W-algebras and their representations*

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1 Introduction to W-algebras

A conformal field theory describes a set of fields which depend on the coordinates $(z, \bar{z})$ on the complex plane, and the space on which these fields act. Every conformal field theory has a subset of fields which are independent of $\bar{z}$, and this is called its chiral algebra.

It would be nice to classify all possible chiral algebras; known examples are affine Lie algebras, the Virasoro algebra and the super-Virasoro algebra. What can be more general than these?

In order to answer this question, we have to have a framework in which to discuss this problem, and some associated notation. One can take different approaches, treating conformal field theory as e.g. a D–module [42], a Vertex operator algebra [42] or the approach we shall follow, a Meromorphic cft [38]. Of these three, the last is the most similar to the physicists' methods for performing calculations in conformal field theory, and so we shall use this, if only because it allows us to prove some results which otherwise are a bit mysterious.

1.1 Meromorphic conformal field theory

Meromorphic conformal field theory is an abstraction and formalisation of the properties of conformal field theories where the only singularities which occur in the operator product expansions are poles. It is not the only form of conformal field theory which may be formulated rigorously, but it has been done so.

The structure of meromorphic conformal field theory (mcft) is developed in [38], and we shall only present a brief summary here. A mcft consists of a Hilbert space $\mathcal{H}$ and a vertex operator map from a dense subspace $\mathcal{F}$ of $\mathcal{H}$ into the space of fields. There are two distinguished states: the vacuum $|0\rangle$ and the 'conformal state' $|L\rangle$, whose vertex operator is the stress-energy tensor of the theory, and whose modes form a copy of the Virasoro algebra.

The vertex operator V is a map $V\colon \mathcal{H} \times \mathbb{C} \to \mathrm{End}(\mathcal{H})$ which has to satisfy the following conditions:

[1] $V(|\phi\rangle, z)|0\rangle = e^{zL_{-1}}|\phi\rangle$

[2] $\langle\phi_1|V(|\psi\rangle, z)|\phi_2\rangle$ *is a meromorphic function*

[3] $\langle\phi_1|V(|\psi\rangle, z)V(|\chi\rangle, z')|\phi_2\rangle$ *is a holomorphic function for* $|z| > |z'|$.

[4] $\langle\phi_1|V(|\psi\rangle, z)V(|\chi\rangle, z')|\phi_2\rangle = \epsilon\langle\phi_1|V(|\chi\rangle, z')V(|\psi\rangle, z)|\phi_2\rangle$ *by analytic continuation, where* $\epsilon = 1$ *unless both* ψ *and* χ *are fermionic, in which case* $\epsilon = -1$

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The distinguished state $|L\rangle$ corresponds to the Stress-Energy tensor,

$$V(|L\rangle, z) \equiv L(z) \equiv \sum_m L_m z^{-m-2} , \tag{1}$$

where the modes L_m satisfy the Virasoro algebra,

$$[\, L_m \,, L_n \,] = \frac{c}{12} m(m^2-1)\delta_{m,-n} + (m-n)\, L_{m+n} , \tag{2}$$

where c is a central element called the central charge. The subset L_{-1}, L_0, L_1 generate an $su(1,1)$ subalgebra which is the Lie algebra of the global Möbius transformations and which leave $|0\rangle$ invariant.

These axioms allow us to prove the operator product expansion, that is

$$V(|\psi\rangle, z) V(|\chi\rangle, z') = V(V(|\psi\rangle, z-z')|\chi\rangle, z') , \tag{3}$$

and to show that $L_{-1} \equiv \partial/\partial z$, that is

$$V(L_{-1}|\psi\rangle, z) = \frac{\partial}{\partial z} V(|\psi\rangle, z) . \tag{4}$$

For a state $|\psi\rangle$ of definite L_0 eigenvalue h, we introduce the mode expansion

$$V(|\psi\rangle, z) = \psi(z) = \sum_m \psi_m \, z^{-m-h} , \tag{5}$$

so that the operator product expansion (3) of two fields $\psi(z)$ and $\phi(z')$ of weights h and h' becomes

$$\psi(z)\, \phi(z') = \sum_m (z-z')^{-m} V(\psi_{-h-m}|\phi\rangle, z') , \tag{6}$$

This now allows us to give a definition of the normal ordered product of $\psi(z)$ and $\phi(z')$ as

$$(\psi\phi)\,(z) = V(\psi_{-h}\phi_{-h'}|0\rangle, z) , \tag{7}$$

with modes

$$(\psi\phi)_m = \sum_{n\le -h} \psi_n \phi_{m-n} + \sum_{n > -h} \phi_{m-n}\psi_n . \tag{8}$$

This is a non-commutative, non-associative operation. There are many other normal ordering prescriptions, which all differ by finite local fields, for example the prescription of Nahm [45] which includes a projection onto the subspace annihilated by L_1.

How does this all work in practice? Let's consider the simplest non-trivial example, being the operator product of $L(z)$ with $L(z')$. Since these fields are both of weight 2, we can immediately use (6) to give

$$L(z)L(z') = \sum_m (z-z')^{-m} V(L_{m-2}L_{-2}|0\rangle, z') . \tag{9}$$

As we shall only be interested in the singular part, we only need to consider the states

$$L_{m-2}L_{-2}\,|0\rangle , \tag{10}$$

for $m > 0$. We use the Virasoro commutation relations to find

$$L_{m-2}L_{-2}\,|0\rangle = \Big(L_{-2}L_{m-2} + \frac{c}{2}\delta_{m,4} + m\, L_{m-4} \Big)\, |0\rangle , \tag{11}$$

$$\begin{array}{c|ccccc} m & 1 & 2 & 3 & 4 & \geq 5 \\ \hline L_{m-2}L_{-2}\,|0\rangle & L_{-1}L_{-2}\,|0\rangle & 2\,L_{-2}\,|0\rangle & 0 & (c/2)\,|0\rangle & 0 \end{array} \tag{12}$$

Combining these results, and remembering that $L_{-1} \equiv \partial/\partial z$, we get

$$L(z)\,L(z') = \frac{c/2}{(z-z')^4} + \frac{2\,L(z')}{(z-z')^2} + \frac{L'(z')}{z-z'} + \text{ regular terms .} \tag{13}$$

Using the standard method for obtaining the commutator of two modes from the double contour integral of the singular part of the operator product expansion, we recover the Virasoro algebra (2).

Taking Wick contractions may be much faster for working out operator product expansions of expressions involving only free fields, but for the more complicated algebras we will have, the method outlined above is the only practical method, and is the method by which we shall work out all operator product expansions.

1.2 What is a W algebra?

We are now in a position to define a W-algebra.

[1] *A W-algebra is a meromorphic conformal field theory.*
[2] *There is a distinguished set of fields $W^a(z)$ which are primary with respect to the Virasoro algebra and have conformal weight h^a*

This means that

$$L(z)\,W^a(z') = \frac{h^a\,W^a(z')}{(z-z')^2} + \frac{W^{a\prime}(z')}{z-z'} + \text{ regular terms ,} \tag{14}$$

or equivalently

$$[\,L_m\,, W^a_n\,] = (m(h^a-1)-n)\,W^a_{m+n}\ . \tag{15}$$

[3] *The Hilbert space $\mathcal{H}$ is spanned by states of the form*

$$W^{a_1}_{-m_1}W^{a_2}_{-m_2}\cdots\cdots W^{a_x}_{-m_x}\ L_{-n_1}\cdots\cdots L_{-n_y}|0\rangle\ , \tag{16}$$

which are ordered, in the sense that we consider all fields of the same type first, $a_i \geq a_{i+1}$, and that the modes are always increasing for fields of the same type, i.e. $n_j \geq n_{j+1} \geq 2$ and $m_j \geq m_{j+1} \geq h^{a_j}$ if $a_j = a_{j+1}$.

We can deduce from (3) that the action of L_n for any n, and of W^a_m for any a and m, on a state of this sort will give a finite sum of states of the same sort, so that this definition is consistent.

Furthermore, (6) and (16) mean that the operator product of two fields which are normal ordered polynomials in L, W^a and their derivatives closes on such normal ordered combinations, so that the operator product algebra of a W-algebra is closed in this sense.

2 Simple W-algebras

The whole subject of W-algebras started with the paper [49] of Zamolodchikov in which he presented the first non-trivial examples. He started by considering all $h^a = 1$, and found that the only algebras which resulted were direct sums of affine Lie algebras. Again, considering only $h^a = 2$ leads to direct sums of commuting Virasoro algebras.

Although considering mixtures of these two, i.e. some $h = 1$ and some $h = 2$ does lead to interesting possibilities [14], the first non-trivial example Zamolodchikov found was when he considered extending the Virasoro algebra by a single field of weight 3. The resulting algebra is known as the W_3 algebra, and since we shall illustrate most of our examples with this algebra, shall go through its derivation in detail in the next section.

2.1 The W_3 algebra

We shall consider a W-algebra with one extra field $W(z)$ of weight 3, so that

$$[\, L_m \, , W_n \,] = (2m - n)\, W_{m+n} \ , \tag{17}$$

and so that the Hilbert space is spanned by states of the form

$$W_{-m_1} \dots W_{-m_j} L_{-n_1} \dots L_{-n_k} |0\rangle \ , \tag{18}$$

where $m_i \geq m_{i+1} \geq 3$ and $n_i \geq n_{i+1} \geq 2$.

Now we can ask what the ope of $W(z)$ with $W(z')$ is, or equivalently the commutator $[\, W_m \, , W_n \,]$. Using (6) we have

$$\begin{aligned} W(z)\, W(z') = {} & \frac{V(W_3 W_{-3}|0\rangle, z')}{(z-z')^6} + \frac{V(W_2 W_{-3}|0\rangle, z')}{(z-z')^5} + \frac{V(W_1 W_{-3}|0\rangle, z')}{(z-z')^4} \\ & + \frac{V(W_0 W_{-3}|0\rangle, z')}{(z-z')^3} + \frac{V(W_{-1} W_{-3}|0\rangle, z')}{(z-z')^2} + \frac{V(W_{-2} W_{-3}|0\rangle, z')}{z-z'} \\ & + \text{ regular terms} \ . \end{aligned} \tag{19}$$

Since the Hilbert space is spanned by the states (18), we can express the states $W_m W_{-3}|0\rangle$ which occur here as follows:

$$\begin{aligned} W_3\, W_{-3}|0\rangle &= \alpha_1 |0\rangle \ , \\ W_2\, W_{-3}|0\rangle &= 0 \ , \\ W_1\, W_{-3}|0\rangle &= \alpha_2 L_{-2}|0\rangle \ , \\ W_0\, W_{-3}|0\rangle &= (\alpha_3 L_{-3} + \alpha_4 W_{-3})\, |0\rangle \ , \\ W_{-1}\, W_{-3}|0\rangle &= (\alpha_5 L_{-4} + \alpha_6 L_{-2} L_{-2} + \alpha_7 W_{-4})\, |0\rangle \ , \\ W_{-2}\, W_{-3}|0\rangle &= (\alpha_8 L_{-5} + \alpha_9 L_{-3} L_{-2} + \alpha_{10} W_{-5} + \alpha_{11} W_{-3} L_{-2})\, |0\rangle \ . \end{aligned} \tag{20}$$

We have the freedom to choose one of the 11 unknowns α_i to fix the scale of $W(z)$, and by convention we choose $\alpha_1 = c/3$.

To fix the remaining 10, we require that $[W_m \, , W_n]$ is antisymmetric in m and n and that the Jacobi identity

$$[\, L_m \, , [\, W_n \, , W_p \,]\,] + [\, W_n \, , [\, W_p \, , L_m \,]\,] + [\, W_p \, , [\, L_m \, , W_n \,]\,] = 0 \ , \tag{21}$$

holds. This results in

$$\begin{gathered}\alpha_2 = 2\,,\ \alpha_3 = 1\,,\ \alpha_4 = \alpha_7 = \alpha_{10} = \alpha_{11} = 0\,,\\ \alpha_5 = \tfrac{3}{5} - \tfrac{6}{5}\tfrac{16}{22+5c}\,,\ \alpha_6 = \tfrac{32}{22+5c}\,,\ \alpha_8 = \tfrac{2}{5} - \tfrac{4}{5}\tfrac{16}{22+5c}\,,\ \alpha_9 = \tfrac{32}{22+5c}\,.\end{gathered} \tag{22}$$

This has fixed all the unknowns in the W_3 commutators, and we still have not checked the Jacobi identity

$$[W_m,[W_n,W_p]] + [W_n,[W_p,W_m]] + [W_p,[W_m,W_n]] = 0\,. \tag{23}$$

However, this holds identically, and so we have arrived at a consistent set of operator product expansions. It is conventional to define

$$\Lambda(z) = (LL)(z) - \frac{3}{10}L''(z)\,, \tag{24}$$

so that the commutation relations which result are

$$\begin{aligned}[\,W_m\,,W_n\,] = {}&\frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0} + \frac{16}{22+5c}\Lambda_{m+n}\\ &+ \frac{(m-n)(2m^2-mn+2n^2-8)}{30}L_{m+n}\,.\end{aligned} \tag{25}$$

Eqn (25), together with (2) and (17), defines the W_3 algebra.

Several comments are now in order:

(1) To check the Jacobi identity (21) in the manner indicated, it is necessary to work out several new commutators, for example

$$[\,L_m\,,\Lambda_n\,] = (3m-n)\,\Lambda_{m+n} + \frac{(22+5c)}{30}m(m^2-1)\,L_{m+n}\,. \tag{26}$$

It is very easy to find this using methods analogous to those which gave (13), and is a useful exercise. To derive this commutator using the mode expansion for $\Lambda(z)$ is possible, but very quickly this method becomes impossible for normal ordered products of three or more fields.
However, in fact there are easier ways to fix 9 of the unknowns in (20), and that is to realise that they are the coefficients of states which are Virasoro descendants of the vacuum or of the Virasoro highest weight state $W_{-3}|0\rangle$; consequently, they can be found readily in terms of α_1 and α_4 using the descent equations of appendix B of [6].

(2) It is already clear that any attempt at classification of W-algebras will be hard, and that there will be many identities amongst W-algebras[1]. Consider what happens to the W_3 algebra at $c = -22/5$. The combination $16/(22+5c)$ diverges, but this is one of the structure constants of the algebra, appearing in (25). It is possible to solve this problem by rescaling W so that we then have the commutation relations

$$\begin{aligned}[\,L_m\,,\tilde W_m\,] &= (2m-n)\,\tilde W_{m+n}\\ [\,\tilde W_m\,,\tilde W_n\,] &= (m-n)\,\Lambda_{m+n}\\ &+\ \tfrac{22+5c}{16}\Big[\tfrac{cm(m^2-1)(m^2-4)}{360}\delta_{m+n,0} + \tfrac{(m-n)(2m^2-mn+2n^2-8)}{30}L_{m+n}\Big]\,,\\ [\,L_m\,,\Lambda_n\,] &= (3m-n)\,\Lambda_{m+n} + \tfrac{22+5c}{16}\big[\ m(m^2-1)\,L_{m+n}\ \big]\,,\\ &\ \vdots\end{aligned} \tag{27}$$

[1] For a fuller list of such relations, see e.g. [8, 9].

and if we continue we shall see that at $c = -22/5$, $\tilde{W}$ and Λ generate an ideal and so must be set to zero in any irreducible representation of the W_3 algebra, so that at $c = -22/5$ the W_3 algebra reduces to the Virasoro algebra.
However, the representation theory of the W_3 algebra naturally includes $c = -22/5$, and all expressions such as character formulae, fusion rules, etc, are regular at $c = -22/5$, so that many theorems which we might like to be true are going to be complicated by such facts.

(3) So far we have the commutation relations (2), (17) and (25) which satisfy the Jacobi identity. Is this sufficient to guarantee that the operator product is associative? In general, the answer is no. Consider the algebra with generators i, j, k and multiplication

$$\begin{aligned} i^2 &= j^2 = k^2 = 1 \,, \\ ij &= -ji = \alpha\, k \,, \\ jk &= -kj = \beta\, i \,, \\ ki &= -ik = \gamma\, j \,. \end{aligned} \tag{28}$$

The Jacobi identity is satisfied, and in fact the commutator algebra is isomorphic to $sl(2)$, but it is an easy exercise to show that this algebra is only associative for $\alpha = \beta = \gamma = \pm 1$.
However, given a commutator algebra satisfying the Jacobi identity, we can define an associative product by constructing the universal enveloping algebra and then saying that the algebra acts in this by multiplication. This is guaranteed to be an associative action.

(4) The W_3 algebra is not a Lie algebra since the commutator (25) includes an infinite sum of bilinears in the modes L_m. It is very hard to work with this algebra as a Lie algebra – it becomes necessary to introduce ever more generators such as Λ_m – but it is quite straightforward as a meromorphic conformal field theory.
In fact, if one wishes to consider a W-algebra in the same way as a Lie algebra as a space of modes with commutation relations, albeit quadratic or higher, it is quite hard to define what the space of modes is. It is clear to a physicist which expressions should make sense and which not – those which have a finite expectation value in any state – but this is rather recursive since this already uses the algebraic structure to evaluate the expectation value. We escape this dilemma by defining the W-algebra as an mcft.

3 Direct searches for W-algebras

Between 1990 and 1992, roughly, people looked for W-algebras, using essentially the method we have seen, that is to consider a set of weights $\{h^a\}$ and then, as we did for the W_3 algebra, consider the most general set of operator product expansions and then, by imposing the Jacobi identity (or some equivalent means), discover whether there are any consistent solutions or not. The following table gives a list of such algebras investigated so far in this way, and we then make comments on the entries in each of the columns of this table.

3.1 Allowed c values

The allowed sets of c values fall into two classes, either an algebra is consistent for all c values, or only a finite set.

A typical example of an algebra which is only consistent for a finite set of c values is the $W(2, 5/2)$ algebra with one extra field $W(z)$ of weight 5/2 which was considered by Zamolodchikov [49]. As for the W_3 algebra, the commutator $[W_m\,, W_n]$ is entirely fixed by requiring that (21) vanishes, but when

Table 1. Constraints on the consistency of W-algebras

$\{h^a\}$	Consistent c–values	Null fields	Names	DS	Coset	Ref.
$3/2$	**all**	**none**	$svir$	$\surd$	$\surd$	49
2	**all**	**none**	$vir \oplus vir$	$\surd$	$\surd$	49
$5/2$	$-13/14$	$11/2, ..$	$W(2,5/2)$			49
3	**all**	**none**	$WA_2 \equiv W_3$	$\surd$	$\surd$	49
$7/2$	$21/22, -19/6, -161/8$		$W(2,7/2)$			7
4	**all**	**none**	$WB_2 \equiv WC_2$	$\surd$	—	12
$9/2$	$25/26, -7/20, -125/22,$ $-279/10, -35$		$W(2,9/2)$			7
5	$6/7, -250/11, -7, 134 \pm 60\sqrt{5}$		$W(2,5)$			$7, 12, 44$
$11/2$	$-217/16$		$W(2,11/2)$			7
6	**all**	**none**	$WG_2 \equiv W(2{,}6)$	$\surd$	—	$7, 30, 44$
$13/2$	$9/34, -611/14, -111/10$		$W(2,13/2)$			7
7	$-25/2$		$W(2,7)$			7, 44
$15/2$	$25/28, -11/38, -39/10,$ $-473/34, -825/16, -59$		$W(2,15/2)$			7
8	$21/2, -1015/2, -224/65, -23,$ $-712/7, -3164/23,$ $350 \pm 252\sqrt{2}, -944/17$		$W(2,8)$			7, 44
9	$-1206/19, -14/11, -208/35,$ $-91/5, -71$		$W(2,9)$			22
10	$8/35, 25/26, -29, -2$		$W(2,10)$			22
11	$-36/13, -1826/23, -24$		$W(2,11)$			22
$3, 3$	**none**					7, 44
$3, 4$	**all**	**none**	$WA_3 \equiv WD_3$	$\surd$	$\surd$	7, 44
$3, 5$	**none**					7, 44
$4, 5/2$	**all**	**none**	$WB(0,2)$	$\surd$	$\surd$	22
$4, 7/2$	$1, -403/22$					22
$4, 4$	$1, -656/11$					7, 44
$4, 9/2$	$1, -141/2, -779/26$					22
$4, 5$	$1, -253/7, -1060/13$					22, 44
$4, 6$	**all**	$10, \ldots$	$Orb(svir)$			44
	all	$10, \ldots$	WD_{-1}	—	$\surd$	44
	all	**none**	WB_3	$\surd$	—	44
	all	**none**	WC_3	$\surd$	—	44
$6, 7/2$	$561/2$					22
$6, 9/2$	$-304/5$					22

Table 2. Constraints on the consistency of W-algebras (cont.)

$\{h^a\}$	Consistent c–values	Null fields	Names	DS	Coset	Ref.
$3,3,3$	$-2,-30$	$6,\ldots$				43
$3,4,5$	**all**	**none**	$WA_4 \equiv W_5$	$\surd$	$\surd$	39
	all	$8,8,\ldots$	$W_{\{2,3\}}$	$\surd$	$\surd$	39
$5,5,5$	-7	$8,\ldots$				43
$7,7,7$	$-25/2$	$10,\ldots$				43
$3,4,5,6$	**all**	**none**	$WA_5 \equiv W_6$	$\surd$	$\surd$	39

we insert this into (23) he found that W_{m+n+p} appears in the right hand side for all values of c except $c=-13/14$, and consequently it is only for this value of c that the Jacobi identity is satisfied.

For the purposes of this table we have allowed $c=-22/5$ as a consistent value for the W_3 algebra, although we have seen earlier that in fact it should be consistently truncated to the Virasoro algebra. This is a generic phenomenon: all the W-algebras with fields of weights greater than 2 have a finite set of c values where a structure constant is singular, and for which one must truncate the spectrum of fields. In fact, it occurs for the $WB(0,2)$ algebra with fields of weight 4 and 5/2 exactly at $c=-13/14$. In this case the structure constant appearing in the operator product $W^4W^4 \sim C^4_{44}W^4$ has a singularity, and so we must re-scale this field. Doing so we in fact remove it entirely from the W-algebra and we arrive at the $W(2,5/2)$ algebra above

The algebras which are consistent for only a finite set of c values are (almost all) uninteresting as W-algebras, as they arise in this way as special cases of the other algebras when the spectrum must be truncated. For example, the $W(2,8)$ algebra with $c=21/22, -944/17$ and $-712/7$ is a truncation of WE_8, at $c=-3164/23$ of WE_7, and all the allowable c values for the algebras $W(2,4,4)$ and $W(2,4,5)$ arise as truncations of the algebra WD_4 and WD_5 respectively [22]. However, they are certainly interesting as rational conformal field theories.

3.2 Null fields

It may happen for an algebra that the right hand side of the Jacobi identity

$$[\,W^a_m\,,[\,W^b_n\,,W^c_p\,]\,] + \text{ cyclic perms.} = X_{m+n+p}\;, \tag{29}$$

does not vanish identically. However, this need not be a problem if the combination X_{m+n+p} can be consistently set to zero. For example, this occurs in the $W(2,5/2)$ algebra at $c=-13/14$ where the Jacobi identity for the field W is broken by a term proportional to

$$U(z) = -28(LW') + 35(L'W) + 4W'''\;. \tag{30}$$

However, this field decouples from all expectation values and should be set to zero in any physical correlation function. Such a field we call a null field, and we indicate the first level h at which such a field occurs, if it is known.

While it might be that this phenomenon is solely due to the fact that the $W(2,5/2)$ algebra is only consistent for one c value, we see that this is not always the case, and that the algebras WD_{-1} and

$W_{2,3}$ which are consistent for all c values are also afflicted in this way, there being fields polynomial in the W-algebra generators which must be set equal to zero for all values of c.

This causes many problems if one attempts to follow a 'classical' route to their representation theory, as the universal enveloping algebra contains a very large non-trivial ideal, much larger than one would like, which should be set to zero for all representations of physical interest.

In all the W-algebras which are defined for all c values and for which the coupling constants have been explicitly found, the coupling constants depend upon c in a typical manner: if the coefficients appearing in the singular part of the operator product expansion are

$$W^a W^b \sim \frac{c}{h^a}\delta^{ab} + \sum C^{ab}_{d_1 d_2 \ldots d_n} (W^{d_1}(W^{d_2}(\ldots W^{d_n}))) + \text{ regular terms } , \tag{31}$$

then we find that

$$C^{ab}_{d_1 d_2 \ldots d_n} \sim \begin{cases} c^{1-n} & \text{No generic null fields} \\ c^{1-n/2} & \text{Generic null fields} \end{cases} \tag{32}$$

This has implications if we try to take the classical limit, which corresponds to $c \to \infty$ (see e.g. [10, 15, 27]).

3.3 Names

There is no consistent convention for naming W–algebras; the names given in the table are some which the reader might expect to meet, but it seems that almost every group has their own favourite convention. For some algebras, the names simply indicate the field content, e.g. $W(2, 5/2)$, but as this does not uniquely specify an algebra (there are four of type $W(2, 4, 6)$) it is not perfect. Other names, such as WA_n, $WB(0, 2)$, indicate that the algebra is a 'Casimir' type algebra, of which more later.

3.4 Constructions

In the final two columns it is indicated whether this algebra is known to be found via either of the two main methods of construction W-algebras, namely the Drinfel'd Sokolov construction and the Coset construction.

Each construction has its own merits, and since the W_3 algebra is our typical example and can be found using both methods, we shall go through these two constructions in some detail in the next two sections.

3.5 Limitations of this method

It is clear that this method can never produce a classification of W-algebras, only find a (hopefully representative) selection of algebras from which one may start to find conjectures and turn these conjectures into theorems.

However, it turns out that in fact the algebras listed in table 1 are not fully representative of W–algebras in general, for the following reason: One of the main limitations of this method is that it is technically hard to deal with more than one field of the same spin, and even harder to deal with W-algebras which include affine Lie subalgebras.

As Zamolodchikov said in his paper [49], it would surely be interesting to investigate algebras with fields of weight 1 and 2; in fact it runs out that the most general constructions of W-algebras now known will typically produce algebras with large affine Lie subalgebras, and that these were not noticed for a long time, simply because they were not looked for. As a result, the early literature is biased towards algebras with no weight 1 fields. This has now been rectified in the light of more recent work.

4 The Coset construction

As explained in Jurgen Fuchs' lectures, we can consider an affine algebra $\hat{g}$ with subalgebra $\hat{h}$, and then the coset algebra $W(\hat{g}/\hat{h})$ is the set of all fields polynomial in the currents of $\hat{g}$ and their derivatives which commute with the currents of $\hat{h}$.

The polynomials in the currents of an untwisted affine algebra always contain a canonical Virasoro algebra $L^g(z)$, which for each semisimple $\hat{g}$ is given by the Sugawara construction

$$L^g(z) = \frac{1}{2(k+h^\vee(g))} \left(J_g^a J_g^a \right)(z) \, , \tag{33}$$

where k is the level of g, $h^\vee$ is the dual Coxeter number of g, and which has central charge c^g,

$$c^g = c(g,k) = \frac{k \dim(g)}{k+h^\vee} \, . \tag{34}$$

Similarly there is a Virasoro algebra $L^h(z)$ corresponding to the subalgebra $\hat{h}$, and the coset algebra contains a Virasoro algebra given by

$$L^{g/h} = L^g - L^h \, . \tag{35}$$

The currents $J_h^a(z)$ are primary fields of weight 1 with respect to both L^g and L^h, hence they commute with $L^{g/h}$, and it is easy to check that $L^{g/h}$ satisfies the Virasoro algebra operator product expansion with

$$c^{g/h} = c^g - c^h \, . \tag{36}$$

Furthermore, given that the vacuum representation of $\hat{g}$ is a mcft, it is easy to check that the coset algebra $W(\hat{g}/\hat{h})$ is also a mcft, with Hilbert space consisting of all states ψ such that $J_m^a \psi = 0$ for J^a in h and $m \geq 0$.

In order for this coset algebra to be a W-algebra, we have to check that $W(\hat{g}/\hat{h})$ is generated by a finite set of fields. For most cases the answer to this question is not known, but it is generally believed that this is *always* the case. While $W(\hat{g}/\hat{h})$ may be generated by a finite set of fields, in most cases this is not freely-generated, i.e. there may be generic null fields of the sort discussed earlier which have to be consistently set to zero.

How can we check whether $W(\hat{g}/\hat{h})$ is a W-algebra? One way is by a counting argument. As explained in Fuchs' lectures, the branching functions of a coset are not necessarily single characters of irreducible representations of the coset algebra, but they are certainly sums of characters. This gives an easy proof that the coset

$$\frac{su(3)_k \oplus su(3)_1}{su(3)_{k+1}} \, , \tag{37}$$

contains the W_3 algebra. In this coset, g is the direct sum $su(3)_k \oplus su(3)_1$ of two affine $su(3)$ algebras at levels k and 1, with currents $J_k^a(z)$ and $J_1^a(z)$ respectively. Then the currents $J_{k+1}^a = J_k^a(z) + J_1^a(z)$ generate an $su(3)$ affine algebra at level $k+1$, which is the subalgebra h.

From the Kac-Weyl formula in Fuchs' lectures, we can calculate the branching function relating the vacuum representations of g and h for large positive integer k. So, if

$$\chi_0^g(q) = \chi_0^h(q) b(q) + \dots \, , \tag{38}$$

where $\chi_0^g(q)$ and $\chi_0^h(q)$ are the characters of the vacuum representations of g and h, then we find that the branching function $b(q)$ is

$$b(q) = 1 + q^2 + 2q^3 + 3q^4 + 4q^5 + \dots \, . \tag{39}$$

We know that the coset algebra must contain the Virasoro algebra, and so in particular the Hilbert space must contain the states

$$|0\rangle\,,\ L^{g/h}_{-2}|0\rangle\,,\ L^{g/h}_{-3}|0\rangle\,,\ L^{g/h}_{-2}L^{g/h}_{-2}|0\rangle\,,\ L^{g/h}_{-4}|0\rangle\,,\ L^{g/h}_{-3}L^{g/h}_{-2}|0\rangle\,,\ L^{g/h}_{-5}|0\rangle\ . \tag{40}$$

If we subtract the contributions of these states from $b(q)$, we see that there are still states at levels 3, 4 and 5 unaccounted for. If we call the state at level 3 $W_{-3}|0\rangle$, then those at levels 4 and 5 must be

$$W_{-4}|0\rangle\,,\ W_{-3}L^{g/h}_{-2}|0\rangle\,,\ W_{-5}|0\rangle\ , \tag{41}$$

and that exhausts $b(q)$ up to level 5. In the operator product of $W(z)$ with $W(z')$, only fields corresponding to states of weight 5 or less can arise; since we have just seen that these states are exactly those which we considered when we constructed the W_3 algebra the first time, associativity of the operator product expansions forces us to conclude that, exactly as before, the field W generates the W_3 algebra.

One must be very careful when applying this counting argument as it may be that 'generic' null vectors, or null vectors at particular c-values imply that there are relations between the states which one would believe were independent. In this case we circumvent the problem as the only null vectors which might exist amongst the states (40), (41) would be Virasoro descendants, and by taking k to be a large positive integer we are guaranteed by the representation theory of the Virasoro algebra that this is not the case. For a fuller discussion of this problem, see e.g. [10, 13].

It is possible to find the explicit construction [2, 3]. If we introduce the totally symmetric $SU(3)$ invariant tensor d_{abc},

$$d_{abc} = \mathrm{Tr}\left(T_a\{T_b,T_c\}\right)\ ,$$

then, up to an overall factor, $W(z)$ is given by

$$\begin{aligned} d_{abc}\Big(&k(3+k)(3+2k)J^a_1J^b_1J^c_1 - 3.4(3+k)(3+2k)J^a_kJ^b_1J^c_1 \\ &+3.4.5(3+k)J^a_kJ^b_kJ^c_1 - 3.5J^a_kJ^b_kJ^c_k\Big)\ . \end{aligned} \tag{42}$$

The central charge of the W_3 algebra so constructed is

$$c = 2\left(1 - \frac{12}{(k+3)(k+4)}\right)\ . \tag{43}$$

4.1 Merits of this construction

It is known that for positive integer k, the vacuum representation of the affine algebras are unitary, and consequently the W_3 algebra has unitary representations for c-values (43), and any unitary representation of $su(3)_k \oplus su(3)_1$ for positive integer k automatically induces a unitary representation of the W_3 algebra. This is the only construction so far that is proven to lead to unitary representations of the W_3 algebra with $c<2$, and it is firmly believed that there are no more unitary representations with $c<2$ than those which arise this way.

Furthermore, it is believed that in this case the branching functions of the coset are actually the characters of irreducible representations of the W_3 algebra, although this is not yet proven. If this is the case, then one can now consider the classification of all the modular invariant partition functions of theories with W_3 symmetry and c given by (43).

If $\mathcal{H}$ is the full Hilbert space of a conformal field theory, including all local fields, not only those in the chiral algebra, then as Fuchs explained, it splits into the direct sum of representations of the left (z dependent) and right ($\bar{z}$ dependent) chiral algebras,

$$\mathcal{H} = \bigoplus n_{ii'} H_i \otimes \bar{H}_{i'} \ , \tag{44}$$

where H_i is the representation space of the left chiral algebra, and $\bar{H}_{i'}$ of the right, and $n_{ii'}$ are non-negative integers, and where the vacuum sector arises only once, i.e. $n_{00} = 1$. Then the partition function on a torus is given by

$$Z = \sum n_{ii'} \chi_i(q) \bar{\chi}_{i'}(\bar{q}) \ , \tag{45}$$

where $\chi_i(q)$ and $\bar{\chi}_{i'}(\bar{q})$ are the characters of these spaces. If the chiral algebra is an $\widehat{su}(2)$ or $\widehat{su}(3)$ current algebra, then all possible sets $n_{ii'}$ have been classified using the fact that Z should be invariant under reparametrisations of the torus, that is if $q = \exp(-2\pi \mathrm{i} \tau)$, then Z should be invariant under $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. For $\widehat{su}(2)$ this is the famous ADE classification of Cappelli et al. [16], and Gannon has found a similar result for $\widehat{su}(3)$ [36].

More recently, the classification of the modular invariant partition functions of theories with W_3 symmetry and c in (43) has been achieved by Gannon and Walton [37]. Again this is of an A-D-E type, by which is meant that there are certain infinite series of modular invariant partition functions for all k, and certain extra discrete invariants, which this time occur for $k = 4, 5, 8, 9, 20$ and 21. In the 'A' type series the chiral algebra is purely the W_3 algebra, whereas in the others it may be increased by the addition of other extra fields.

This is another case where the classification of conformal field theories using W-algebras becomes inherently difficult! These extra cases with increased symmetry naturally fall into the W_3 algebra classification, but yet their algebras are larger – similarly they will certainly also fall into the classification of modular invariant partition functions for larger algebras which have had their spectrum truncated.

5 The Drinfel'd–Sokolov construction

The classical Drinfel'd–Sokolov construction, which is the basis of the quantum Hamiltonian reduction construction, is an old construction from the theory of classical integrable systems [20]. For the sake of brevity we shall simply state the ingredients and the method; for more details and the connection to W-algebras there are many reviews available e.g. [26].

Consider a set of currents $j^a(x)$ in a finite dimensional Lie algebra g with $x \in S^1$, whose Poisson brackets satisfy an affine Lie algebra,

$$\{ j^a(x), j^b(y) \} = \delta(x-y) f^{ab}_c j^c(y) + g^{ab} \delta'(x-y) \ . \tag{46}$$

Then consider the matrix J and functional X where

$$J = j^a(z) T_a \ , \ \ X = \int_0^{2\pi} \mathrm{d}x \ j^a(x) f_a(x) \ . \tag{47}$$

Then the action of X on J is a gauge transformation:

$$\begin{gathered} \delta_X J \equiv \{ J, X \} = [\, \hat{X} \, , J \,] + \hat{X}' \ , \quad \text{where} \quad \hat{X} = f_a(x) \, T_a \ , \\ [\, T^a, T^b \,] = f^{ab}_c T^c \ , \quad [\, T^a, T_c \,] = -f^{ab}_c T_b \ . \end{gathered} \tag{48}$$

We now consider various gauge-fixings and the space of functionals which are invariant under the residual gauge symmetry. For our purposes it will be sufficient to consider constructions based on an embedding

$$sl(2) \hookrightarrow g \ , \tag{49}$$

and further consider only those embeddings for which g decomposes into integer spins representations of $sl(2)$. If the generators of the $sl(2)$ are I^+, I^0, I^- then we can decompose g into eigenspaces of I^0,

$$g = \oplus g_m \text{ where } [\, I^0 \,, g_m \,] = m \, g_m \tag{50}$$

and also split g into three parts

$$g = g_- \oplus g_0 \oplus g_+ \text{ where } g_+ = \oplus_{m>0} \, g_m \ , \quad g_- = \oplus_{m<0} \, g_m \ , \tag{51}$$

Now we choose to fix the currents in g_- so that

$$J_+ = I_+ \ . \tag{52}$$

This leaves a residual g_- gauge symmetry generated by the Poisson brackets with the constrained currents. The polynomials in the remaining currents which are invariant under these residual gauge transformations will satisfy a classical W-algebra. We can always choose to gauge fix in the so-called 'highest weight gauge' by taking

$$J_{\text{fix}} = I_+ + \sum X^a W^a \ , \tag{53}$$

where X^a are highest weights of spin i^a for the $sl(2)$, and the W^a will have conformal weight $i^a + 1$. Let's do this explicitly in the simplest case where $g = sl(2)$.

5.1 DS construction for $g = sl(2)$

We take $sl(2) \hookrightarrow sl(2)$ with

$$I^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ I^0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \ , \ I^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \tag{54}$$

and define I^a so that $\text{Tr}(I^a I_b) = \delta^a_b$. If we put

$$J = j^a I_a = \begin{pmatrix} j^0 & j^- \\ j^+ & -j^0 \end{pmatrix} \ , \ X = \int_0^{2\pi} \mathrm{d}x \ f_+(x) j^+(x) + f_0(x) j^0(x) + f_-(x) j^-(x) \tag{55}$$

we find that

$$\{\, J \,, X \,\} = \left[\, \hat{X} \,, J \,\right] + \hat{X}' \ , \quad \text{where } \hat{X} = f_a(x) I^a = \begin{pmatrix} f_0/2 & f_+ \\ f_- & -f_0/2 \end{pmatrix} \ , \tag{56}$$

is consistent with the fundamental Poisson brackets

$$\begin{aligned} \{\, j^0(x) \,, j^\pm(y) \,\} &= \pm j^\pm(y) \, \delta(x-y) \ , \\ \{\, j^+(x) \,, j^-(y) \,\} &= 2\, j^0(y) \, \delta(x-y) + \delta'(x-y) \ , \\ \{\, j^0(x) \,, j^0(y) \,\} &= (1/2) \, \delta'(x-y) \ . \end{aligned} \tag{57}$$

These Poisson brackets are indeed a classical affine $\hat{su}(2)$ algebra, as we can check by considering the Poisson brackets of the modes

$$j^a_m = \int_0^{2\pi} \mathrm{d}x\, j^a(x) \exp(-\mathrm{i}mx)\ . \tag{58}$$

Now consider constraining J to be in the form

$$J_{\text{cons}} = \begin{pmatrix} j^0 & 1 \\ j^+ & -j^0 \end{pmatrix}\ , \tag{59}$$

with the residual gauge symmetry generated by $j^-(x)$,

$$J \mapsto AJA^{-1} + A' \quad \text{where } A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\ . \tag{60}$$

Using this residual gauge symmetry we can always transform J into the form

$$J_{\text{fix}} = \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \quad \text{where } l = j^+ + (j^0)^2 + (j^0)'\ . \tag{61}$$

Then, putting $l_m = \int_0^{2\pi} \mathrm{d}x\, l(x) \exp(-\mathrm{i}mx)$, it is an easy exercise to show that l_m the Poisson bracket algebra of the l_m is the Virasoro algebra, (up to a constant shift in l_0)

$$\{\, l_m\,, l_n \,\} = \frac{\mathrm{i}m^3}{2}\delta_{m+n,0} + \mathrm{i}(m-n)\, l_{m+n}\ . \tag{62}$$

Note that by setting $j^+(x)$ to zero, we recover a classical free field construction, $l(x) = (j^0)^2 + (j^0)'$.
In the general case $sl(2) \hookrightarrow g$, as we said before, it is possible to fix J in the form

$$J_{\text{fix}} = I_- + \sum X^a W^a\ , \tag{63}$$

where X_a is a highest weight of the $sl(2)$ of spin i_a and W^a is a primary field of weight $i_a + 1$. The embeddings (49) have been classified by Dynkin [18], and it is straightforward to find the spectrum of weights $\{h^a\}$ given an embedding.

For example, the embeddings of $sl(2) \hookrightarrow sl(n)$ are characterised by the partitions of n positive integers[2] and it is only the trivial partition of n which leads to no weight 1 fields in the resulting W-algebra. All other W-algebras obtained from the Drinfel'd-Sokolov construction based on $sl(n)$ will have at least a $u(1)$ subalgebra. For Drinfel'd-Sokolov constructions based on other algebras, there are more possibilities for $sl(2)$ embeddings which result in no weight 1 fields (see [15] for a full list) but still the embeddings with weight 1 fields greatly outweigh those without.

For each simple Lie algebra there is a canonical $sl(2)$ embedding with no singlets, and that is the principal embedding. In this case the spins of the representations appearing in the decomposition of the Lie algebra are exactly the exponents of the Lie algebra. For these embeddings the W-algebras are those originally found by Drinfel'd and Sokolov and are also known as the 'Casimir' W-algebras.

It is an easy exercise to show that the classical W-algebra obtained by this method from $sl(3)$ with the choice

$$I^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\ ,\ I^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\ ,\ I^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\ , \tag{64}$$

is the classical W_3 algebra.

[2] If we want to ensure that there are only integer spin i_a representations of $sl(2)$ more constraints are necessary

5.2 Quantisation of the Drinfel'd-Sokolov construction

The quantisation of the Drinfel'd-Sokolov construction has a somewhat chequered history, but it is now clear that the correct way is by the BRST method, and when applied to the Drinfel'd-Sokolov construction, this now goes under the name of 'Quantum Hamiltonian Reduction'. Again, there are several nice articles on this method as applied to W-algebras, e.g. [11,28,31], and I shall simply present an outline of the method and the results that it gives.

For notation, we shall take J^a to be currents in g, J^α to be currents in g_- which are to be constrained, and χ^α to be the value to which they are constrained. Then according to the BRST method, we introduce two fermionic fields $b^\alpha(z)$ and $c_\alpha(z)$ for each field to be constrained, with operator products

$$b^\alpha(z)\, c_\beta(z') = \frac{\delta^\alpha_\beta}{z-z'} \;, \tag{65}$$

and then form the operator

$$Q = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left\{ \, (J^\alpha - \chi^\alpha) c_\alpha - \frac{1}{2} f^{\alpha\beta}_{\gamma} \, (b^\gamma \, (c_\alpha c_\beta)) \, \right\} \;. \tag{66}$$

It is a simple exercise that $Q^2 = 0$. the W algebra is then the mcft with space of states

$$\ker Q \, / \, \mathrm{Im} Q \;. \tag{67}$$

Two important simplifications can be made:

(1) We introduce new currents $\hat{J}^a(z)$,

$$\hat{J}^a(z) = J^a(z) + f^{a\beta}_{\gamma} \, (\, b^\gamma c_\beta \,)\,(z) \;, \tag{68}$$

since these combinations will occur in the final answer.

(2) We note that Q can be split in two as $Q = Q_0 + Q_1$,

$$Q_0 = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left\{ \, J^\alpha c_\alpha - \frac{1}{2} f^{\alpha\beta}_{\gamma} \, (b^\gamma \, (c_\alpha c_\beta)) \, \right\} \;,\quad Q_1 = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} (-\chi^\alpha c_\alpha) \;. \tag{69}$$

As explained in [11], the calculation of $(\ker Q / \mathrm{Im} Q)$ can be split into separate calculations for the space of constrained currents and their associated b ghosts, and for the rest. The cohomology on the space of constrained currents and b ghosts is trivial and so we only need worry about the remaining currents, that is the c ghosts and the unconstrained currents.

De Boer and Tjin have found that this cohomology is identical as a vector space (i.e. ignoring the algebra structure) to the classical case, i.e. there is an independent field for each highest weight representation of $sl(2)$ in the decomposition of g, and furthermore these fields have the form

$$W^a = \hat{J}^a + \ldots + W[\hat{J}^a_{(0)}] \;, \tag{70}$$

where $\hat{J}^a(z)$ is the field corresponding to the highest weight X^a, and $W[\hat{J}^a_{(0)}]$ is an expression in the currents of g_0. It is further possible to show that this algebra closes as a W-algebra, although the structure constants have to be found explicitly in each case.

Some comments:

(1) One immediate consequence is that we know the relation between the levels of the affine Lie subalgebras which arise in the W-algebra. These correspond to singlets under the $sl(2)$, and so formula (70) reduces to

$$W^a = \hat{J}^a \ . \tag{71}$$

The only change in these currents from Quantum Hamiltonian reduction is the addition of ghost contributions which change the levels of each semisimple component by amounts which can be determined.

(2) Since the currents $\hat{J}^a$ still satisfy the relations of an affine Lie algebra with a shifted central term, we see that nothing is altered by restricting the expressions (70) to the term only involving the currents in g_0. If g_0 were an Abelian algebra, as happens for the principal embeddings, then this would mean that the currents in g_0 were free fields, and we would find a free-field construction of W-algebras. In fact many W-algebras were first discussed by Fateev and Luk'yanov exactly in terms of their free field constructions [23, 24]. However, in the general case g_0 is not Abelian, and so we find a construction of the W-algebra in the currents of some affine Lie algebra, generalising the free field construction.

5.3 Representations of W-algebras from quantum Hamiltonian reduction

Clearly we can consider the space (67) based on any representation of $\hat{g}$, and each of these cohomology space will be a representation of the relevant W-algebra. Frenkel, Kac and Wakimoto did this for various representations of $\hat{g}$, and found a consistent set of conjectures for characters of irreducible representations of W_3 minimal model representations.

The minimal models are those for which the sum in the partition function (45) has only a finite number of $n_{i,i'}$ non-zero; this is a very strong requirement, and is only possible for a discrete set of c values. Using their conjectures, Frenkel et al. found the minimal models of the W_3 algebras have

$$c = 2\left(1 - \frac{12(p-q)^2}{pq}\right) \ , \tag{72}$$

where p and q are coprime positive integers, greater than 2. The representations which occur in these models are labelled by four integers $[a, b; c, d]$ with $1 \le a, b, c, d$; $a+b \le p-1$; $c+d \le q-1$ and subject to the identifications

$$[a, b; c, d] \equiv [b, (p-a-b); d, (q-c-d)] \equiv [(p-a-b), a; (q-c-d), d] \ , \tag{73}$$

so that the number of different representations is $(p-1)(p-2)(q-1)(q-2)/3$, We shall leave a precise description of these representations and their definitions until section 6.4.

For $q = p+1$ we see that the c values (72) are the same as the series (43) obtained from the coset construction with $p = k+3, q = k+4$. In this case, the characters that Frenkel et al conjectured are the same as the branching functions of the coset model. Frenkel et al also computed the fusion rules of the W_3 algebra via the Verlinde formula [47], using the transformations of their characters under the modular group generated by S and T. However, although it is firmly believed to be the case, I think it is fair to say that there is still no rigorous proof that these expressions are the characters of the irreducible representations of the W_3 algebra.

6 Representations of the W_3 algebra and correlation functions

6.1 Introduction

All this discussion so far has been a bit academic if the study of these symmetry algebras does not help one work out correlation functions, which are the basic physical quantities of a conformal field theory. In this section, we hope to show that indeed it is possible to use W-algebra symmetry to find correlation functions, that this is a mathematically interesting problem, and that it is substantially harder than the corresponding problem for the Virasoro algebra, i.e. for theories with pure conformal symmetry. Purely for reasons of presentation, these discussions will be limited to the W_3 algebra, but they can just as easily be applied to any algebra.

6.2 A highest weight representation of the W_3 algebra

A highest weight representation of the W_3 algebra is a space on which L_0 is diagonalisable, and for which it has a minimal eigenvalue h, say. If the representation is to be irreducible, since W_0 and L_0 commute, this space must be one-dimensional, spanned by a state $|h,w\rangle$ for which

$$\begin{array}{ll} L_m|h,w\rangle = 0, m>0\ , & L_0|h,w\rangle = h\,|h,w\rangle\ , \\ W_m|h,w\rangle = 0, m>0\ , & W_0|h,w\rangle = w\,|h,w\rangle\ , \end{array} \tag{74}$$

and the whole space is generated by the action of the modes W_m and L_m on this state. We shall denote an irreducible representation variously by $L_{h,w}, L_{[ab;cd]}$ and L_a.

Since the whole conformal field theory has two chiral algebras, we should really consider fields $\Phi_{h,w;\bar{h},\bar{w}}(z,\bar{z})$, and states which carry representations of both left and right chiral algebras, but we shall essentially ignore the $\bar{z}$ dependence and suppress the dependence on h' and w'. Accordingly, we shall loosely say that the state $|h,w\rangle$ corresponds to a field $\phi_{h,w}(z)$. Then the first things we would like to know about this field is its operator product with $L(z)$ and $W(z)$. We assume that the rules are the same as those of mcft, and write

$$\begin{aligned} L(z)\phi(z') &= \frac{V(L_0|h,w\rangle,z')}{(z-z')^2} + \frac{V(L_{-1}|h,w\rangle,z')}{z-z'} + O(1) \\ &= \frac{h\phi(z')}{(z-z')^2} + \frac{\phi'(z')}{z-z'} + O(1) \end{aligned} \tag{75}$$

$$\begin{aligned} W(z)\phi(z') &= \frac{V(W_0|h,w\rangle,z')}{(z-z')^3}\frac{V(W_{-1}|h,w\rangle,z')}{(z-z')^2} + \frac{V(W_{-2}|h,w\rangle,z')}{z-z'} + O(1) \\ &= \frac{w\phi(z')}{(z-z')^3} + \frac{\hat{W}_{-1}\phi'(z')}{(z-z')^2} + \frac{\hat{W}_{-2}\phi'(z')}{z-z'} + O(1)\ . \end{aligned} \tag{76}$$

However this is a real mess – we have had to introduce two new fields $\hat{W}_{-1}\phi(z)$ and $\hat{W}_{-2}\phi(z)$ and a priori we do not have any real understanding of their nature. It would be very nice if these fields could be interpreted geometrically, but for the moment we shall have to be happy with their algebraic properties.

Using these two operator product expansions, we can find the commutators of the modes L_m and W_m with $\phi(z)$,

$$[\,L_m\,,\phi(z)\,] = \left[h\,z^m(m+1) + z^{m+1}\partial/\partial z\right]\,\phi(z)\ , \tag{77}$$

$$[\,W_m\,,\phi(z)\,] = \left[w\,z^m\tfrac{(m+1)(m+2)}{2} + z^{m+1}(m+2)\,\hat{W}_{-1} + z^{m+2}\,\hat{W}_{-2}\right]\phi(z)\ . \tag{78}$$

Note that, just as $[L_{-1}, \phi(z)] = \phi'(z)$, so $[W_{-2}, \phi(z)] = \hat{W}_{-2}\phi(z)$; it is very tempting to consider W_{-2} as a derivative in an extra direction, especially given that $[L_{-1}, W_{-2}] = 0$, but I know of no sensible way of incorporating such an interpretation of W_{-2} in the quantum case.

Also note that, given (77) and (78), we can find linear combinations of the L_m and W_m which have simpler commutation relations with the field $\phi(z)$:

$$[L_m - zL_{m-1}, \phi(z)] = h\, z^m\, \phi(z) , \tag{79}$$

$$[W_m - 2zW_{m-1} + z^2 W_{m-2}, \phi(z)] = w\, z^m\, \phi(z) . \tag{80}$$

One way to see this easily is to note that by multiplying (75) and (76) by $(z - z')$ and $(z - z')^2$ respectively, we find only a simple pole on the right hand side of these equations, and taking modes of both sides, we recover (79) and (80). It turns out that these new equations are really all we need to answer the next question – what is the operator product expansion of two primary fields $\phi_{h,w}(z)$ and $\phi_{h',w'}(z')$?

6.3 The operator product of two W_3–primary fields

The most natural view of the operator product of two fields $\phi_a(z)$ and $\phi_b(z')$ is as a function of z and z' taking values in representations labelled by c,

$$\phi_a(z)\phi_b(z') \sim \sum_{c,n} (z - z')^{h_c - h_a - h_b + n} V(|c;n\rangle, z') , \tag{81}$$

where $|c;n\rangle$ is a state in the representation c of L_0 eigenvalue $h_c + n$.

If we know a basis $\{|i\rangle\}$ for the representation c at level n, and the inner product matrix on this space, we see that we can write

$$|c;n\rangle = |i\rangle\, M^{-1}_{ij}\, \langle j|\, \phi_a(1)\, |b\rangle , \quad \text{where } M_{ij} = \langle\, i\,|\, j\,\rangle . \tag{82}$$

Two comments follow immediately:

(1) If M_{ij} has zero eigenvectors, i.e. there are null vectors in the representation c, then the operator product (81) is only defined modulo these null vectors.
(2) There is an alternative basis in which $\langle j|\phi_a(1)|b\rangle$, or equivalently $\langle b|\phi_a(1)|i\rangle$, is very easy to work out, namely one which uses the linear combinations of the modes in (79) and (80) – this is essentially the observation of Feigin and Fuchs in [29]. Let's investigate this basis in more detail.

For simplicity, let's consider trying to work out $\langle j|\phi_a(1)|b\rangle$. Using the combinations

$$l_m = L_m - 2L_{m-1} + L_{m-2} , \quad w_m = W_m - 3W_{m-1} + 3W_{m-2} - W_{m-3} , \tag{83}$$

$$h_1 = -L_{-2} + L_{-1} , \qquad w_1 = -W_{-3} + 2W_{-2} - W_{-1} , \tag{84}$$

$$h_\infty = L_{-2} - 2L_{-1} + L_0 , \quad w_\infty = W_{-3} - 3W_{-2} + 3W_{-1} - W_0 , \tag{85}$$

we see that we can choose a basis of the representation c consisting of states of the form

$$w_{m_1} \ldots w_{m_2} l_{n_1} \ldots l_{n_2} (w_1)^a (w_\infty)^b (h_1)^c (h_\infty)^d W_{-1}^e |c\rangle . \tag{86}$$

where $m_i \le m_{i+1} < 0$, $n_i \le n_{i+1} < 0$. The advantage of this basis is that

$$\langle b|\,\phi_a(1)\,w_m = \langle b|\,\phi_a(1)\,l_m = 0\ ,\quad m<0\ , \tag{87}$$
$$\langle b|\,\phi_a(1)\,w_1 = w_a\langle b|\,\phi_a(1)\ ,\qquad \langle b|\,\phi_a(1)\,h_1 = h_a\langle b|\,\phi_a(1)\ , \tag{88}$$
$$\langle b|\,\phi_a(1)\,w_\infty = w_b\langle b|\,\phi_a(1)\ ,\qquad \langle b|\,\phi_a(1)\,h_\infty = h_b\langle b|\,\phi_a(1)\ . \tag{89}$$

so that it is trivial to work out any inner product. If we take any vector i and put it in the basis (86), then it is clear that, for some coefficients α_n,

$$\langle b|\,\phi_a(1)\,|i\rangle = \sum_{c,n}\alpha_n\langle b|\,\phi_a(1)\,(W_{-1})^n|c\rangle\ . \tag{90}$$

Some comments:

(1) If $\langle b|\,\phi_a(1)\,(W_{-1})^n|c\rangle$ is not zero for any n, then the three point coupling of the representations a, b and c can a priori depend on infinitely many unknown constants, or in the language of fusion coefficients introduced by Fuchs, $N_{abc}=\infty$
(2) Conversely, if there is some level n for which $W_{-1}^n|c\rangle$ is linearly dependent on the other states at this level, we see that $N_{abc}\le n$. As it turns out, we are very lucky, since Bajnok [4] and Furlan et al. [33] both showed that if ever the inner product matrix M_{ij} has a zero eigenvector at some level n, then necessarily this vector must be of the form $W_{-1}^n|c\rangle+\ldots$ Furthermore, the vanishings of the determinant of M_{ij} are also known explicitly, so that we know all representations L_c for which the fusion coefficients N_{abc} must be finite.
(3) Mathematically, the space of independent couplings $\langle b|\,\phi_a(1)\,(W_{-1})^n|c\rangle$ to an irreducible representation L_c appearing in (90) is the same as the quotient of the space L_c by the relations

$$w_m\,\psi = l_m\,\psi = 0\ ,\quad m<0\ , \tag{91}$$
$$(w_1-w_a)\,\psi = 0\ ,\qquad (h_1-h_a)\,\psi=0\ , \tag{92}$$
$$(w_\infty-w_b)\,\psi = 0\ ,\qquad (h_\infty-h_b)\,\psi=0\ . \tag{93}$$

6.4 Quasi-rational representations

Thus far we have implicitly considered using all of (87), (88) and (89) to help us evaluate a three point point function. For some representations L_c this will still leave an infinite number of unknowns

$$\langle b|\,\phi_a(1)\,(W_{-1})^n\,|c\rangle\ , \tag{94}$$

for any representations L_a, L_b. More interesting are the cases where (94) is zero unless h_a, h_b, w_a and w_c obey some constraints. This would be possible if singular vectors in the representation L_c allow us to simplify

$$w_1\psi \quad\text{and}\quad h_1\psi\ , \tag{95}$$

using only (91) and (93) without the use of (92). That this does happen was first shown by Bajnok et al in [5], and we shall reproduce their results here, albeit in a somewhat modified form. Representations L_c for which this happens, i.e. for which only a finite number of representations L_a can couple via (94) have also been studied by Nahm, and are also called quasi-rational representations [46].

We first introduced the parametrisation of W_3 algebra representations by four integers in the discussion of minimal models in section 5.3, but it is possible to consider the representations $[rs;tu]$ of the minimal models with c no longer a minimal value. If we parametrise c as

$$c = 50 - \frac{24}{t} - 24t \;, \tag{96}$$

then the representation $[ab;cd]$ has weights given by

$$\begin{aligned} h &= \frac{1}{3\,t}\left((at-c)^2 + (at-c)(bt-d) + (bt-d)^2 - 3(t-1)^2\right) \;, \\ w &= \frac{1}{27\,t^{3/2}}(at-c-bt+d)(2at+bt-2c-d)(at+2bt-c-2d) \;. \end{aligned} \tag{97}$$

The minimal models are given by (96), (97) with $t = p/q$. The simplest representation $[11;11]$ is the vacuum, with $h = w = 0$. The next simplest is $[11;12]$, and in the next section we shall investigate the fusion of this representation.

6.5 The fusion of the $[11;12]$ representation

The special property of this representation is that it has two independent singular vectors, i.e. zero eigenvectors of the inner product matrix, at levels 1 and 2, which are

$$\left(W_{-1} - \left(\tfrac{\sqrt{t}}{2} - \tfrac{5}{6\sqrt{t}}\right)\right)|11;12\rangle \;, \tag{98}$$

$$\left(L_{-1}L_{-1} + \tfrac{2}{3t}\,L_{-2} + \sqrt{t}\,W_{-2}\right)|11;12\rangle \;. \tag{99}$$

Using these two singular vectors, and their descendants, it is possible to reduce the whole space $L_{[11;12]}$ to a three dimensional space, using only (91) and (93). Since (92) still has to hold, it must be the case that h_1 and w_1 are diagonalisable on this three dimensional space, and in this way they are determined by our choice of h_b and w_b.

Explicitly, we can choose a basis of the representation $L_{[11,12]}$ modulo the constraints (87) and (89) as

$$|11;12\rangle \;,\quad L_{-1}\,|11;12\rangle \;,\quad W_{-2}\,|11;12\rangle \;. \tag{100}$$

In this basis, we do indeed find that h_1 and w_1 can be represented by the matrices

$$h_1 = h_b + \begin{pmatrix} \frac{-4+3t}{3t} & \frac{2(-4+3t+3th_b)}{9t^2} & \frac{-8+6t+21th_b-9t^2h_b}{27t^{5/2}} - \frac{w_b}{t} \\ 1 & 0 & \frac{-14+9t+3th_b}{9t^{3/2}} \\ 0 & \frac{1}{\sqrt{t}} & \frac{7-3t}{3t} \end{pmatrix} \tag{101}$$

$$\begin{aligned} &w_1 = w_b + \\ &\begin{pmatrix} \frac{(5-3t)(3t-4)}{27\,t^{3/2}} & \frac{-2(3t-4)^2+9h_bt(1-t)}{27\,t^{5/2}} + \frac{w_b}{t} & \frac{-2\left((3t-4)^2-h_bt(11-9t-3h_bt)\right)}{27\,t^3} + \frac{2\,w_b}{3\,t^{3/2}} \\ \frac{5-3t}{3\sqrt{t}} & \frac{82-81t+18t^2-9h_bt}{27t^{3/2}} & \frac{92-102t+27t^2-t(39-18t)h_b}{27\,t^2} + \frac{w_b}{\sqrt{t}} \\ -1 & \frac{-2+t}{t} & \frac{-68+54t-9t(t-h_b)}{27\,t^{3/2}} \end{pmatrix} \end{aligned} \tag{102}$$

We can check explicitly that h_1 and w_1 commute, and can be diagonalised simultaneously. As a result we find that, given h_b and w_b, there are only three possible choices for h_a and w_a which are consistent with the decoupling of the singular vectors (98) and (99) from correlation functions. While the eigenvalues of the matrices (101) and (102) are not themselves very revealing, when we use the parametrisation (97) we find that the three point coupling

$$\langle 11; ab|\, \phi_{[11;cd]}\, (W_{-1})^n\, |11; 12\rangle \ , \tag{103}$$

vanishes unless

$$(c, d) \in \{(a, b+1), (a+1, b-1), (a-1, b)\} \ . \tag{104}$$

These are exactly the fusion rules for a tensor product of the $\bar{3}$ representation of $sl(3)$ with a general $sl(3)$ representation and agree with the fusion rules found by Frenkel et al [31] in the minimal case.

However, note that although the three point coupling (103) vanishes unless (104) holds, it does not mean that it must be non-zero in any actual conformal field theory.

Furthermore, it is not even necessarily the case that the three point coupling of three representations is determined by the coupling of the highest weight fields. For example, consider the representation $L_{[11;22]}$; in this case $L_{[11;22]}$ quotiented by (91) and (93) is 8 dimensional, and roughly speaking this representation behaves in a similar fashion to the adjoint representation of $sl(3)$. As a result, the operator product

$$\phi_{[11;22]}(z)\, \phi_{[11;22]}(z') \to L_{[11;22]} \ , \tag{105}$$

has two independent couplings, determined by

$$\langle 11; 22|\, \phi_{[11;22]}(1)\, |11; 22\rangle \quad \text{and} \quad \langle 11; 22|\, \phi_{[11;22]}(1)\, W_{-1}\, |11; 22\rangle \ . \tag{106}$$

It may even happen that at some c values there are extra singular vectors which further truncate this fusion, e.g. at $c = 4/5$ the first coupling is identically zero, and the second coupling is free.

6.6 Quasi-finite representations

As a final step we can ask ourselves if there are any representations for which we need only impose (87) for the fusion to be fully determined, in which case there will only be a finite set of $\{h_a, h_b, w_a, w_b\}$ for which the fusion is non-vanishing. The answer is yes, and these are the minimal model representations, which are the last subject in this section.

The minimal model representations L_c have the (defining) property that N_{abc} is zero unless L_a and L_b are in some finite set. For this reason they are also called quasi-finite representations. This property was proven for the minimal model Virasoro representations by Feigin and Fuchs in [29].

For the W_3 algebra, a representation is quasi-finite if the space L_c quotiented by (91) is finite dimensional, i.e. if only a finite number of the states

$$W_{-3}^a W_{-2}^b W_{-1}^c L_{-2}^d L_{-1}^e |c\rangle \ , \tag{107}$$

are linearly independent modulo l_m and w_m. If this is the case, then we can directly see that the operators h_1, w_1, h_∞, and w_∞ commute modulo l_m and w_m and hence may be simultaneously diagonalised[3].

It is believed that this property holds for exactly the minimal model representations with c given by (72) and weights h, w given by (73).

[3] It is possible that the common eigenvectors do not span the full space, but it is believed that this is not the case. For some worked examples, see [48].

6.7 Conclusions on fusion

It is an ambitious program to prove that the minimal model representations indeed have this beautiful property. It has been completed for the Virasoro algebra by Feigin and Fuchs in [29] but for this they needed to know the whole structure of the Virasoro Verma modules.

For a long time there were not even consistent conjectures for the structure of W_3 Verma module representations, but there have recently appeared some very beautiful conjectures. These have been put forward by de Vos and van Driel in [19], and give the multiplicities of irreducible representations appearing in the composition series of Verma module representations in terms of Kazhdan-Lusztig polynomials for cosets of the affine Weyl group of the affine algebra appearing in the DS construction before Hamiltonian reduction, which in this case is $a_2^{(1)} \equiv \hat{su}(3)$.

At the very least, this definition of fusion gives a way to derive the fusion rules for any W-algebra, by suitable generalisations and can be further generalised to give 4,5,6,... point function fusion rules, and is the only known way to find differential equations for correlation functions of W-algebra primary fields, simply using the algebraic structure of a W-algebra[4]. For the first calculation performed in this way, see [5].

Another very promising route to studying fusion of W-algebra primary fields is that developed by Gaberdiel [34], which attacks the problem of fusion from the opposite end, that is it attempts to decompose the tensor product of two fields into representations of a single copy of the W-algebra. Of course this idea is not essentially new, as neither is the method given here, and in many ways the calculations reduce to essentially the same steps, but it has some advantages when the theory has unexpected features, as we can read in Gaberdiel and Kausch [35].

Finally, when L_c is the vacuum representation, the space of states $\langle b|$ which can couple to $\phi_a(1)\,|c\rangle$ is clearly one-dimensional – only the field conjugate to ϕ_a can have a non-zero three point function with the vacuum

$$\langle \bar{a}|\, \phi_a(1)\,|0\rangle \ ,$$

so that the space of states in the vacuum representation modulo (91) counts the allowed representations in the conformal field theory. For the quasi-finite representations we have been talking about in this section, this space is finite dimensional and has been studied in greater detail by Zhu [50]. He has shown that one can also put an algebra structure on this space, (and hence it is also known as Zhu's algebra), and he has further shown that any representation of this algebra induces a representation of the full chiral algebra.

[4] It may be possible to find differential equations using the methods of Furlan et al [32].

7 Conclusions

Clearly there are many interesting topics which have not been covered in these lectures for lack of time, and so I would like to finish off by listing some of these here.

(1) It was an assumption right from the beginning that we were only interested in W-algebras with a finite number of generating fields. However, there is a vast literature dealing with W-algebras with an infinite number of generating fields. There are at least two such algebras, which are commonly known as W_∞ and $W_{1+\infty}$ which are closely related to the W-algebras we have been studying. These are very important as they may arise as the symmetry algebra in physical models, for example in the quantum Hall effect [17]. The representation theory of these algebras has been studied in great detail, [1]. However, it is a surprising fact that each unitary representation of these algebras may be identified as a unitary representation of some standard W-algebra to which the larger algebra truncates at that value of c.

(2) Another assumption we have made is that our algebras all contain the Virasoro algebra as a subalgebra. It is equally possible to consider W-algebras which contain some superconformal algebra as a subalgebra, e.g. the $N = 1$ or $N = 2$ super Virasoro algebras. These have been investigated in a similar fashion, with searches for solutions to the Jacobi identities, coset constructions and especially with quantum Hamiltonian reductions based on affine super Lie algebras.

(3) Several times during these lectures I have mentioned that there are a large number of possible identifications between W-algebras which occur for specific c values. The most systematic investigation of these identifications have been carried out by Blumenhagen et al and by Hornfeck in [8,9,39–41]. During this work they also uncovered a large class of W-algebras which are not freely generated, i.e. have generic singular vectors, and which had hitherto been thought of as somehow exceptional. In fact they now appear as very regular and indeed to be in some way dual to the W-algebras we have been looking at, in that each acts as the 'unifying algebra' for the other class. It seems quite likely that the story of W-algebra truncations and identifications is not yet over.

(4) An outstanding problem is that of finding explicit formulae for the fully local correlation functions of W-primary fields. The first problem is that the presence of multiple independent three-point couplings between W-primary fields means that the evaluation of four and higher point couplings does not simply reduce to the description of a simple set of 'coupling constants', i.e. the three-point couplings between W-primary fields. One must also take into account the presence of couplings via W-descendant fields. A further problem is that the 'Dotsenko–Fateev' type integrals which should be the building blocks of these correlation functions are of new types and are not amenable to the same sorts of methods as were used for the Virasoro correlation functions.

(5) Finally I should like to stress again that many results which are proven for the Virasoro algebra are only conjectured for W-algebras, even for just the W_3 algebra. It seems to me that the results of de Vos and van Driel [19] would be the best route to understanding the structure of W-algebra representations, and from there going on to such more complicated topics as fusion etc.

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References

[1] H. Awata, M. Fukuma, Y. Matsuo, and S. Odake, *Character and determinant formulae of quasifinite representation of the $W_{1+\infty}$ algebra,* Commun. Math. Phys. **172** (1995) 377, `hep-th/9405093`;
V. Kac and A. Radul, *Representation theory of the vertex algebra $W_{1+\infty}$*, `hep-th/9512150`.

[2] F.A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, *Extensions of the Virasoro algebra constructed from Kac-Moody algebras using higher order Casimir invariants*, Nucl. Phys. B304 (1988) 348.

[3] F.A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, *Coset construction for extended Virasoro algebras*, Nucl. Phys. B304 (1988) 371.

[4] Z. Bajnok, *Singular vectors of the WA_2 algebra*, Phys. Lett. **B329** (1994) 225, `hep-th/9403032`.

[5] Z. Bajnok, L. Palla and G Takács, *A_2 Toda theory in reduced WZNW framework and the representations of the W algebra*, Nucl. Phys. **B385** (1992) 329, `hep-th/9206075`.

[6] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B241** (1984) 333.

[7] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel, and R. Varnhagen. *W-algebras with two and three generators*, Nucl. Phys. **B361** (1991) 255.

[8] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, and R. Hübel, *Unifying W-algebras*, Phys. Lett. **B322** (1994) 51, `hep-th/9404113`

[9] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, and R. Hübel, *Coset realisation of unifying W-algebras*, Int. J. Mod. Phys. **A10** (1995) 2367, `hep-th/9406203`

[10] J. de Boer, L. Feher and A. Honecker, *A class of W algebras with infinitely generated classical limit*, Nucl. Phys. **B420** (1993) 409, `hep-th/9312049`.

[11] J. de Boer and T. Tjin, *The relation between quantum W algebras and Lie algebras*, Commun. Math. Phys. **160** (1994) 317, `hep-th/9302006`

[12] P. Bouwknegt, *Extended Conformal algebras*, Phys. Lett. **207B** (1988) 295.

[13] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, Phys. Rept. **223** (1993) 183.

[14] P. Bowcock, *Representation theory of a W algebra from generalised DS reduction*, Durham preprint DTP-94-5, `hep-th/9403157`

[15] P. Bowcock and G.M.T. Watts, *On the classification of quantum W–algebras*, Nucl. Phys. B379 (1992) 63, `hep-th/9111062`.

[16] A. Cappelli, C. Itzykson and J.-B. Zuber, *The A-D-E classification of minimal and $a_1^{(1)}$ conformal field theories*, Commun. Math. Phys. **113** (1987) 1.

[17] A. Cappelli, C.A. Trugenberger and G.R. Zemba, *Stable hierarchical quantum Hall fluids as $W_{1+\infty}$ minimal models*, Nucl. Phys. **B448** (1995) 470, `hep-th/9502021`; *$W_{1+\infty}$ minimal models and the hierarchy of the quantum Hall effect*, Nucl. Phys. Proc. Suppl. **45A** (1996) 112.

[18] E.B. Dynkin, Transl. Am. Math. Soc. Series 2, 6 (1957) 112.

[19] P. van Driel and K. de Vos, *The Kazhdan-Lusztig conjecture for W-algebras*, Bonn preprint BONN-TH-95-14, `hep-th/9508020`.

[20] V.G. Drinfel'd and V.V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, J. Sov. Math. **30** (1985) 1975.

[21] W. Eholzer, M. Flohr, A. Honecker, R. Hübel, W. Nahm and R. Varnhagen, *Representations of W-algebras with two-generators and new rational models,* Nucl. Phys. **B383** (1992) 249.

[22] W. Eholzer, A. Honecker and R. Hübel, *How complete is the classification of W-symmetries?*, Phys. Lett. **B308** (1993) 42, `hep-th/9302124`.

[23] V.A. Fateev and S.L. Luk'yanov, *The models of two-dimensional conformal quantum field theory with Z_n symmetry*, Int. J. Mod. Phys. **A3** (1988) 507.

[24] V.A. Fateev and S.L. Luk'yanov, *Additional symmetries and exactly-soluble models in two-dimensional conformal field theory*, Sov. Sci. Rev. **A15** (1990) 1.

[25] V.A. Fateev and A.B. Zamolodchikov, *Conformal quantum field theory models in two dimensions having Z_3 symmetry*, Nucl. Phys. **B280** [FS18] (1987) 644.

[26] L. Feher, L. O'Raifeartaigh, P. Ruelle, I.Tsutsui and A. Wipf, *On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories*, Phys. Rep. **222** (1992) 1.
[27] L. Feher, L O'Raifeartaigh and I. Tsutsui, *The vacuum preserving Lie algebra of a classical W algebra*, Phys. Lett. **B316** (1993) 275, `hep-th/9307190`
[28] B. Feigin and E. Frenkel, *Quantization of the Drinfeld-Sokolov reduction*, Phys. Lett. **B246** (1990) 75.
[29] B.L. Feigin and D.B. Fuchs, *On the cohomology of some nilpotent subalgebras of Kac-Moody and the Virasoro algebras*, J. Geom. Phys. **5** (1988) 209.
[30] J.M. Figueroa-O'Farrill and S. Schrans, *The spin 6 extended conformal algebra*, Phys. Lett. **B245** (1990) 471.
[31] E.V. Frenkel, V. Kac and M. Wakimoto, *Characters and fusion rules for W algebras via quantized Drinfeld-Sokolov reductions,* Commun. Math. Phys. **147** (1992) 295.
[32] P. Furlan, A.C. Ganchev, R. Paunov and V.B. Petkova, *Solutions of the Knizhnik-Zamolodchikov equation with rational isospins and the reduction to the minimal models*, Nucl. Phys. **B394** (1993) 665, `hep-th/9201080`.
[33] P. Furlan, A.C. Ganchev and V.B. Petkova, *Singular vectors of W algebras via DS reduction of* $A_2^{(1)}$ Nucl. Phys. **B431** (1994) 622, `hep-th/9403075`.
[34] M.R. Gaberdiel, *Fusion rules of chiral algebras*, Nucl. Phys. **B417** (1994) 130, `hep-th/9309105`; *Fusion in conformal field theory as the tensor product of the symmetry algebra*, Int. J. Mod. Phys. **A9** (1994) 4619, `hep-th/9307183`.
[35] M.R. Gaberdiel and H.G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. **B386** (1996) 131, `hep-th/9606050`; *Indecomposable fusion products*, Nucl. Phys. **B477** (1996) 293, `hep-th/9604026`.
[36] T. Gannon, *The classification of affine* $su(3)$ *modular invariant partition functions*, Commun. Math. Phys **161** (1994) 233, `hep-th/9212060`.
[37] T. Gannon and M.A. Walton, *On the classification of diagonal coset modular invariants*, Commun. Math. Phys **173** (1995) 175, `hep-th/9407055`
[38] P. Goddard, *Meromorphic Conformal Field Theory, in:* Infinite Dimensional Lie Algebras and Lie Groups, ed. V. G. Kac, World Scientific, 1989, CIRM-Luminy July conference on Infinite dimensional Lie algebras and Lie Groups, Marseilles 1988.
[39] K. Hornfeck, *W-algebras with set of primary fields of dimensions (3,4,5) and (3,4,5,6)*, Nucl. Phys. **B407** (1993) 237, `hep-th/9212104`.
[40] K. Hornfeck, *Classification of structure constants for W-algebras from highest weights*, Nucl. Phys. **B411** (1994) 307, `hep-th/9307170`.
[41] K. Hornfeck, *W-algebras of negative rank*, Phys. Lett. **B343** (1995) 94, `hep-th/9410013`.
[42] Y.-Z. Huang and J. Lepowsky, *On the* $\mathcal{D}$*-module and formal variable approaches to vertex algebras*, in: 'Topics in Geometry: In Memory of Joseph D'Atri', ed. S. Gindikin, Progress in Nonlinear Differential Equations, Vol. 20, Birkhauser, Boston, (1996) 175, `q-alg/9603020`.
[43] H.G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields*, Phys. Lett. **B259** (1991) 448.
[44] H.G. Kausch and G.M.T. Watts, *A study of W-algebras using Jacobi identities*, Nucl. Phys. **B354** (1991) 740.
[45] W. Nahm, *Chiral algebras of two-dimensional chiral field theories and their normal ordered products*, in 'Recent developments in conformal field theories' S. Randjbar-Daemi et al. eds, (World Scientific 1990) 81.
[46] W. Nahm, *Quasi-rational fusion products*, Int. J. Mod. Phys. **B8** (1994) 3693, `hep-th/9402039`.
[47] E. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nucl. Phys. **B300** [FS22] (1988) 360.
[48] G.M.T. Watts, Fusion in the W_3 algebra, Commun. Math. Phys. **171** (1995) 87, `hep-th/9403163`.
[49] A.B. Zamolodchikov, *Infinite additional symmetries in two-dimensional conformal quantum field theory*, Theor. Mat. Fiz. **65** (1985) 347.
[50] Y.-C. Zhu, *Vertex operator algebras, elliptic functions and modular forms,* Ph.D. thesis, Yale University, 1990;

I.B. Frenkel and Y.-C. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras,* Duke Math. J. **66** (1992), 123.