x)a(x')\rangle = \frac{1}{2}\mathrm{sign}(x-x')\,.$$

The corresponding operator a acts on the vacua as $a|0,\pm\rangle = |0,\mp\rangle$.
Starting with a free boundary condition we perturb it by switching on a boundary magnetic field h that couples to the boundary spin operator: $\sigma_B(x) = i(\psi(x)+\bar{\psi}(x))a(x)$.

$$S = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy[\psi\partial_{\bar{z}}\psi+\bar{\psi}\partial_{z}\bar{\psi}]+\int_{-\infty}^{\infty} dx\Big[-\frac{i}{4\pi}\psi\bar{\psi}+\frac{1}{2}a\partial_x a+ih(\psi+\bar{\psi})a\Big]\,.$$

This model is Gaussian. It is arguably the simplest analytically solvable model for QFT that interpolates between two *non-trivial* fixed points. At $h = \pm\infty$ we arrive at a fixed spin boundary condition.

The model can be solved via a Bogolyubov transformation. It is convenient to put the system on a strip of width L



We get a discrete state space described as a Fock space.

For a free spectator the unperturbed theory is described by Neveu-Schwarz fermions. The Fock space is built using the basis

$$a^{\dagger}_{k_1+1/2}\dots a^{\dagger}_{k_N+1/2}|0\rangle\,, \quad \text{N–even}\,, \qquad a^{\dagger}_{k_1+1/2}\dots a^{\dagger}_{k_N+1/2}|a\rangle\,, \quad \text{N–odd}\,.$$

The creation and annihilation operators that diagonalise the perturbed Hamiltonian are

$$\begin{aligned} b^{\dagger}_{h,n} &= \sum_{k=0}^{\infty}(A_{n,k}a^{\dagger}_{k+1/2} + B_{n,k}a_{k+1/2}) + \frac{a}{f_n}\,, \\ b_{h,n} &= \sum_{k=0}^{\infty}(A^{*}_{n,k}a_{k+1/2} + B^{*}_{n,k}a^{\dagger}_{k+1/2}) + \frac{a}{f_n} \end{aligned}$$

where $A_{n,k}$ and $B_{n,k}$ are *infinite* matrices. This transformation is realised by a unitary operator for all finite values of the coupling h.

The Bogolyubov transformation is singular at the IR fixed point $h = \pm\infty$

The limit $h \to \infty$ gives well-defined transformation matrices but it is no longer invertible (we can no longer get the boundary fermion a) and the vacuum vector has infinite norm. We have what is called an improper Bogolyubov transformation which is very common in QFT. This is an important discontinuity reflecting the jump in UV behaviour. It has other guises, some to be mentioned later.

Construction of the RG operator

The boundary RG operator acts between two separate Fock spaces:

$$\hat{\psi}_{0,\lambda} : \mathcal{F}^h \to \mathcal{F}^0$$

and satisfies the intertwining property for the respective Heisenberg algebras:

$$\hat{\psi}_{0,\lambda} b^{\dagger}_{h,n} = \Big[\sum_{k=0}^{\infty}(A_{n,k}a^{\dagger}_{k+1/2} + B_{n,k}a_{k+1/2}) + \frac{a}{f_n}\Big]\hat{\psi}_{0,\lambda}$$

and similarly for the annihilation operator.
I calculated the matrix elements of $\hat{\psi}_{0,\lambda}$ explicitly up to an overall factor given by a certain determinant of an infinite-dimensional matrix.

Aren't any calculation in Gaussian models trivial?

Any quantity in a Gaussian model can be calculated via a Gaussian integral

$$\int \prod_i \frac{d\phi_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi_i A^{ij}\phi_j + J^i\phi_i} = (\det(A))^{-1/2} e^{\frac{1}{2}J^i A^{-1}_{ij} J^j}$$

One needs to invert infinite dimensional matrices and calculate determinants of infinite matrices.

What do we learn from the explicit construction of $\hat{\psi}_{0,\lambda}$?

- In the $h \to \infty$ limit we get

$$\lim_{h\to\infty} \mathcal{N}(h)\hat{\psi}_{0,\lambda} = \sigma$$

  where $\mathcal{N}(h)$ is a divergent numerical factor (multiplicative renormalisation), and σ is a conformal primary of dimension $1/16$ – the lowest dimension boundary condition changing operator between the free and fixed boundary conditions.

- We can study the first correction to σ near $h = \infty$:

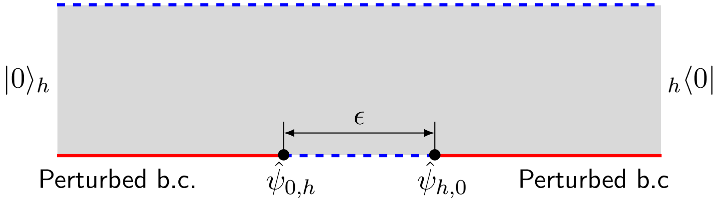
$$\hat{\psi}_{0,\alpha} \sim \exp\Big(-\delta_h \partial_\tau\Big)\hat{\psi}_{0,\infty}$$

$$\delta_h \equiv \frac{\ln(h^2 L/\pi))}{2(\pi h)^2}\,.$$

Norm of the vacuum and OPE of $\hat{\psi}$ with itself

$$ {}_h\langle 0|0\rangle_h = \lim_{\epsilon\to 0}\langle \hat{\psi}_{0,h}(-\frac{\epsilon}{2},0)\hat{\psi}_{h,0}(\frac{\epsilon}{2},0)\rangle = 1\,. $$

This equals 1 because the Bogolyubov transformation at hand is proper and $\hat{\psi}_{0,h}(0,0)$ is a unitary operator. We illustrate this on the following figure.



Similarly we can swap the operators $\hat{\psi}_{h,0}$, $\hat{\psi}_{0,h}$ and obtain a representation of the norm squared of $|0\rangle$ as represented in the perturbed theory Fock space. That norm squared also equals to 1.

Variational method

To make use of RG operators in general situation, when we cannot solve the perturbed theory analytically we employ the variational method.
The idea is to take a family of trial states:

$$|x_1\,, \dots\,, x_n\rangle$$

parameterised by some parameters: $x_1\,, \dots\,, x_n$ and then find an approximation to the vacuum by minimising the average energy

$$\frac{\langle x_1\,, \dots\,, x_n|H|x_1\,, \dots\,, x_n\rangle}{\langle x_1\,, \dots\,, x_n|x_1\,, \dots\,, x_n\rangle}$$

The last published scientific paper of R.P. Feynman was

"Difficulties in Applying the Variational Principle to Quantum Field Theories"

(published in Variational Calculations in Quantum Field Theory, pp. 28-40 (1988))

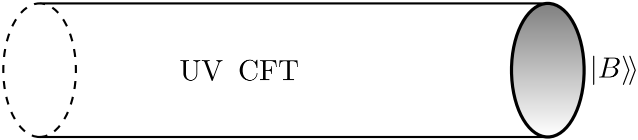
Feyman noted 3 difficulties

- Sensitivity to high frequencies (the UV divergences)
- Hard to come up with anything but Gaussian states if we want the norm to be calculable exactly
- Need to be able to calculate the average via functional integral; that's again is difficult as we don't want to use perturbation theory to have enough precision

Cardy's variational ansatz

Cardy, 2017 proposed an ansatz for a vacuum state in a massive RG flow. His ansatz uses a boundary state $|B\rangle\rangle$ smoothed out by a translation:

$$|B,\tau\rangle = e^{-\tau \hat{H}^{\rm UV}_{\rm CFT}}|B\rangle\rangle\,.$$



where the choice of conformal boundary condition B and the parameter τ are to be minimised over. Cardy has used this ansatz to map out the massive flows in the Virasoro minimal models in terms of conformal boundary states. The latter can be thought to label infrared massive phases.

For a theory on a circle of circumference $L \to \infty$ with a perturbed Hamiltonian

$$\hat{H} = \hat{H}_{\rm CFT} + \sum_i \lambda^i \int \hat{\phi}_i dx$$

we calculate, in the limit of large L:

$$\frac{\langle B, \tau | \hat{H} | B, \tau \rangle}{\langle B, \tau | B, \tau \rangle} \to \boxed{E_B \equiv \frac{\pi c}{24(2\tau)^2} + \sum_i \lambda^i \frac{A_B^i}{(2\tau)^{\Delta_i}}}$$

where A_B^i stands for a 1-point function of ϕ_i on a disc with boundary condition B.

Variational ansatz for boundary RG flows

Consider now a boundary RG flows between two BCFTs. Let

$$\hat{\psi} : \mathcal{H}^{\mathrm{IR}} \to \mathcal{H}^{\mathrm{UV}}$$

be the RG operator for this flow. The vacuum state of the infrared BCFT as a state in the UV BCFT is then

$$\hat{\psi}(0)|0\rangle_{\mathrm{IR}} \in \mathcal{H}^{\mathrm{UV}} \, .$$

It has an infinite norm because the OPE of $\hat{\psi}$ with its conjugate is singular

$$\hat{\psi}^{\dagger}(\tau)\hat{\psi}(0) \sim \frac{1}{\tau^{2\Delta}} + \dots$$

Following Cardy's idea we smear the vacuum vector by shifting the position of $\hat{\psi}$:

$$|\tau, \hat{\psi}\rangle = \hat{\psi}(-\tau)|0\rangle_{\rm IR}$$

This is a variational ansatz that depends on

- Choice of the conformal boundary condition describing the IR fixed point
- Choice of the RG operator $\hat{\psi}$
- The smoothing parameter τ

For a perturbed boundary Hamiltonian on a strip

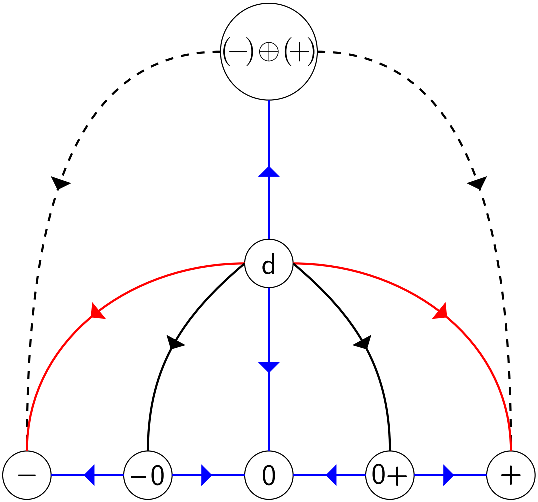
$$\hat{H} = \hat{H}_{\rm BCFT} + \sum_i \lambda^i \psi_i(0)$$

the variational energy at $L \to \infty$ is

$$\boxed{E_{\hat{\psi}}(\tau) = \frac{\Delta}{\tau} + \sum_i \lambda^i \left(\frac{C_{\hat{\psi}\hat{\psi}\psi_i}}{C_{\hat{\psi}\hat{\psi}1}}\right) \frac{1}{(\tau/2)^{\Delta_i}}}$$

Boundary RG flows in tricritical Ising model

Tricritical Ising model is a Virasoro unitary minima model with central charge $c = 7/10$. It has 6 elementary conformal boundary conditions labeled as: $(d), (0), (0+), (-0), (+), (-)$. The phase diagram of RG flows originating from elementary boundary conditions was put together by I. Affleck, 2000.



$(-)\oplus(+)$
d
$-$
-0
0
$0+$
$+$

Some conjectures

For boundary RG flows triggered by strictly relevant operators, i.e. $\Delta_i < 1$ we conjecture that

Conjecture 1 The RG operator $\hat{\psi}$ is always a primary.

Conjecture 2 For a flow triggered by operators ψ_i the IR boundary condition and the RG operator must be such that $C_{\hat{\psi},\hat{\psi},\psi_i} \neq 0$. In other words we must have in the OPE terms

$$\hat{\psi}^{[\mathrm{UV},\mathrm{IR}]}(0)\hat{\psi}^{[IR,UV]}(\tau) \sim \frac{\psi_i}{\tau^{2\Delta-\Delta_i}}$$

for each i. (A weaker form may refer to the case when only one primary perturbation ψ_i is present.)

Both of these conjectures can be substantiated by the use of variational ansatz.

Conjecture 3 Perhaps the leading irrelevant operator χ^{IR} along which the boundary RG flow arrives at the IR fixed point is contained in the OPE of $\hat{\psi}$ taking in the other order:

$$\hat{\psi}^{[\mathrm{IR},\mathrm{UV}]}(0)\hat{\psi}^{[\mathrm{UV},\mathrm{IR}]}(\tau) \sim \frac{\chi^{\mathrm{IR}}}{\tau^{2\Delta-\Delta_{\chi}}} \ ?$$

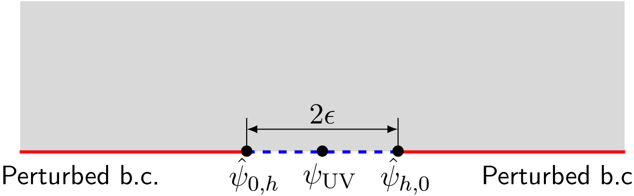
Note that the stress-energy tensor is always contained in such OPE. While we set up the problem so that we do know the perturbing operator in the UV we rarely know the leading irrelevant operator in the IR.

Open problems

- It would be desirable to demonstrate the conjectured statements without relying on the variational method.
- Understand the variational method (for bulk and boundary flows better). While Cardy's ansatz solves the last two problems mentioned by Feynman it is unclear whether it also resolves the first problem (sensitivity to UV behaviour).
- Understand the role of off-diagonal terms in the variational ansatz. If we take them into account can the method handle the critical lines separating basins of attraction of different fixed points?
- Understand transport of local operators by RG interfaces

Transport of local operators

While the operator $\hat{\psi}_{0,h}$ establishes a mapping between states in the unperturbed and perturbed theories, since at finite h the usual CFT state-operator correspondence does not work, this mapping does not automatically establish a mapping of local operators. To establish the latter we can surround a UV CFT operator $\psi_{\rm UV}$ by $\hat{\psi}_{0,h}$ and its conjugate taken some finite distance ϵ away from $\psi_{\rm UV}$.



One should then examine the limit $\epsilon \to 0$.