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Conformal Field Theory (for string theorists)

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Abstract

A write up of about ten lectures on conformal field theory given as part of a first semester course on string theory.

Contents

1 Opening Remarks

To date in this class, string theory boils down to the study four free (quadratic) quantum field theories: one for the X fields, one for the ψ fields, one for the bc ghost system, and one for the $\beta\gamma$ ghost system. We saw a BRST action that coupled the X and ψ fields to world-sheet supergravity, and hence required the presence of the bc and $\beta\gamma$ ghosts in addition to some auxiliary fields d and Δ and also ghosts for the Weyl and super-Weyl symmetry. After some elementary path-integral manipulations, these extra fields dropped out, and we were left with simple, quadratic actions for the remaining X, ψ, bc and $\beta\gamma$ fields on a flat world sheet $h_{ab} = \eta_{ab}$.

The full quantum world-sheet supergravity action had a number of symmetries which are no longer evident in the gauge fixed action for X, ψ, bc and $\beta\gamma$. Among other symmetries, the full quantum action had world-sheet diffeomorphism invariance, $\sigma \to \sigma'(\sigma)$. Under diffeomorphisms, the metric changes in the usual way

$$h'_{cd}(\sigma') = \frac{\partial\sigma^a}{\partial\sigma'^c}\frac{\partial\sigma^b}{\partial\sigma'^d}h_{ab}(\sigma) \ . \tag{1}$$

Another symmetry was world-sheet Weyl invariance, $h_{ab} \to \Lambda(x) h_{ab}$. There were then corresponding rules for how the fields X, ψ, bc, and $\beta\gamma$ transform under diffeomorphisms and Weyl scaling. We gauge fixed by choosing a flat world-sheet metric $h_{ab} = \eta_{ab}$. However, this gauge fixing is not complete. There are residual gauge transformations that are a combination of a diffeomorphism and a Weyl scaling that leave the metric η_{ab} invariant. These residual gauge transformations are called conformal transformations:

Definition. A *conformal transformation* is a map on coordinates $\sigma \to \sigma'$ that preserves the metric up to a scale factor

$$h'_{ab}(\sigma) = \Lambda(\sigma) h_{ab}(\sigma) \ .$$

Example. In the case when $h_{ab} = \eta_{ab}$, two conformal transformations are

- Elements of the Poincaré group (Lorentz group and translations) for which $\Lambda = 1$.
- Dilations $x \to \lambda x$, $\lambda \in \mathbb{R}$, for which $\Lambda = \lambda^2$.

Note in the Euclidean case, $h_{ab} = \delta_{ab}$, the Lorentz group is replaced by rotations. Both rotations and dilations manifestly preserve the angles between vectors, motivating the choice of word "conformal", which means preserving angles.

Remark. The set of conformal transformations $\mathcal{C}$ forms a group when the transformation $\sigma \to \sigma'$ is invertible.

Definition. A *conformal field theory* is a quantum field theory which has $\mathcal{C}$ as a classical symmetry of the action.

Almost all the quantum field theories we study, when coupled to gravity, will be diffeomorphism invariant. The litmus test for figuring out when a quantum field theory in a fixed background space-time is a conformal field theory is then the presence of local Weyl invariance. Perhaps as a result, in the literature there is a certain carelessness and interchanging in the use of the words Weyl scaling and conformal transformation. We will try to be careful here.

Having fixed $h_{ab} = \eta_{ab}$, the field theories for X, ψ, bc, and $\beta\gamma$ become examples of conformal field theories. In fact, we will eventually see they are essentially all the same conformal field theory, just expressed in different variables. We can therefore use the extensive and highly developed machinery of conformal field theory to systematize our understanding of these four systems. The goal of these lectures will be four-fold:

1. To replace the cumbersome oscillator algebra manipulations with (in our view) more elegant operator product expansions.
2. To streamline calculations involving the BRST symmetry.
3. To understand how a quantum anomaly in the classical conformal symmetry restricts the types of consistent string theories.
4. To set up machinery for string scattering calculations.

References

These lecture notes draw largely from chapters 2, 3, 6, 8, and 10 of Polchinski's classic string theory text book [1]. I have also drawn on early chapters in Di Francesco, Sénéchal, and Mathieu's classic work on conformal field theory [2] and P. van Nieuwenhuizen's unpublished string theory lecture notes [3]. Another nice publicly available reference I found are unpublished notes by M. Kreuzer [4].

2 Conformal Transformations in One and Two Dimensions are Special

In one dimension, any diffeomorphism $y(x)$ is conformal with $g'_{yy} = \frac{\partial x}{\partial y}\frac{\partial x}{\partial y} g_{xx}$.

In two dimensions, for convenience, consider the Euclidean case $h_{ab} = \delta_{ab}$.[1] We take advantage of complex numbers:

$$z = \sigma^1 + i\sigma^2 \ , \quad \bar{z} = \sigma^1 - i\sigma^2 \ , \tag{2}$$

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \ , \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \ . \tag{3}$$

The world sheet metric then has components

$$h_{z\bar{z}} = h_{\bar{z}z} = \frac{1}{2} \ , \quad h_{zz} = h_{\bar{z}\bar{z}} = 0 \ . \tag{4}$$

In complex coordinates, any holomorphic transformation $z \to w(z)$ along with its anti-holomorphic counterpart $\bar{z} \to \bar{w}(\bar{z})$ is conformal:

$$h'_{w\bar{w}}(w,\bar{w}) = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial \bar{w}} h_{z\bar{z}}(z,\bar{z}) \quad \text{where} \quad \Lambda = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial \bar{w}} \ . \tag{5}$$

In more than two dimensions, the set of conformal transformations is far smaller. It is generated by

- translations: $x^\mu \to x^\mu + a^\mu$.
- dilations: $x^\mu \to a x^\mu$.
- rigid rotations: $x^\mu \to M^\mu{}_\nu x^\nu$.
- special conformal transformations:

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x - b^2 x^2} \ .$$

In d Euclidean dimensions, these transformations generate a connected part of the Lorentz group $SO^+(1,d+1)$. This group forms an important subgroup of the conformal transformations in $d=2$, where it is isomorphic to the set of Moebius transformations on the complex plane, $SO^+(1,3) =$

[1] To return to the Lorentzian case, one can make the Wick rotation $\sigma^0 = -i\sigma^2$.

$PSL(2,\mathbb{C})$. In particular, translations, dilations, rotations, and special conformal transformations on the plane combine to give the transformation rule

$$z \to \frac{az+b}{cz+d} \ , \tag{6}$$

where a, b, c, and $d \in \mathbb{C}$. Without further conditions on a, b, c, and d, this map would be in $GL(2,\mathbb{C})$. However, as multiplying a, b, c, and d by an overall scale factor does not change the transformation rule, we are free to set $ad-bc=1$ and restrict to $SL(2,\mathbb{C})$. Furthermore, the map is invariant under the sign flip $(a,b,c,d) \to (-a,-b,-c,-d)$, which restricts the group to $PSL(2,\mathbb{C})$.

We can also consider the corresponding Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ for $PSL(2,\mathbb{C})$. This Lie algebra has the generators conventionally labeled

$$L_{-1} = \partial_z \ , \quad L_0 = z\partial_z \ , \quad L_1 = z^2\partial_z \ . \tag{7}$$

(There is another copy of $\mathfrak{sl}(2,\mathbb{C})$ generated by complex conjugates of L_0, L_{-1}, and L_1.) The operator L_{-1} generates a translation, L_0 a combination of dilation and rotation, and L_1 a special conformal transformation. These generators satisfy the standard $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra

$$[L_0, L_{-1}] = -L_{-1} \ , \quad [L_0, L_1] = L_1 \ , \quad [L_{-1}, L_1] = 2L_0 \ . \tag{8}$$

(In quantum mechanics, we might make the replacements $L_0 \to J_z$, $L_{-1} \to J_-$, and $L_1 \to J_+$.)

An infinite dimensional representation of this algebra is furnished by the monomials z^n where

$$L_0 z^n = nz^n \ , \quad L_{\pm 1} z^n = nz^{n\pm 1} \ . \tag{9}$$

At first sight there is something a bit odd about this representation; under what inner product do the eigenvectors z^n have finite norm and is there a notion of Hermiticity? To obtain an inner product, we make the transformation $z = e^{-it}$. Under this transformation, we find the new generators

$$L_{-1} = -ie^{-it}\partial_t \ , \quad L_0 = -i\partial_t \ , \quad L_1 = -ie^{it}\partial_t \ . \tag{10}$$

There is then an obvious inner product based on the orthogonality of Fourier modes on the circle,

$$\int_0^{2\pi} (e^{int})^\dagger (e^{imt}) \mathrm{d}t = 2\pi\delta_{n,m} \ , \tag{11}$$

and under which L_0 is now clearly Hermitian. Back in the z coordinate, interestingly, this inner product corresponds to a contour integral along the curve $|z| = 1$. This so-called plane to cylinder map $z = e^{it}$ along with corresponding contour integrals will play a key role as we go forward.

3 Correlation Functions are Highly Constrained by Conformal Symmetry

The transformation properties of fields fix two and also three point functions up to some undetermined constants. Previously in the class, we saw examples of how X and ψ transform infinitesimally under such conformal transformations. The finite versions of those rules are as follows:

$$\partial_z X'(z,\bar{z}) = \left(\frac{\partial w}{\partial z}\right) \partial_w X(w,\bar{w}) \ , \tag{12}$$

$$\psi'(z) = \left(\frac{\partial w}{\partial z}\right)^{1/2} \psi(w) \ . \tag{13}$$

The fields ∂X and ψ are examples of primary fields. More generally we have the definition:

Definition. For any meromorphic map $z \to w(z)$, a primary field satisfies the transformation rule

$$\phi'(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \ . \tag{14}$$

The quantity h is called the holomorphic conformal dimension, $\bar{h}$ the anti-holomorphic conformal dimension. The quantities $h + \bar{h} = \Delta$ are the conformal (or scaling) dimension and $h - \bar{h} = s$ the spin.

Applying this definition to our two examples, we find that $h = 1$ and $\bar{h} = 0$ for ∂X while $h = 1/2$ and $\bar{h} = 0$ for ψ. Reassuringly, the fermion has spin one half and an object with a world-sheet vector index has spin one. Moreover, the conformal dimension Δ is equal to the naive engineering dimension in both cases, as it should be for free fields.

The notion of quasi-primary will also be important for us. A quasi-primary satisfies this transformation rule (14) but only for the Moebius transformation $PSL(2,\mathbb{C})$. We will see later that the stress-tensor is an important example of a quasi-primary field that is not also primary.

The conformal symmetry fixes the form of the two-point correlation function of quasi-primary fields. To keep the formulae simple, we will focus on a case where $\bar{h} = 0$. The transformation rule (14) on the fields imply that for the correlation function

$$\left(\frac{\partial w_1}{\partial z_1}\right)^{-h_1} \left(\frac{\partial w_2}{\partial z_2}\right)^{-h_2} \langle \phi_1(z_1)\phi_2(z_2)\rangle = \langle \phi_1(w_1)\phi_2(w_2)\rangle \ . \tag{15}$$

I have removed the primes on the right hand side because I have assumed that the vacuum state in which I evaluate the correlation function is invariant under the map $w(z)$. Thus the quantities in brackets on the left and right hand side should have the *same* functional form.

First consider translations $w = z + b$. Translation invariance[2] of the vacuum and locality imply that $\langle \phi_1(z_1)\phi_2(z_2)\rangle = f(z_1 - z_2)$. Next consider dilations/rotation, $w = az$. We find from the constraint (15) that $a^{-h_1-h_2} f(z_1 - z_2) = f(az_1 - az_2)$, which implies that $f(z) = c_{12} z^{-h_1-h_2}$ where c_{12} is independent of z_1 and z_2. Finally, consider an inversion $w = 1/z$, which implies

$$\left(-\frac{1}{z_1^2}\right)^{-h_1} \left(-\frac{1}{z_2^2}\right)^{-h_2} \frac{c_{12}}{(z_1 - z_2)^{h_1+h_2}} = \frac{c_{12}}{(1/z_1 - 1/z_2)^{h_1+h_2}} \ .$$

This constraint can only be satisfied if the two-point correlation function vanishes $c_{12} = 0$ or if $h_1 = h_2$. We find the result

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{c_{12}\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}{(z_1 - z_2)^{2h_1}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}_1}} \ , \tag{16}$$

[2]Note that certain vacua we consider later, in particular the vacua for the bc-system, are not translation invariant. They have operators inserted at $z = 0$ and $z \to \infty$.

where we now give the general case $\bar{h} \neq 0$ as well. Applying this result to the X and ψ fields, we obtain

$$\langle \partial X(z,\bar{z}) \partial X(z',\bar{z}') \rangle \sim \frac{1}{(z-z')^2} \ , \tag{17}$$

$$\langle \psi(z)\psi(z') \rangle \sim \frac{1}{z-z'} \ . \tag{18}$$

Integrating the first relation twice and inserting a conventional normalization we have that

$$\langle X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \rangle = -\frac{\alpha'}{2}\eta^{\mu\nu} \log|z-z'|^2 \ . \tag{19}$$

4 Noether's Theorem

Combining complex analysis with Noether's theorem and corresponding Ward identities will let us replace commutator and oscillator algebras with (in our view) more elegant operator product expansions. To that end, consider the following transformation rule on a quantum field:

$$\phi'(\sigma) = \phi(\sigma) + \rho(\sigma)\delta\phi(\sigma) \ , \tag{20}$$

where $\rho(\sigma) \ll 1$ is a small parameter. When $\rho(\sigma)$ is constant, the transformation is assumed to be a symmetry of the action. Through Ward identities, this symmetry constrains the form of correlation functions. Consider first the one point function of an operator $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \int [\mathrm{d}\phi] e^{-S[\phi]} \mathcal{O}(\sigma') \ . \tag{21}$$

Invariance under change of variables means

$$0 = \int [\mathrm{d}\phi'] e^{-S[\phi']} \mathcal{O}(\sigma') - \int [\mathrm{d}\phi] e^{-S[\phi]} \mathcal{O}(\sigma') \ .$$

Assuming that the field ϕ' is related to ϕ via the transformation rule and applying Noether's theorem, one then finds

$$0 = \frac{1}{2\pi i} \int \mathrm{d}^2\sigma\, \rho(\sigma) \langle (\nabla_a j^a(\sigma)) \mathcal{O}(\sigma') \rangle + \langle \delta\mathcal{O}(\sigma') \rangle \ ,$$

where $j^a(\sigma)$ is the conserved current associated with the global symmetry. The factor of $1/(2\pi i)$ out front is simply a convenient normalization for $j^a(\sigma)$. We now take a very particular form for $\rho(\sigma)$, that it's a constant $\epsilon \ll 1$ in a region R that includes σ' and zero elsewhere. Using Stoke's Theorem, we find an alternate expression for $\langle \delta O(\sigma') \rangle$:

$$0 = -\frac{\epsilon}{2\pi} \oint_{\partial R} \langle (j_z \mathrm{d}z - j_{\bar{z}} \mathrm{d}\bar{z}) \mathcal{O}(z',\bar{z}') \rangle + \langle \delta\mathcal{O}(z',\bar{z}') \rangle \ . \tag{22}$$

In complex coordinates, current conservation is the condition $\partial_{\bar{z}} j_z + \partial_z j_{\bar{z}} = 0$. However, in many cases of interest j_z is holomorphic and each term in the current conservation condition vanishes independently.[3] Then we can use the residue theorem:

$$\mathrm{Res}_{z\to z'}\, j(z)\mathcal{O}(z',\bar{z}') + \mathrm{Res}_{\bar{z}\to\bar{z}'}\, \tilde{j}(\bar{z})\mathcal{O}(z',\bar{z}') = \frac{1}{i\epsilon}\delta\mathcal{O}(z',\bar{z}')\ , \tag{23}$$

where we have defined $j(z) \equiv j_z(z,\bar{z})$ and $\tilde{j}(\bar{z}) \equiv j_{\bar{z}}(z,\bar{z})$. In other words, singularities of coincident operators are telling us about transformation rules for $\mathcal{O}$. There is also now the intriguing possibility of starting with a holomorphically conserved current instead of with a transformation rule. Given such a current, we can now use this result to deduce a transformation rule on the field.

We have removed the expectation values in the relation (23). The reason is that, revisiting the arguments above, we are free to include any number of additional operator insertions in the path integral, so long as they are not in the region R. The relation (23) will continue to hold with these additional insertions. Thus equality holds as an operator equality inside a correlation function, as long as the other operators do not become coincident with $\mathcal{O}(z',\bar{z}')$ or $j(z)$.

We can push this formalism further and reformulate equal time commutators of conserved charges $[Q_1, Q_2]$ in terms of singularities that appear as conserved currents approach each other, $\mathrm{Res}_{z_1\to z_2} j_1(z_1) j_2(z_2)$. To make this reformulation, we first have to introduce the notion of radial quantization, where time runs radially outward from $z = 0$. A constant time slice is then a circle with a constant value of $|z|$. (After a plane to cylinder map $z = e^w$, time propagation looks more familiar. On the cylinder, one considers a CFT on a spatial circle, and time propagation is along the cylinder.) In this picture of radial quantization, the relation between the conserved charge Q and the current $j(z)$ is

$$Q(C) = \oint_C \frac{\mathrm{d}z}{2\pi i} j(z)\ . \tag{24}$$

Consider three concentric circles, C_1, C_2, and C_3 about the origin with increasing radii (equivalently larger times). We claim the operator

$$Q_1(C_3)Q_2(C_2) - Q_1(C_1)Q_2(C_2) \tag{25}$$

is the equal time commutator of the two Q_j. When inserted in the path integral, the contour integrals can be written in a simpler form due to time ordering, which in this context is radial ordering:

$$Q_1Q_2 - Q_2Q_1 = [Q_1, Q_2]\ . \tag{26}$$

But given the expression as a contour integral, we can deform the contour

$$[Q_1, Q_2] = \oint_{C_2} \frac{\mathrm{d}z_2}{2\pi i} \mathrm{Res}_{z_1\to z_2} j_1(z_1) j_2(z_2)\ . \tag{27}$$

[3]There is a subtlety here. Holomorphicity in this business is something that occurs after one has applied the classical equations of motion. Here however, we are attempting to say something about quantum field theory and path integrals. By holomorphic, we don't mean j_z as an off-shell operator is holomorphic. We mean when $j_z(z,\bar{z})$ appears inside a correlation function and is not coincident with other operator insertions, the correlation function will depend only on z, not on $\bar{z}$.

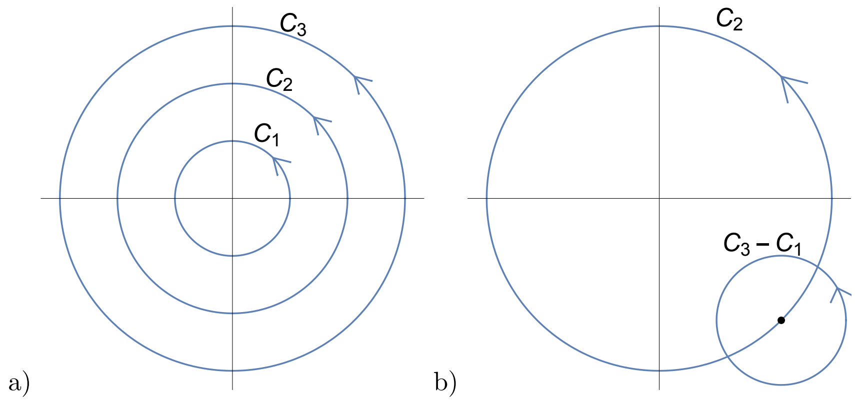

Figure 1: Contours used for integrating holomorphic currents $j_1(z)$ and $j_2(z)$ to get conserved charges Q_1 and Q_2.

The deformation is show in figure 1. In other words, singularities in the limit $z_1 \to z_2$ contain the information of the commutator algebra of conserved charges! Using the contour integral deformation argument, we can also re-interpret the previous calculation of $\delta\mathcal{O}(z,\bar{z})$:

$$[Q, \mathcal{O}] = \frac{1}{i\epsilon}\delta\mathcal{O} \ . \tag{28}$$

5 Conformal Anomaly

As our first application of this reformulation of Noether's theorem and Ward identities, consider the stress tensor. In the Euclidean setting, we identify the stress-tensor as a variation of the classical action with respect to the background metric as follows[4]

$$T^{ab} \equiv -\frac{2}{\sqrt{h}}\frac{\delta S}{\delta h_{ab}} \ . \tag{29}$$

Under an infinitesimal local Weyl transformation, the variation in the metric is $\delta h_{ab} = \epsilon h_{ab}$. From the definition of the stress tensor, if the action has local Weyl invariance, then it follows that the trace of the stress tensor must vanish,

$$T^a{}_a = h^{ab}T_{ab} = 0 \ . \tag{30}$$

In complex coordinates, vanishing of the trace means that $T_{z\bar{z}} = 0 = T_{\bar{z}z}$.

The consequence of diffeomorphism invariance, on the other hand, is that the stress tensor is covariantly conserved, $\nabla_a T^{ab} = 0$. In flat space, this condition reduces to $\partial_a T^{ab} = 0$. In complex coordinates $\partial_z T^{zz} = 0 = \partial_{\bar{z}} T^{\bar{z}\bar{z}}$, given that $T^{z\bar{z}}$ and $T^{\bar{z}z}$ both vanish for a theory that is additionally locally Weyl scale invariant. Lowering the indices, we find the holomorphicity conditions

$$\partial_{\bar{z}} T_{zz} = 0 = \partial_z T_{\bar{z}\bar{z}} \ . \tag{31}$$

[4]In our conventions, for a single particle in classical mechanics, the Euclidean Lagrangian is kinetic plus potential energy. In the Lorentzian setting, where the Lagrangian is kinetic minus potential energy, the minus sign would be absent.

We will follow convention here and introduce a rescaled version of the stress tensor:

$$T(z) \equiv -2\pi T_{zz} \ , \tag{32}$$

$$\tilde{T}(\bar{z}) \equiv -2\pi T_{\bar{z}\bar{z}} \ . \tag{33}$$

(The 2π will cancel a corresponding 2π in the Cauchy residue formula.) Given such a holomorphic operator, we can build a large set of holomorphically conserved currents

$$j(z) = iv(z)T(z) \ , \quad \tilde{\jmath}(\bar{z}) = iv(z)^* \tilde{T}(\bar{z}) \ , \tag{34}$$

where $v(z)$ is any meromorphic function. Using the relation (23), we can then study the associated symmetry transformations of the fields.

Consider first our X CFT. The Euclidean action is

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{h} h^{ab} \partial_a X \cdot \partial_b X \ . \tag{35}$$

From the definition of the stress tensor, we find

$$T^{ab} = \frac{1}{2\pi\alpha'} \left[\partial^a X \cdot \partial^b X - \frac{1}{2} h^{ab} h^{cd} \partial_c X \cdot \partial_d X \right] \ , \tag{36}$$

or in components

$$T(z) \ = \ -\frac{1}{\alpha'} \partial X \cdot \partial X \ , \tag{37}$$

$$\tilde{T}(\bar{z}) \ = \ -\frac{1}{\alpha'} \bar{\partial} X \cdot \bar{\partial} X \ . \tag{38}$$

Now we saw previously that

$$\langle \partial X_\mu(z) \partial X^\mu(z') \rangle = -\frac{D\alpha'/2}{(z-z')^2} \ , \tag{39}$$

where D is the number of space time dimensions. Thus, in the stress tensor, we need to regulate the divergence as $z \to z'$. One common prescription is normal ordering. Our stress tensor is really

$$\begin{aligned} T(z) \ &= \ -\frac{1}{\alpha'} {:}\partial X_\mu(z) \partial X^\mu(z){:} & (40) \\ &\equiv \ -\frac{1}{\alpha'} \lim_{z \to z'} \left[\partial X_\mu(z) \partial X^\mu(z') + \frac{D\alpha'/2}{(z-z')^2} \right] \ . & (41) \end{aligned}$$

Given this normal ordering prescription, we can then study how X transforms in response to symmetries associated with $j(z) = iv(z)T(z)$. The first step is to use Wick's theorem to consider the singular terms in $T(z)X^\mu(z')$ as $z \to z'$:

$$\begin{aligned} T(z) X^\mu(0) \ &\sim \ -\frac{2}{\alpha'} {:}\partial X^\nu(z) \partial X_\nu(z){:}\, X^\mu(0) + \ldots \\ &\sim \ -\frac{2}{\alpha'} \partial_z \left[-\frac{\alpha'}{2} \eta^\mu_\nu \log |z|^2 \right] \partial X^\nu(z) + \ldots \\ &\sim \ \frac{1}{z} \partial X^\mu(0) + \ldots \end{aligned} \tag{42}$$

The ellipsis indicates terms which are not singular in the limit $z \to 0$. This expression is frequently referred to as an operator product expansion (OPE). We will shortly give a more formal definition below. From the residue relation (23), we then obtain

$$\frac{1}{i\epsilon}\delta X^\mu = iv(z)\partial X^\mu + iv(z)^*\bar{\partial} X^\mu \ . \tag{43}$$

This rule is recognizable as an infinitesimal coordinate transformation with

$$z' = z + \epsilon v(z) \ , \tag{44}$$

the finite version of which is $z' = w(z)$ for a meromorphic function w. But such coordinate transformations are precisely the conformal transformations we discussed earlier, which leave the metric invariant up to a local scale factor. We may then tentatively conclude that $v(z)T(z)$ generates conformal transformations.

As OPEs will be of central importance, it is useful to get some additional practice with simple examples. Let us consider then the operator $\partial X^\mu(z)$ instead of $X^\mu(z)$. In this case, we obtain

$$\begin{aligned} T(z)\partial X^\mu(0) &= -\frac{2}{\alpha'}\partial_z\partial_w\left[-\frac{\alpha'}{2}\eta^\mu_\nu \log|z-w|^2\right]\partial X^\nu(z) + \dots \\ &= \frac{1}{(z-w)^2}\partial X^\mu(z) + \dots \\ &= \frac{1}{(z-w)^2}\partial X^\mu(w) + \frac{1}{z-w}\partial^2 X^\mu(w) + \dots \end{aligned} \tag{45}$$

From this OPE, we can deduce the transformation rule for ∂X^μ, namely

$$\delta\partial X^\mu = -\epsilon v(z)\partial^2 X^\mu - \epsilon(\partial v)(\partial X^\mu) \ . \tag{46}$$

Let us compare this transformation rule with what we would expect for a primary operator (for simplicity with $\bar{h} = 0$):

$$\mathcal{O}'(z') = \left(\frac{\partial z'}{\partial z}\right)^{-h}\mathcal{O}(z) \ , \tag{47}$$

where the infinitesimal version is given by $z' = z + \epsilon v(z)$. Then

$$\begin{aligned} \delta\mathcal{O} &= \mathcal{O}'(z) - \mathcal{O}(z) \\ &= \mathcal{O}'(z' - \epsilon v(z)) - \mathcal{O}(z) \\ &= \mathcal{O}'(z') - \epsilon v(z)\partial_{z'}\mathcal{O}'(z') - \mathcal{O}(z) \\ &= (1 + \epsilon\partial_z v)^{-h}\mathcal{O}(z) - \epsilon v(z)\partial_z\mathcal{O}(z) - \mathcal{O}(z) \\ &= -h\epsilon(\partial v)\mathcal{O} - \epsilon v\partial\mathcal{O} \end{aligned} \tag{48}$$

The relation of this transformation rule to the OPE suggests the following alternate definition of a primary field. It is a field which has the following OPE with the stress tensor:

$$T(z)\mathcal{O}(0) \sim \frac{h}{z^2}\mathcal{O}(0) + \frac{1}{z}\partial\mathcal{O}(0) + \dots \tag{49}$$

I do not want to delve too deeply into when a product of two nearby operators can be expressed as an (infinite) sum of local operators. In the Lorentzian case, there are obvious subtleties involved with what exactly is meant by "near". Even in the Euclidean case, such sums can have various pathologies. However, in the CFT case, for the operators we study, it is almost always the case that the OPE will take the (schematic) form

$$\mathcal{O}(z)\mathcal{O}'(w) = \sum_{k=-N}^{\infty} (z-w)^k \mathcal{O}_k(w) \ . \tag{50}$$

Exercise 1. Verify that $h = \bar{h} = \frac{\alpha'}{4}k^2$ for the operator $\mathcal{O} = :e^{ik\cdot X}:$.

Using the OPE machinery and the residue relation (23), we can check if $T(z)$ itself is a primary field, at least for the X system:

$$\begin{aligned}
T(z)T(w) &\sim \frac{1}{\alpha'^2} :\partial X^\mu \partial X_\mu(z)::\partial X^\nu \partial X_\nu(w): \\
&\sim \frac{2}{\alpha'^2}(-\partial_z\partial_w \eta^\mu_\nu \frac{\alpha'}{2}\log|z-w|^2)(-\partial_z\partial_w\eta^\nu_\mu \frac{\alpha'}{2}\log|z-w|^2) \\
&\quad + \frac{4}{\alpha'^2}(-\partial_z\partial_w\eta^\mu_\nu \frac{\alpha'}{2}\log|z-w|^2):\partial X^\nu(z)\partial X_\mu(w): \\
&\sim \frac{\eta^\mu_\mu}{2}\frac{1}{(z-w)^4} - \frac{2}{\alpha'}\frac{1}{(z-w)^2}:\partial X^\mu(z)\partial X_\mu(w): + \ldots \\
&\sim \frac{D}{2}\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) + \ldots
\end{aligned} \tag{51}$$

While here D is the number of X fields, more generally for a two dimensional CFT, the coefficient of the leading $1/(z-w)^4$ term in the OPE of two stress tensors is identified with the central charge c of the theory. (In this case $c = D$.) The central charge is an obstruction to the stress-tensor transforming as a primary field:

$$\delta T(z) = -\epsilon \frac{c}{12}(\partial^3 v) - 2\epsilon(\partial v)T - \epsilon v \partial T \tag{52}$$

However, since $\partial^3 v$ vanishes for $v(z) = 1$, z, or z^2, i.e. the generators of $\mathfrak{sl}(2,\mathbb{R})$, the stress tensor does have a simple transformation rule under elements of $PSL(2,\mathbb{C})$ and is an example of a quasi-primary field.

The finite form of the transformation rule for the stress tensor is

$$\left(\frac{\partial z'}{\partial z}\right)^2 T'(z') = T(z) - \frac{c}{12}\{z', z\} \tag{53}$$

where the quantity on the right is the Schwarzian derivative

$$\{f, z\} \equiv \frac{2(\partial_z^3 f)(\partial_z f) - 3(\partial_z^2 f)^2}{2(\partial_z f)^2} \ , \tag{54}$$

Exercise 2. Verify that the finite form of the transformation rule above has the correct infinitesimal form. Also verify that the finite form of the transformation composes correctly.

A Famous Calculation: The Equation of State of a CFT

Consider a cylinder parametrized by $w = \sigma^1 + i\sigma^2$ where σ^2 is periodic with period β: $\sigma^2 + \beta \sim \sigma^2$. Eventually, we will be able to interpret σ^2 a Euclidean time coordinate and $\beta = 1/T$ as the inverse temperature, but for now we can treat β as just some length scale characterizing the cylinder. There is a plane to cylinder transformation given by the exponential map $z = e^{2\pi w/\beta}$. Let us see how the stress tensor behaves with respect to this transformation:

$$\left(\frac{\partial z}{\partial w}\right)^2 T(z)_{\rm pl} = T(w)_{\rm cyl} - \frac{c}{12}\{z, w\} \ . \tag{55}$$

Plugging in the exponential map yields

$$\begin{aligned} \left(\frac{2\pi}{\beta}\right)^2 z^2 T(z)_{\rm pl} &= T(w)_{\rm cyl} - \frac{c}{12}\frac{2(\partial_w^3 z)(\partial_w z) - 3(\partial_w^2 z)^2}{2(\partial_w z)^2} \\ &= T(w)_{\rm cyl} - \frac{c}{12}\left(\frac{2\pi}{\beta}\right)^2 \frac{2-3}{2} \ . \end{aligned} \tag{56}$$

Given the symmetries of the plane, it seems reasonable to assume that in the vacuum state $\langle T(z)\rangle_{\rm pl} = 0$. It follows from the Schwarzian derivative then that

$$\langle T(w)\rangle_{\rm cyl} = -\frac{c}{24}\left(\frac{2\pi}{\beta}\right)^2 \ . \tag{57}$$

Translating back to a rectilinear coordinate system, we obtain

$$T^{22} = -T^{11} = \frac{1}{2\pi}(T(z) + \tilde{T}(\bar{z})) = -\frac{c}{24\pi}\left(\frac{2\pi}{\beta}\right)^2 \ . \tag{58}$$

We can interpret this result in one of two ways. If we think of σ^1 as the Euclidean time coordinate and the CFT as living on a circle of circumference β, then Wick rotating to Minkowski signature, we obtain a negative Casimir energy

$$T^{tt} = -T^{11} = -\frac{\pi c}{6\beta^2} \ . \tag{59}$$

Alternatively, we can treat σ^2 as a Euclidean time direction, in which case $\beta = 1/T$ is interpreted as an inverse temperature. In this case, Wick rotating back, we get a positive thermal energy density

$$T^{tt} = -T^{22} = \frac{\pi c T^2}{6} \ . \tag{60}$$

The Trace Anomaly

The presence of a nonzero central charge c in a conformal field theory is intimately related to the presence of a trace anomaly: the trace of the stress tensor will not vanish on a curved manifold but instead is given by

$$\langle T^\mu_\mu \rangle = \frac{c}{24\pi} R \ . \tag{61}$$

Classically, the right hand side should be zero, but the measure in the path integral may not respect the Weyl scaling symmetry and there can be an anomaly. The form of the right hand side is fixed by symmetry. It must be a scalar quantity under the action of diffeomorphisms that has scaling weight two – the same as the stress tensor in two dimensions. For CFT, the only candidate is the Ricci scalar.

There are two points of view regarding this trace anomaly. From the point of view of conformal field theory, gravity on the world sheet is not dynamical and the anomaly is global. The quantity c tells us interesting things about the properties of the CFT, for example as we saw above, the equation of state. From the point of view of string theory, where world sheet gravity is dynamial (in a sense), the anomaly is a gauged anomaly and indicates a pathology in the theory. It indicates that the string theory will depend in a nontrivial way on the world sheet metric h_{ab} even though the choice of world sheet metric should have been a gauge choice. In the context of string theory, it had better be that at the end of the day the total central charge c should vanish.

In this section, we will demonstrate that the trace anomaly in the stress tensor implies the Schwarzian derivative transformation rule. The proof begins with the statement that mixed partial derivatives commute:

$$\begin{aligned} \langle T^{ab}(x) T^c_c(x')\rangle \sqrt{h(x)}\sqrt{h(x')} &= 2\frac{\delta}{\delta h_{ab}(x)} \langle T^c_c(x')\rangle \sqrt{h(x')} \\ &= h_{cd}(x') 2 \frac{\delta}{\delta h_{cd}(x')} \langle T^{ab}(x)\rangle \sqrt{h(x)} \ . \end{aligned} \tag{62}$$

I am renaming the world-sheet coordinates x in this section because I want to reserve the letter σ for Weyl transformations. In the second equality, we can replace the variation with respect to $h_{cd}(x')$ with an equivalent variation with respect to a Weyl scaling parameter σ. Under an infinitesimal Weyl scaling, $\delta h_{cd} = 2h_{cd}\delta\sigma$, and thus

$$2\frac{\delta}{\delta h_{ab}(x)} \langle T^c_c(x')\rangle \sqrt{h(x')} = \frac{\delta}{\delta \sigma(x')} \langle T^{ab}(x)\rangle \sqrt{h(x)} \ . \tag{63}$$

Plugging the result for the trace anomaly into the left hand side, we will solve this functional differential equation for $\langle T^{ab}(x)\rangle$ and thereby establish the Schwarzian derivative transformation rule. To solve the differential equation, we use dimensional regularization and work in $n = 2+\epsilon$ dimensions. The first observation is that R transforms nicely with respect to Weyl variations in $2+\epsilon$ dimensions. The variation of the Ricci scalar with respect to the metric is familiar from the derivation of Einstein's equations:

$$\frac{\delta}{\delta h_{ab}} \int \mathrm{d}^n x \sqrt{h} R = (-R^{ab} + \frac{1}{2} h^{ab} R)\sqrt{h} \ . \tag{64}$$

Restricting to Weyl variations, we then obtain

$$\frac{\delta}{\delta \sigma} \int \mathrm{d}^n x \sqrt{h} R = (n-2) R \sqrt{h} \ . \tag{65}$$

In other words, R behaves like an eigenvector with eigenvalue $(n-2)$ under Weyl transformations. We can therefore replace the trace of the stress tensor in the differential equation (63) with a Weyl variation of the trace anomaly:

$$\frac{c}{24\pi(n-2)} 2 \frac{\delta}{\delta h_{ab}(x)} \left(\frac{\delta}{\delta\sigma(x')} \int \mathrm{d}^n x \sqrt{h} R \right) = \frac{\delta}{\delta \sigma(x')} \langle T^{ab}(x)\rangle \sqrt{h(x)} \ . \tag{66}$$

We can now functionally integrate with respect to σ' on both sides to obtain

$$\begin{aligned}\langle T^{ab}(x)\rangle\sqrt{h(x)} &= \frac{c}{12\pi(n-2)}\frac{\delta}{\delta h_{ab}(x)}\int \mathrm{d}^n x' \sqrt{h(x')}R(x') \\ &= \frac{c}{12\pi(n-2)}\left[-R^{ab}+\frac{1}{2}g^{ab}R\right]\sqrt{h(x)}\ , \end{aligned} \tag{67}$$

where in the last line we are again deriving Einstein's equations from the Einstein-Hilbert action. I've dropped a constant of integration on both sides. More precisely, we should think about integrating σ from some reference metric h^0_{ab} to the metric of interest h_{ab}. We are really computing the change in the stress tensor as we scale from one metric to another. However, things are simple in two dimensions because every metric is Weyl equivalent to the flat metric $h_{ab} = e^{2\sigma}\delta_{ab}$. Moreover, if we are computing the one point function of the stress tensor in the vacuum, it is reasonable to assume that $\langle T^{ab}\rangle_{\delta_{ab}} = 0$, as we did in the previous section. Then we can take the constant of integration in the solution (67) to vanish.

We have to be careful in evaluating the solution (67) because Einstein's equations vanish identically in $d = 2$ dimensions. Continuing to work in $n = 2+\epsilon$ dimensions, we have that for a metric of the form $h_{ab} = e^{2\sigma}\delta_{ab}$ the relevant curvatures are

$$R_{ab} = (2-n)[\partial_a\partial_b\sigma - (\partial_a\sigma)(\partial_b\sigma)] - \delta_{ab}(\partial^2\sigma + (n-2)(\partial\sigma)^2)\ , \tag{68}$$

$$R = -e^{-2\sigma}[2(n-1)\partial^2\sigma + (n-1)(n-2)(\partial\sigma)^2]\ . \tag{69}$$

We are implicitly contracting indices in the quantities $(\partial^2\sigma)$ and $(\partial\sigma)^2$ with the flat metric δ_{ab}. The Einstein tensor reduces to

$$R_{ab} - \frac{1}{2}h_{ab}R = (n-2)\left([-\partial_a\partial_b\sigma + (\partial_a\sigma)(\partial_b\sigma)] + \delta_{ab}\left[\partial^2\sigma + \frac{1}{2}(n-3)(\partial\sigma)^2\right]\right)\ . \tag{70}$$

Carefully taking the limit $n \to 2$, we obtain our result for the stress tensor

$$\langle T_{ab}(x)\rangle = \frac{c}{12\pi}\left[\partial_a\partial_b\sigma - (\partial_a\sigma)(\partial_b\sigma) - \delta_{ab}\left(\partial^2\sigma - \frac{1}{2}(\partial\sigma)^2\right)\right]\ . \tag{71}$$

In our complex coordinates, we can write instead

$$\langle T(z)\rangle = -\frac{c}{6}\left[\partial_z^2\sigma - (\partial_z\sigma)^2\right]\ . \tag{72}$$

As we discussed, after gauge fixing to δ_{ab}, the residual symmetry of the CFT is a combination of a Weyl scaling and diffeomorphism that leaves the metric invariant:

$$h_{z\bar{z}} = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial \bar{w}}e^{2\sigma(z)}h_{z\bar{z}}\ , \tag{73}$$

or equivalently

$$\sigma = \frac{1}{2}\log\frac{\partial w}{\partial z} + \frac{1}{2}\log\frac{\partial \bar{w}}{\partial \bar{z}}\ . \tag{74}$$

Under the Weyl rescaling, we find that the new stress tensor becomes

$$\begin{aligned}
\langle T(z) \rangle_{h_{ab}} &= -\frac{c}{6}\left[\frac{1}{2}\partial_z \frac{\partial_z^2 w}{\partial_z w} - \frac{1}{4}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2\right] \\
&= -\frac{c}{6}\left[\frac{1}{2}\frac{\partial_z^3 w}{\partial_z w} + \left(-\frac{1}{2} - \frac{1}{4}\right)\frac{(\partial_z^2 w)^2}{(\partial_z w)^2}\right] \\
&= -\frac{c}{12}\left[\frac{2(\partial_z^3 w)(\partial_z w) - 3(\partial_z^2 w)^2}{(\partial_z w)^2}\right] . \qquad (75)
\end{aligned}$$

Finally, we need to perform a diffeomorphism associated with the map $z \to w(z)$:

$$\langle T(w) \rangle_{\delta_{ab}} = \left(\frac{\partial z}{\partial w}\right)^2 \langle T(z) \rangle_{h_{ab}} . \qquad (76)$$

At the end of the day, we recover the Schwarzian derivative formula for the transformation of the stress tensor, starting from a background where $\langle T(z) \rangle_{\delta_{ab}} = 0$:

$$\left(\frac{\partial w}{\partial z}\right)^2 \langle T(w) \rangle = -\frac{c}{12}\{w, z\} . \qquad (77)$$

6 Path Integral Approach

We return to a study of the four CFTs relevant for string theory: the X, ψ, bc and $\beta\gamma$ systems. Our plan in this section is four-fold. From the action, we will derive the OPEs of these fundamental fields. Next we will derive/recall the form of the stress tensor. Using the stress tensor and building block OPEs, we will verify that the fields have the correct scaling dimensions h and $\bar{h}$. Finally, by considering the OPE of the stress tensor with itself, we will derive expressions for the central charges. As output, we will see that both the bosonic string and the spinning string have vanishing total central charge c and thus have no Weyl anomaly.

The X System

We begin with the X system. We already argued for the singularity in the OPE of two X^μ fields based on conformal symmetry alone. However, there is a small hole in the logic that we need to fill: that the normalization of the OPE is consistent with our conventional normalization of the action of the X^μ system. That action, in complex coordinates, is

$$S = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2 z\, \partial X \cdot \bar{\partial} X , \qquad (78)$$

where the measure $\mathrm{d}^2 z = 2\mathrm{d}^2\sigma$.

Consider the expectation value of the following composite operator $\mathcal{O}[X]$:

$$\langle \mathcal{O}[X] \rangle = \int [\mathrm{d}X] e^{-S} \mathcal{O}[X] , \qquad (79)$$

The integral of a total derivative must vanish, even in the path integral context, and so

$$\begin{aligned} 0 &= \int [\mathrm{d}X] \frac{\delta}{\delta X_\mu(z,\bar z)} \left(e^{-S} \mathcal{O}[X] \right) \\ &= \int [\mathrm{d}X] \left(- \frac{\delta S}{\delta X_\mu(z,\bar z)} \mathcal{O}[X] + \frac{\delta \mathcal{O}[X]}{\delta X_\mu(z,\bar z)} \right) e^{-S} \\ &= - \left\langle \frac{\delta S}{\delta X_\mu(z,\bar z)} \mathcal{O}[X] \right\rangle + \left\langle \frac{\delta \mathcal{O}[X]}{\delta X_\mu(z,\bar z)} \right\rangle \\ &= \frac{1}{\pi\alpha'} \left\langle \mathcal{O}[X] \partial\bar\partial X^\mu(z,\bar z) \right\rangle + \left\langle \frac{\delta \mathcal{O}[X]}{\delta X_\mu(z,\bar z)} \right\rangle \ . \end{aligned} \tag{80}$$

If $X_\mu(z,\bar z)$ does not appear in $\mathcal{O}[X]$, then

$$\left\langle \mathcal{O} \partial\bar\partial X(z,\bar z) \right\rangle = 0 \ . \tag{81}$$

In this sense, then, we claim that the equation of motion $\partial\bar\partial X(z\bar z) = 0$ holds as an operator equation at the quantum level. Of course, if $\mathcal{O}[X]$ does depend on $X_\mu(z,\bar z)$, then things are different. One may consider the special case where $\mathcal{O}[X] = X^\nu(z',\bar z')$, in which case we find

$$\frac{1}{\pi\alpha'} \partial_z \partial_{\bar z} X^\mu(z,\bar z) X^\nu(z',\bar z') + \delta^{\mu\nu} \delta^2(z-z', \bar z - \bar z') = 0 \ , \tag{82}$$

which will also hold as an operator equation inside expectation values $\langle \cdots \rangle$ provided no other operators get too close. To determine the OPE of two X^μ fields, we now integrate this expression. Recall from electricity and magnetism that a point charge $\rho = \delta(\sigma^1, \sigma^2)$ in two dimensions has a logarithmic potential function $\phi = -\log r$ where $r^2 = (\sigma^1)^2 + (\sigma^2)^2$. Another way of expressing this relation is Poisson's equation, which in d spatial dimensions takes the form

$$\nabla^2 \phi = -\operatorname{Vol}(S^{d-1}) \rho \ . \tag{83}$$

In complex coordinates, $\nabla^2 = 4\partial\bar\partial$ and $\delta^2(\sigma^1,\sigma^2) = 2\delta^2(z,\bar z)$, and we find then that

$$\partial\bar\partial \log|z|^2 = 2\pi\delta^2(z,\bar z) \ . \tag{84}$$

We conclude that the normalization we asserted before is indeed correct,

$$X^\mu(z,\bar z) X^\nu(z',\bar z') \sim -\frac{\alpha'}{2} \eta^{\mu\nu} \log|z-z'|^2 \ . \tag{85}$$

For the X system, we already wrote down the stress tensor and (assuming the above normalization) used the OPEs to verify that X^μ has conformal weights $h = \bar h = 0$ and that each X^μ field contributes one unit to the central charge. We can thus move on.

The bc System

Consider the one derivative action

$$S = \frac{1}{2\pi g} \int \mathrm{d}^2 z \, b \bar\partial c \ , \tag{86}$$

where b and c are anti-commuting. By dimensional analysis, it must be that for this free field theory, $h_b + h_c = 1$. Let us then parametrize our ignorance by setting $h_b = \lambda$ and $h_c = 1 - \lambda$.

I claim this action actually encodes both the bc ghost system and the ψ system. That it encodes the ghost system is clear from (2.155) and (2.171) in PvN's notes, provided we set $g = 1$. In this case, since b naturally has two lower world-sheet indices and c one upper index, it is natural to take $h_c = -1$ and $h_b = 2$. To see that the general bc system encodes (two copies) of the ψ system, we need to do a little more work. We split the fermionic fields into pieces

$$b = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \ , \quad c = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \ . \tag{87}$$

The action is then

$$S = \frac{1}{4\pi g} \int \mathrm{d}^2 z (\psi_1 \bar{\partial} \psi_1 + \psi_2 \bar{\partial} \psi_2) \ . \tag{88}$$

Up to a factor of i, this expression is precisely two copies of (3.37) in PvN's notes with the identification $g = l^2 = 2\alpha'$. (In contrast, to recover the fermions with Polchinski's normalization, we should instead set $g = 1$.) In this case, $h_c = h_b = \frac{1}{2}$.

To obtain the OPE of the $b(z)$ and $c(z)$ fields, we follow precisely the same steps that we did with the X system. The equations of motion are $\bar{\partial} c(z) = 0 = \bar{\partial} b(z)$. It then follows from the path integral that

$$\bar{\partial} c(z) b(0) = 2\pi g \, \delta^2(z, \bar{z}) \ . \tag{89}$$

From differentiating the OPE of the two X fields, it is straightforward to deduce that $\bar{\partial} z^{-1} = 2\pi \delta^2(z, \bar{z})$ or

$$c(z_1) b(z_2) \sim \frac{g}{z_{12}} \ . \tag{90}$$

Because the b and c fields anti-commute, we also have the relation

$$b(z_1) c(z_2) = -c(z_2) b(z_1) \sim \frac{g}{z_{12}} \ . \tag{91}$$

In the fermionic case that $h_b = h_c = \frac{1}{2}$, this OPE also implies the two-point function

$$\langle c(z_1) b(z_2) \rangle = \frac{g}{z_{12}} \ . \tag{92}$$

For more general λ, the nature of the vacuum state introduces some subtleties that we will return to later.

For the stress-tensor, comparing with (2.185) in PvN's notes (where g is set equal to one) we see that for the ghost system, we can write

$$T(z) = :(\partial b) c: - 2\partial(:bc:) \ . \tag{93}$$

For the fermions, from (3.67) in PvN's notes, we have instead that

$$T(z) = -\frac{1}{2g} \left(:\psi_1 \partial \psi_1: + :\psi_2 \partial \psi_2: \right) \ , \tag{94}$$

or in terms of the bc fields

$$T(z) = \frac{1}{g}\left[:(\partial b)c: - \frac{1}{2}\partial(:bc:)\right] \ . \tag{95}$$

The natural generalization seems to be (and indeed is as we will verify by taking the appropriate OPEs)

$$T(z) = \frac{1}{g}\left[:(\partial b)c: - \lambda\partial(:bc:)\right] \ . \tag{96}$$

We consider three OPEs, $T(z)$ with $b(w)$, $c(w)$, and $T(w)$. From the OPE with $b(w)$,

$$\begin{aligned} T(z)b(w) &\sim (\partial_z b(z))\frac{1}{z-w} - \lambda\partial_z\left(b(z)\frac{1}{z-w}\right) \\ &\sim \frac{\lambda}{(z-w)^2}b(z) + \frac{(1-\lambda)}{z-w}\partial_z b(z) \\ &\sim \frac{\lambda}{(z-w)^2}b(w) + \frac{1}{z-w}\partial b(w) \ , \end{aligned} \tag{97}$$

we verify that $b(w)$ is a primary field with conformal dimension $h_b = \lambda$. From the OPE with $c(w)$,

$$\begin{aligned} T(z)c(w) &\sim -\left(\partial_z\frac{1}{z-w}\right)c(z) + \lambda\partial_z\left(\frac{c(z)}{z-w}\right) \\ &\sim \frac{1-\lambda}{(z-w)^2}c(z) + \frac{\lambda}{z-w}\partial c(z) \\ &\sim \frac{1-\lambda}{(z-w)^2}c(w) + \frac{\partial c(w)}{z-w} \ , \end{aligned} \tag{98}$$

we verify that $c(w)$ is a primary field with conformal dimension $h_c = 1-\lambda$. To obtain the central charge, we look at the leading singularity in the $T(z)T(w)$ OPE:

$$\begin{aligned} T(z)T(w) &\sim \frac{1}{g^2}\left[:(\partial b)c: - \lambda\partial(:bc:)\right](z)\left[:(\partial b)c: - \lambda\partial(:bc:)\right](w) \\ &\sim \left(\partial_z\frac{1}{z-w}\right)\left(\partial_w\frac{1}{z-w}\right) - \lambda\partial_w\left(\frac{1}{z-w}\partial_z\frac{1}{z-w}\right) \\ &\quad -\lambda\partial_z\left(\frac{1}{z-w}\partial_w\frac{1}{z-w}\right) + \lambda^2\partial_z\partial_w\left(\frac{1}{z-w}\right)^2 + \ldots \\ &\sim \frac{1}{(z-w)^4}\left(-1+3\lambda+3\lambda-6\lambda^2\right) + \ldots \end{aligned} \tag{99}$$

where we have left the subleading yet still singular terms as an exercise for the reader to compute. From the leading $(z-w)^{-4}$ term, however, we can read off the central charge of the bc system

$$c_{bc} = -12\lambda^2 + 12\lambda - 2 = -3(2\lambda-1)^2 + 1 \ . \tag{100}$$

For the bc ghost system with $\lambda = 2$, we obtain $c_{bc} = -26$. Meanwhile for two copies of the ψ system, with $\lambda = \frac{1}{2}$, we obtain $c_{bc} = 1$, or $c_\psi = 1/2$ for each copy.

The $\beta\gamma$ System

We consider the analog of the bc system above for commuting fields β and γ. At the penalty of introducing a parameter $\epsilon = \pm 1$ whose sign depends on whether bc commute or anti-commute, one can treat both cases simultaneously. For clarity, I find it simpler to separate the two cases. We shall nevertheless be brief here, and leave most of the relevant checks as an exercise for the reader. The action is

$$S = \frac{1}{2\pi g} \int \mathrm{d}^2 z \, \beta \bar{\partial} \gamma \ , \tag{101}$$

where again we parametrize our ignorance by setting $h_\beta = \lambda$ and $h_\gamma = 1 - \lambda$. The Ward identity following from the equations of motion is

$$\partial_{\bar{z}} \gamma(z) \beta(0) = 2\pi g \delta^2(z, \bar{z}) \ . \tag{102}$$

As before, this relation can be integrated to give the singularity in the OPE:

$$\gamma(z)\beta(0) \sim \frac{g}{z} \ . \tag{103}$$

Now however because β and γ commute, we get instead

$$\beta(z)\gamma(0) \sim -\frac{g}{z} \ . \tag{104}$$

The stress tensor has exactly the same form as before

$$T(z) = :(\partial\beta)\gamma: - \lambda\partial(:\beta\gamma:) \ . \tag{105}$$

The central charge is then minus what it was before

$$c_{\beta\gamma} = 3(2\lambda - 1)^2 - 1 \ . \tag{106}$$

(The computation is essentially identical to the bc system. The minus sign is the difference in sign between the OPE for $b(z)c(0)$ and the OPE for $\beta(z)\gamma(0)$.) Through their index structure, for the ghost system $h_\beta = 3/2$ while $h_\gamma = -1/2$. In this case, the central charge works out to be $c_{\beta\gamma} = 11$.

Exercise 3. Calculate the singular terms in the $T(z)T(0)$ operator product expansion both for the bc system and for the $\beta\gamma$ system. Assume $h_b = h_\beta = \lambda$ and $h_c = h_\gamma = 1 - \lambda$.

Canceling the Trace Anomaly

We considered two types of string theory, the bosonic string and the spinning string. We can see now that in each case, the total central charge vanishes. Thus in each case, there is no Weyl anomaly.

	X	ψ	bc	$\beta\gamma$	c_{tot}
bosonic	$26 \cdot 1$	0	$1 \cdot (-26)$	0	0
spinning	$10 \cdot 1$	$10 \cdot \frac{1}{2}$	$1 \cdot (-26)$	$1 \cdot 11$	0

7 BRST meets CFT

We argued in the earlier part of the course that the BRST charge Q_B should be nilpotent: $Q_B^2 = 0$. We would like in this section to use OPE techniques to verify nilpotency. As a bonus, we will see again that the bosonic string and spinning string are consistent only in $D = 26$ and $D = 10$ space time dimensions, respectively.

The BRST current in the bosonic string case is

$$j_B(z) = cT^m(z) + \frac{1}{2}cT^g(z) + \tilde{\alpha}_B \partial^2 c \ . \tag{107}$$

In the spinning string case, we have instead

$$j_B(z) = cT^m(z) + \frac{1}{2}cT^g(z) + \tilde{\alpha}_S \partial^2 c + \gamma J^m(z) + \frac{1}{2}\gamma J^g(z) \ . \tag{108}$$

Here the superscript m stands for matter fields – X and ψ – while g stands for ghosts – bc and $\beta\gamma$. We will write the full stress tensor $T(z)$ and supercurrent $J(z)$ in the spinning string case only. The way to divide it up into matter and ghost pieces should be obvious at this point, as should the way to restrict it to the purely bosonic case:

$$\begin{aligned} T(z) &= -\frac{1}{\alpha'}\partial X \cdot \bar{\partial} X - \frac{1}{2}\psi \cdot \partial\psi + (\partial b)c - 2\partial(bc) + (\partial\beta)\gamma - \frac{3}{2}\partial(\beta\gamma) \ , & (109) \\ J(z) &= i\left(\frac{2}{\alpha'}\right)^{1/2} \psi \cdot \partial X - \frac{1}{2}(\partial\beta)c + \frac{3}{2}\partial(\beta c) - 2b\gamma \ . & (110) \end{aligned}$$

(We are using Polchinski's normalization of the ψ fields here.) Note that the $\partial^2 c$ term we added to the current j_B reflects a certain choice. As a total derivative, it will not contribute to the total charge Q_B. The form of such a derivative correction is highly constrained by dimensional analysis – it must contain one derivative, one c field, and something else that will make the total dimension of the term equal to one. One can think of this term as arising from ordering ambiguity in composite operators making up the first few terms in $j_B(z)$. It turns out that this $\partial^2 c$ term is necessary in order that $j_B(z)$ transform as a primary operator in CFT language, as we will see.

As Q_B is an anti-commuting operator, we can check nilpotency by looking at

$$\{Q_B, Q_B\} = \oint \frac{\mathrm{d}w}{2\pi i} \mathrm{Res}_{z\to w} j_B(z) j_B(w) \ .$$

We will check the bosonic case and leave the spinning string as a (rather long) exercise. Since we are only interested in the residue, we will focus on the $1/(z-w)$ term in the OPE only. We will divide the computation up into four pieces:

$$\begin{aligned} j_B(z)j_B(w) &= \left[c(T^m + \frac{1}{2}T^g) + \tilde{\alpha}_B\partial^2 c\right](z)\left[c(T^m + \frac{1}{2}T^g) + \tilde{\alpha}_B\partial^2 c\right](w) \\ &\sim c(z)c(w)T^m(z)T^m(w) + \left(T^m(w)\frac{1}{2}c(z)T^g(z)c(w) - (z \leftrightarrow w)\right) \\ &\quad + \frac{\tilde{\alpha}_B}{2}\left[c(z)T^g(z)\partial^2 c(w) - (z \leftrightarrow w)\right] + \frac{1}{4}c(z)T^g(z)c(w)T^g(w) \ . \end{aligned} \tag{111}$$

The first piece expands to give

$$\begin{aligned} c(z)c(w)T^m(z)T^m(w) &\sim c(z)c(w)\left[\frac{c_m}{2(z-w)^4} + \frac{2}{(z-w)^2}T^m(w) + \frac{1}{z-w}\partial T^m(w)\right] \\ &\sim \ldots + \frac{1}{z-w}\left[\partial T^m c^2 + 2T^m(\partial c)c + \frac{c_m}{12}(\partial^3 c)c\right](w) \ . \end{aligned} \tag{112}$$

The first term above cancels on its own because $c(w)^2 = 0$. We will see momentarily how the remaining two terms combine with the rest of the OPE.

For the second piece, consider first the composite operator

$$cT^g = c[(\partial b)c - 2\partial(bc)] = 2c(\partial c)b \ . \tag{113}$$

The second piece thus expands to give

$$\begin{aligned} T^m(w)\frac{1}{2}c(z)T^g(z)c(w) - (z \leftrightarrow w) &\sim T^m(w)c(z)(\partial c(z))b(z)c(w) - (z \leftrightarrow w) \\ &\sim \frac{1}{z-w}(T^m(w)c(\partial c)(z) + T^m(z)c(\partial c)(w)) \\ &\sim 2T^m(w)\frac{c(w)(\partial c(w))}{z-w} \ , \end{aligned} \tag{114}$$

which cancels agains the second term in the expression (112).

Next consider the third piece. Using again the expression (113), the third term can be written

$$\begin{aligned} \frac{\tilde{\alpha}_B}{2}\left[cT^g(z)\partial^2 c(w) - (z \leftrightarrow w)\right] &\sim \frac{\tilde{\alpha}_B}{2}\left[2c(\partial c)b(z)\partial^2 c(w) - (z \leftrightarrow w)\right] \\ &\sim \tilde{\alpha}_B\left[c(\partial c)(z)\left(\partial_w^2\frac{1}{z-w}\right) - (z \leftrightarrow w)\right] \ . \end{aligned} \tag{115}$$

The second term, with z and w swapped, will not give rise to a simple pole proportional to $1/(z-w)$. The first term will however:

$$\begin{aligned} \frac{\tilde{\alpha}_B}{2}\left[cT^g(z)\partial^2 c(w) - (z \leftrightarrow w)\right] &\sim \ldots + \frac{\tilde{\alpha}_B}{z-w}\partial^2(c(\partial c)(w)) \\ &\sim \ldots + \frac{\tilde{\alpha}_B}{z-w}\partial(c(\partial^2 c)(w)) \ . \end{aligned} \tag{116}$$

As it's proportional to $\tilde{\alpha}_B$, reassuringly it's a total derivative and will not affect the nilpotency of Q_B^2.

Last but not least, we consider the fourth piece, the OPE involving the two ghost stress tensors:

$$\begin{aligned} {:}c(\partial c)b(z){:}{:}c(\partial c)b(w){:} \sim\ & \left(\frac{1}{z-w}\right)^2\partial c(z)\partial c(w) - \left(\partial_z\frac{1}{z-w}\right)\left(\frac{1}{z-w}\right)c(z)\partial c(w) \\ & - \left(\frac{1}{z-w}\right)\left(\partial_w\frac{1}{z-w}\right)\partial c(z)c(w) + \left(\partial_z\frac{1}{z-w}\right)\left(\partial_w\frac{1}{z-w}\right)c(z)c(w) \\ & + \frac{1}{z-w}(\partial c)b(z)c(\partial c)(w) - \left(\partial_z\frac{1}{z-w}\right)cb(z)c(\partial c)(w) \\ & - \left(\partial_w\frac{1}{z-w}\right)c(\partial c)(z)cb(w) + \frac{1}{z-w}c(\partial c)(z)(\partial c)b(w) \ . \end{aligned} \tag{117}$$

The second set of four terms all come from single contractions and vanish trivially by the anti-commutativity of the b and c fields. The first set of four terms come from double contractions and simplify to give the $1/(z-w)$ term

$${:}c(\partial c)b(z){:}{:}c(\partial c)b(w){:} \sim \ldots + \frac{1}{z-w}\left[\left(1+\frac{1}{2}\right)(\partial^2 c)(\partial c)(w) - \left(\frac{1}{2}+\frac{1}{6}\right)(\partial^3 c)c(w)\right] \ . \tag{118}$$

We can rewrite the first term using the total derivative $\partial(c(\partial^2 c))$ that we found in evaluating the contraction of the matter stress tensor portion with the ghost stress tensor (114):

$$:c(\partial c)b(z)::c(\partial c)b(w): \quad \sim \quad \ldots + \frac{1}{z-w}\left[-\frac{3}{2}\partial(c(\partial^2 c))(w) - \left(\frac{3}{2}+\frac{2}{3}\right)(\partial^3 c)c(w)\right] . \qquad (119)$$

Assembling the four pieces (112), (114), (116), and (119), we obtain

$$j_B(z)j_B(w) \sim \ldots + \frac{1}{z-w}\left[\frac{c_m - 26}{12}(\partial^3 c)c + \left(\tilde{\alpha}_B - \frac{3}{2}\right)\partial(c(\partial^2 c))\right](w) \qquad (120)$$

Thus we find that the BRST charge is nilpotent if the matter sector has total central charge $c_m = 26$, or equivalently if bosonic string theory lives in 26 space time dimensions. There is also a total derivative term which we do not need to vanish for nilpotency. However, if we can choose $\tilde{\alpha}_B = 3/2$, in which case the residue vanishes completely. This choice has the advantage that it makes $j_B(z)$ a primary operator, as we now check:

$$\begin{aligned} T(z)j_B(w) \quad &\sim \quad T^m(z)c(w)T^m(w) + T^g(z)c(w)\left(T^m(w) + \frac{1}{2}T^g(w)\right) + T^g(z)\tilde{\alpha}_B\partial^2 c(w) \\ &\sim \quad c(w)\left(\frac{c_m}{2(z-w)^4} + \frac{2T^m(w)}{(z-w)^2} + \frac{\partial T^m(w)}{z-w}\right) \\ &\quad +T^m(w)\left(-\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)}\right) + \frac{1}{2}T^g(z)c(w)T^g(w) \\ &\quad +\tilde{\alpha}_B\partial_w^2\left(-\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w}\right) . \end{aligned} \qquad (121)$$

The one nontrivial term here is

$$\begin{aligned} \frac{1}{2}T^g(z)c(w)T^g(w) \quad &\sim \quad \left[(\partial b)c - 2\partial(bc)\right](z)\, c(\partial c)b(w) \\ &\sim \quad \left(\partial_z\frac{1}{z-w}\right)\frac{1}{z-w}\partial c(w) - \left(\partial_z\frac{1}{z-w}\right)\left(\partial_w\frac{1}{z-w}\right)c(w) \\ &\quad -2\partial_z\left(\frac{1}{(z-w)^2}\right)\partial c(w) + 2\partial_z\left(\frac{1}{z-w}\partial_w\frac{1}{z-w}\right)c(w) \\ &\quad +\frac{1}{2}\left(\frac{cT^g(w)}{(z-w)^2} + \frac{\partial(cT^g)(w)}{z-w} +\right) . \end{aligned} \qquad (122)$$

where the last line comes just from the single contractions. Assembling the pieces produces

$$\begin{aligned} T(z)j_B(w) \quad &\sim \quad \frac{c(w)}{(z-w)^4}\left(\frac{c_m}{2} - 5 - 6\tilde{\alpha}_B\right) + \frac{\partial c(w)}{(z-w)^3}(3 - 2\tilde{\alpha}_B) \\ &\quad +\frac{j_B(w)}{(z-w)^2} + \frac{\partial j_B(w)}{z-w} . \end{aligned} \qquad (123)$$

Indeed, $j_B(z)$ is a primary operator with conformal scaling dimension $h = 1$ provided $\tilde{\alpha}_B = 3/2$ and $c_m = 26$.

With a little more work, one can also recover the BRST transformation rules. We will leave that as an exercise for the interested reader:

$$j_B(z)b(0) \quad \sim \quad \frac{3}{z^3} + \frac{1}{z^2}j^g(0) + \frac{1}{z}T(0) \qquad (124)$$

$$j_B(z)c(0) \quad \sim \quad \frac{1}{z}c\partial c(0) \qquad (125)$$

$$j_B(z)O(0) \quad \sim \quad \frac{h}{z^2}cO(0) + \frac{1}{z}\left(h(\partial c)O(0)) + c\partial O(0)\right) , \qquad (126)$$

where we have introduced the ghost current $j^g(z) = -{:}bc{:}$ and an arbitrary primary field $O(z)$. The simple poles reflect the BRST transformation rules.

Exercise 4. In the bosonic string, compute the singular terms in the OPE of the BRST current $j_B(z)$ with $b(w)$, $c(w)$, and with a primary field $O(w)$ of weight h.

A similar exercise for the spinning string demonstrates that $c_m = 15$ and hence that the spinning string exists in 10 target space-time dimensions. Really, this check on the space-time dimensions is redundant having already ensured that the Weyl anomaly vanishes. Remember that in constructing the BRST symmetry, we built in a Weyl scaling symmetry.

Exercise 5. *(Lengthy)* In the spinning string, verify that $j_B(z)j_B(w)$ has no residue at the simple pole $z = w$ when $c_m = 15$. Compute also the OPE of $T(z)$ with $j_B(w)$. What value of $\tilde{\alpha}_S$ is required for $j(z)$ to be a primary field?

8 From Operators to States: The Vacuum

There are a number of subtleties associated with how to define states (and in particular the vacuum state) in CFT that we have thus far largely been able to sweep under the rug.[5] To understand these subtleties, we make a mode decomposition of the fields and think instead about creation and annihilation operators. From the point of view of the closed string theory (and also the doubled version of the open string), it makes perhaps more sense to work on a cylinder. The natural string vacuum will be the one associated to the cylinder. From the point of view of CFT, the plane is a more symmetric starting point. We saw already some nontrivial things happen when we map back and forth, $z = e^w$, $w = \sigma^2 - i\sigma^1$. Here z parametrized the plane while w parametrizes the cylinder. (We have taken a somewhat strange complex structure on the cylinder in order to eliminate some troublesome factors of $\sqrt{-1}$.)

Recall that for a holomorphic primary field, we have the transformation rule

$$O(z) = \left(\frac{\partial z}{\partial w}\right)^{-h} O'(w) = z^{-h} O'(w) \ . \tag{127}$$

On the string worldsheet, given translation invariance in time and space, it makes sense to consider a Fourier decomposition

$$O'(w) = \sum_{n=-\infty}^{\infty} e^{n(i\sigma^1 - \sigma^2)} O_n \ , \tag{128}$$

where we have the Fourier modes

$$O_n = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma^1 e^{-in\sigma^1} O'(\sigma^1, 0) \ . \tag{129}$$

[5]One place where these subtleties threatened to derail the lectures was an apparent conflict between the OPE of the bc fields and the claim that two point correlation functions in CFT are only nonzero when the scaling dimensions of the operators involved are the same. Purposefully, I did not put brackets around the bc OPE, and after this part of the lecture series, we will hopefully see why.

Employing the transformation rule, we find that in the z plane, we can write

$$O(z) = \sum_{n=-\infty}^{\infty} \frac{O_n}{z^{n+h}} \ , \quad O_n = \oint_{|z|=1} \frac{\mathrm{d}z}{2\pi i z} z^{n+h} O(z) \ . \tag{130}$$

(There is a minus sign in changing from the σ^1 variable to the z variable that cancels against the fact that the $\mathrm{d}\sigma^1$ integral goes clockwise in the z plane while the contour integral is usually taken to go counter-clockwise.)

There is a tension emerging here. On the cylinder, it makes sense to think of O_n as raising operators for $n < 0$ and as lowering operators for $n > 0$. However, on the plane, there have appeared some non-homogenous factors of z^h that seem to make it more natural to split the operators around $n = -h$ rather than around $n = 0$. To define the path integral, we need boundary conditions as $\sigma^2 \to -\infty$ on the cylinder or correspondingly $r \to 0$ on the plane (in radial quantization). These boundary conditions conventionally define an initial state. We can imagine changing that state by inserting an operator at $r = 0$. The vacuum $|1\rangle$ in the z-plane naively should correspond to inserting the identity operator at $r = 0$. More generally, we can imagine inserting any local operator at the origin and producing a corresponding state. For a primary operator, we have $O(0)|1\rangle = |O\rangle$. But now in order for this expression to be well defined and nonsingular, from the sum (130) it had better be true that

$$O_n|1\rangle = 0 \ , \quad n > -h \ . \tag{131}$$

Unlike the cylinder vacuum where O_n is an annihilation operator for any $n > 0$, on the plane there is (depending on the sign of h) a set of lowering operators which fail to annihilate the z-plane vacuum or a set of raising operators which do annihilate it.

Interestingly, using the Cauchy residue theorem, we can replace $\partial^n O(0)|1\rangle$, for $n = 0, 1, 2, \ldots$, up to a combinatorial factor, with a corresponding mode operator

$$\begin{aligned} \partial^n O(0)|1\rangle &= n! \oint \frac{\mathrm{d}z}{2\pi i z^{n+1}} O(z)|1\rangle \\ &= n!\, O_{-h-n}|1\rangle \ . \end{aligned} \tag{132}$$

This contour integral gives an alternate way of understanding the condition (131). By construction, $O(z)|1\rangle$ is assumed to be a well defined state in the limit $z \to 0$ with no singular behavior. The contour integral for $n < 1$ must then vanish because it no longer has a simple pole. With the definition of the O_n, the vanishing in turn implies the condition (131).

Let's consider two examples to start – the X system and the bc system – both of which are of key importance for the bosonic string. We will then move on to states in the ψ and $\beta\gamma$ system in the context of the spinning string.

For the X field, the raising and lowering operators are conventionally denoted α_m and are given a normalization such that

$$\partial X(z) = -i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_m \frac{\alpha_m}{z^{m+1}} \ , \tag{133}$$

where $[\alpha_m, \alpha_n] = m\delta_{m+n,0}$. As $\partial X(z)$ is a primary field of conformal scaling dimension $h = 1$, we find that

$$\alpha_m|1\rangle = 0 \quad \text{for} \quad m > -1 \ . \tag{134}$$

In this case, things remain in our comfort zone, and $|1\rangle$ is the usual vacuum for a scalar field.

The bc system is the first case where things are strange. The conformal scaling dimensions are $h_b = 2$ and $h_c = -1$. Thus according to our rules

$$b_m|1\rangle = 0 \quad \text{if} \quad m \geq -1 \ , \tag{135}$$

while

$$c_m|1\rangle = 0 \quad \text{if} \quad m \geq 2 \ . \tag{136}$$

As the anti-commutation relations are $\{b_m, c_n\} = \delta_{m+n,0}$, we would expect the usual vacuum in this system (i.e. the vacuum on the cylinder) to be annihilated instead by all b_n and c_n with $n > 0$. We then need to specify how b_0 and c_0 act. As they satisfy the same anti-commutation relations $\{b_0, c_0\} = 1$ as a pair of fermionic creation and annihilation operators, the vacuum on the cylinder is two-fold degenerate: $|\uparrow\rangle$ and $|\downarrow\rangle$. Thinking of b_0 as a lowering operator and c_0 as a raising operator, we then choose

$$c_0|\uparrow\rangle = 0 \ , \quad c_0|\downarrow\rangle = |\uparrow\rangle \ , \tag{137}$$

$$b_0|\downarrow\rangle = 0 \ , \quad b_0|\uparrow\rangle = |\downarrow\rangle \ . \tag{138}$$

Given that c_0 and c_1 do not annihilate $|1\rangle$, the relation between these degenerate vacua and the z-plane vacuum is then

$$|1\rangle = b_{-1}|\downarrow\rangle \ , \tag{139}$$

or equivalently $c_1|1\rangle = |\downarrow\rangle$.

There is a small fly in the ointment. Recall that the Hermitian conjugate of these mode operators is given by replacing the mode number with minus the mode number: $b_n^\dagger = b_{-n}$ and $c_n^\dagger = c_{-n}$. If we were then to insert $1 = \{b_0, c_0\}$ into the naive inner product

$$\langle\downarrow|\downarrow\rangle = \langle\downarrow|b_0 c_0 + c_0 b_0|\downarrow\rangle \ , \tag{140}$$

we would find that $\langle\downarrow|\downarrow\rangle = 0$. The way to remedy the situation is to modify the inner product such that the conjugate state to $|\downarrow\rangle$ is

$$\langle\downarrow|^c = \langle\uparrow| = \langle\downarrow|c_0 \ . \tag{141}$$

Bizarrely and also truly, the inner product on the z-plane then needs to be modified by the introduction of *three* c fields!

$$\langle 1|^c = \langle 1|c_{-1}c_0c_1 \ . \tag{142}$$

At this point, we can try to resolve the apparent contradiction between the fact that the OPE of a b and c field has a $1/z$ singularity with the fact that their correlation function in vacuum $\langle b(z)c(0)\rangle$ should vanish by the $PSL(2,\mathbb{C})$ subgroup of the conformal symmetry group. The key point is that we assumed the $|1\rangle$ vacuum in discussing the relation between conformal symmetry and the two-point functions. Using the relation (132), we can make the replacement

$$c_{-1}c_0c_1|1\rangle = \frac{1}{2}(\partial^2 c(0))(\partial c(0))c(0)|1\rangle \ . \tag{143}$$

We can then compare the right hand side with the Taylor expansion

$$\begin{aligned} c(z_2)c(z_1)c(0) &= (c_0 + z_2\partial c(0) + \frac{1}{2}z_2^2\partial^2 c(0) + \ldots)(c_0 + z_1\partial c(0) + \frac{1}{2}z_1^2\partial^2 c(0) + \ldots)c(0) \\ &= \frac{1}{2}z_1 z_2(z_2 - z_1)(\partial^2 c(0))(\partial c(0))c(0) + \ldots \ . \end{aligned} \tag{144}$$

The simplest correlation function in which we would need to use the OPE of a b and c field is thus secretly a five point correlation function,

$$\langle 1|b(w)c(z)c(z_2)c(z_1)c(0)|1\rangle \ , \tag{145}$$

not a two point function.

Exercise 6. Compute $\langle 1|b(w)c(z)c_{-1}c_0c_1|1\rangle$. What happens to this two-point function under inversion $w \to 1/w$ and $z \to 1/z$.

The mode expansion for the ψ and $\beta\gamma$ systems depend on whether we are in the R or NS sector. Recall the three cases:

$$\psi(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi_r}{z^{r+1/2}} \ , \tag{146}$$

$$\beta(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{r+3/2}} \ , \tag{147}$$

$$\gamma(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{r-1/2}} \ . \tag{148}$$

In the R sector $\nu = 0$ and the field is periodic, while in the NS sector $\nu = 1/2$ and the field is anti-periodic. After the transformation from the cylinder to the plane, the NS sector (perhaps surprisingly) becomes nicer than the R sector. A factor of $\sqrt{z}$ in the conformal transformation removes the branch cut from the anti-periodic boundary conditions whereas in the R sector, the $\sqrt{z}$ leads to a branch cut.

The vacuum on the cylinder in the NS and R sector is defined via the usual conditions

$$\psi_r|0\rangle_{\rm NS} = \beta_r|0\rangle_{\rm NS} = \gamma_R|0\rangle_{\rm NS} = 0 \qquad \text{if} \qquad r = \frac{1}{2}, \frac{3}{2}, \ldots \tag{149}$$

$$\psi_r|0\rangle_{\rm R} = \gamma_r|0\rangle_{\rm R} = \beta_r|0\rangle_{\rm R} = 0 \qquad \text{if} \qquad r = 1, 2, \ldots \tag{150}$$

Like b_0 and c_0, the modes ψ_0, β_0, and γ_0 need special consideration. We will take γ_0 to be a creation operator and so assume $\beta_0|0\rangle_{\rm R} = 0$.

In contrast, on the z-plane, the vacuum is naturally defined through (131). In the NS sector, we find the conditions

$$\begin{aligned} \psi_r|1\rangle &= 0\ , \quad r=\frac{1}{2},\frac{3}{2},\dots && (151)\\ \beta_r|1\rangle &= 0\ , \quad r=-\frac{1}{2},\frac{1}{2},\frac{3}{2},\dots && (152)\\ \gamma_r|1\rangle &= 0\ , \quad r=\frac{3}{2},\frac{5}{2},\dots && (153) \end{aligned}$$

Thus $|1\rangle$ and $|0\rangle_{\rm NS}$ agree for the ψ system, while they do not agree for the the $\beta\gamma$ system. Unfortunately, it's less clear at this point how to relate $|1\rangle$ and $|0\rangle_{\rm NS}$ for the $\beta\gamma$ system than it was for the bc system.

In the R sector, the $|1\rangle$ state is defined by regularity to be

$$\begin{aligned} \psi_r|1\rangle &= 0\ , \quad r=0,1,\dots\ , && (154)\\ \beta_r|1\rangle &= 0\ , \quad r=-1,0,1,\dots\ , && (155)\\ \gamma_r|1\rangle &= 0\ , \quad r=1,2,\dots\ . && (156) \end{aligned}$$

At this point, it is not clear how the operator-state correspondence should function. The relation (132) would seem to relate raising and lowering operators to fractional derivatives involving $\partial^{1/2}$. As we will see, the resolution involves a new concept, bosonization.

8.1 Bosonization

In 1+1 dimensions, fermions and bosons are not so different. Let $H(z)$ be the holomorphic part of a scalar field (with $\alpha'=2$). The OPE of two such fields is then

$$H(z)H(0)\sim -\log z\ , \tag{157}$$

i.e. the holomorphic part of the OPE of $X(z,\bar z)X(0)$ with $\alpha'=2$. It turns out then that the normal ordered operator $:e^{iH(z)}:$ is very similar to a fermionic field $\psi(z)$. We find the OPE's:

$$\begin{aligned} e^{iH(z)}e^{-iH(0)} &\sim \frac{1}{z}\ , && (158)\\ e^{iH(z)}e^{iH(0)} &\sim O(z)\ , && (159)\\ e^{-iH(z)}e^{-iH(0)} &\sim O(z)\ . && (160) \end{aligned}$$

We recognize the OPE of the bc systems with $b(z)=e^{iH(z)}$ and $c(z)=e^{-iH(z)}$. Moreover, with the usual stress tensor for $H(z)$, the operators $e^{\pm iH(z)}$ should have conformal weight $h=1/2$. Thus, we recognize $e^{\pm iH(z)}=\frac{1}{\sqrt{2}}(\psi^1\pm i\psi^2)$ as linear combinations of the usual fermionic fields on the spinning string. We should check though that $e^{iH(z)}$ and $e^{-iH(z')}$ anti-commute at equal times $|z|=|z'|$. First consider the anti-commutator

$$\begin{aligned} [H(z),H(z')] &= -\log(z-z')+\log(z'-z)\ , && (161)\\ &= \log(-1)\\ &= \pi i\ . \end{aligned}$$

Next we use the Campbell-Baker-Hausdorff formula in the special case where the relevant double commutators vanish, $[,[,]]=0$:

$$e^{t(X+Y)} = e^{tX}e^{tY}e^{-\frac{t^2}{2}[X,Y]} \ , \tag{162}$$

From this relation, it follows that

$$\begin{aligned} e^{tX}e^{tY} &= e^{t(X+Y)}e^{\frac{t^2}{2}[X,Y]} \\ &= e^{tY}e^{tX}e^{t^2[X,Y]} \ . \end{aligned} \tag{163}$$

In our case, $X = H(z)$, $Y = \pm H(z)$ and $t = i$. As a result $e^{t^2[X,Y]} = -1$, and the operators $e^{\pm iH(z)}$ do indeed satisfy the relevant anti-commutation relation.

We will perform one more computation which will let us relate the momentum current associated with $H(z)$ to the fermion number current associated with $\psi(z)$ (or equivalently ghost current associated with b and c). We would like to match the OPE of $e^{iH((z)}$ and $e^{-iH(-z)}$ to that of $b(z)$ with $c(-z)$. Unlike in previous computations, we will expand here around the midpoint $z = 0$:

$$\begin{aligned} :e^{iH(z)}::e^{-iH(-z)}: &\sim \frac{1}{2z}:e^{iH(z)}e^{-iH(-z)}: \\ &\sim \frac{1}{2z}:e^{iH(0)+iz\partial H(0)+i\frac{z^2}{2}\partial^2 H(0)}e^{-iH(0)+iz\partial H(0)-i\frac{z^2}{2}\partial^2 H(0)}: \\ &\sim \frac{1}{2z}:e^{2iz\partial H(0)}+O(z^3): \\ &\sim \frac{1}{2z}+i\partial H(0)+2zT_H(0)+O(z^2) \ . \end{aligned} \tag{164}$$

Exercise 7. Demonstrate the first relation in the computation above.

We can then compare that expression with

$$\begin{aligned} b(z)c(-z) &\sim \frac{1}{2z}+:b(z)c(-z): \\ &\sim \frac{1}{2z}+:b(0)c(0):+:[(\partial b)c(0)-b(\partial c)(0)]:z \ . \end{aligned} \tag{165}$$

where the linear term in z is twice the stress-tensor T_{bc} for the bc system when $\lambda = 1/2$. We can thus make the identifications between stress tensors $T_H = T_{bc}$ and conserved currents $i\partial H = :bc:$.

Exercise 8. The general bc system with arbitrary weights $h_b = \lambda$ and $h_c = 1-\lambda$ can also be bosonized. Show that the bosonized system is the linear dilaton CFT with stress tensor

$$T_H = -\frac{1}{2}(\partial H)^2 + \alpha\partial^2 H \ .$$

Show that e^{kH} is a conformal primary with respect to T_H and determine its conformal scaling dimension h. What is the relation between α and λ?

8.2 R Sector Fermions

Our first application of the bosonization technology will be to the R sector fermions $\psi(z)$. Let us first reorganize the fermionic modes

$$b(z) = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2) = \sum_{n\in\mathbb{Z}}\frac{\psi_n}{z^{n+1/2}} \ , c(z) = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2) = \sum_{n\in\mathbb{Z}}\frac{\tilde{\psi}_n}{z^{n+1/2}} \ . \tag{166}$$

On the cylinder, the R sector ground state is thus defined by the conditions

$$\psi_n|0\rangle_R = \tilde{\psi}_n|0\rangle_R = 0 \ , \quad n = 1, 2, 3, \dots \ . \tag{167}$$

The definition is incomplete as we still must decide what to do with the ψ_0 and $\tilde{\psi}_0$ modes. In ten dimensions, we would have five pairs of such modes, which anti-commute among themselves. For simplicity, let us focus on just one such pair. The situation is completely analogous to the *bc* system we studied earlier. We have that $\{\psi_0, \tilde{\psi}_0\} = 1$ and the ground state $|0\rangle_R$ becomes degenerate:

$$\psi_0|\downarrow\rangle = 0 \ , \quad \psi_0|\uparrow\rangle = |\downarrow\rangle \tag{168}$$

$$\tilde{\psi}_0|\downarrow\rangle = |\uparrow\rangle \ , \quad \tilde{\psi}_0|\uparrow\rangle = 0 \ . \tag{169}$$

We can then use bozonization to figure out the relation between $|\downarrow\rangle$ and $|1\rangle$. Let us assume there is an operator $\mathcal{A}_\downarrow$ which does the job, $|\downarrow\rangle = \mathcal{A}_\downarrow|1\rangle$. From the definition of the $|\downarrow\rangle$ vacuum, we must find the following leading singular behavior as the operators $\psi(z)$ and $\tilde{\psi}(z)$ get close to $\mathcal{A}_\downarrow(0)$:

$$\psi(z)\mathcal{A}_\downarrow(0)|1\rangle = \sum_{n=-\infty}^{-1} \frac{\psi_n}{z^{n+1/2}}\mathcal{A}_\downarrow(0)|1\rangle = O(z^{1/2}) \ , \tag{170}$$

$$\tilde{\psi}(z)\mathcal{A}_\downarrow(0)|1\rangle = \sum_{n=-\infty}^{0} \frac{\tilde{\psi}_n}{z^{n+1/2}}\mathcal{A}_\downarrow(0)|1\rangle = O(z^{-1/2}) \ . \tag{171}$$

These OPEs suggest that $\mathcal{A}_\downarrow(z) = e^{iH(z)/2}$. Correspondingly, we could define $|\uparrow\rangle = \mathcal{A}_\uparrow(0)|1\rangle$ and $\mathcal{A}_\uparrow(z) = e^{-iH(z)/2}$.

In ten dimensions, we really have five pairs of fermionic operators. To map from the cylinder vacuum to the z-plane vacuum, we could define a product of such exponentials of holomorphic scalar fields

$$\Theta_s = \exp\left(i \sum_{a=1}^{5} s_a H^a \right) \ , \tag{172}$$

where $s_a = \pm 1/2$ depending on which of the degenerate R-vacua one is interested in. The holomorphic scalar fields are then given by the following linear combinations of fermionic fields

$$e^{\pm iH^0} = \frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1) \ , \tag{173}$$

$$e^{\pm iH^a} = \frac{1}{\sqrt{2}}(\psi^{2a} \pm i\psi^{2a+1}) \ , \quad a = 1, 2, 3, 4 \ . \tag{174}$$

Note that to be compatible with Lorentzian signature in space-time, the defintition of H^0 involves some extra signs and i's.

The observant reader will at this point complain that an operator $e^{\pm iH^a}$ has no reason to anti-commute with an operator $e^{\pm iH^b}$ when $a \neq b$, and yet they should if these operators really define fermions. Cocycles can be constructed to fix this problem, but time is short, and we will not develop cocycles further.

8.3 The $\beta\gamma$ System

The last order of business is to understand the vacuum in the $\beta\gamma$ system, both in the R and NS sectors. The solution involves a further wrinkle. As β and γ already commute, a bosonization procedure replacing them with a single holomorphic scalar would lead to a CFT with the wrong statistics.[6] The solution involves introducing a holomorphic scalar and an anti-commuting bc system. To avoid confusion, we will relabel $b \to \eta$ and $c \to \xi$. The holomorphic scalar we call ϕ. The following composites of η, ξ and ϕ have the correct OPEs to replace β and γ:

$$\beta(z) = :e^{-\phi(z)}\partial\xi(z): \ , \quad \gamma(z) = :e^{\phi(z)}\eta(z): \ . \tag{175}$$

$$\begin{aligned}
\beta(z)\beta(0) = :e^{-\phi(z)}\partial\xi(z)::e^{-\phi(0)}\partial\xi(0): &\sim \frac{1}{z}O(z) = O(1) \ , \\
\beta(z)\gamma(0) = :e^{-\phi(z)}\partial\xi(z)::e^{\phi(0)}\eta(0): &\sim z\left(-\frac{1}{z^2}\right) = -\frac{1}{z} \ , \\
\gamma(z)\gamma(0) = :e^{\phi(z)}\eta(z)::e^{\phi(0)}\eta(0): &\sim \frac{1}{z}O(z) = O(1) \ .
\end{aligned} \tag{176}$$

We can also try to check that the conformal weights work out correctly. To that end, we should first try to identify the matching between the stress tensors:

$$\begin{aligned}
\gamma(z)\beta(-z) &= \frac{1}{2z} + :\gamma(z)\beta(-z): \\
&= \frac{1}{2z} + :\left(\gamma(0) + z\partial\gamma(0) + \ldots\right)\left(\beta(0) - z\partial\beta(0) + \ldots\right): \\
&= \frac{1}{2z} + :\gamma(0)\beta(0): + z:\left(-\gamma\partial\beta(0) + (\partial\gamma)\beta(0)\right): + \ldots \ , \\
&= \frac{1}{2z} + j_{\beta\gamma}(0) - 2z(T_{\beta\gamma}(0) + \partial j_{\beta\gamma}(0)) + \ldots
\end{aligned} \tag{177}$$

We have identified $j_{\beta\gamma} = :\beta\gamma:$ and $T_{\beta\gamma} = :(\partial\beta)\gamma: - \frac{3}{2}\partial(:\beta\gamma:)$. We now compare this OPE with the corresponding OPE after the "bosonization" procedure

$$\begin{aligned}
:e^{\phi(z)}\eta(z): :e^{-\phi(-z)}\partial\xi(-z): &= (2z):e^{\phi(z)}e^{-\phi(-z)}:\left(\frac{1}{(2z)^2} + :\eta(z)\partial\xi(-z):\right) \\
&= 2z:e^{2z\partial\phi(0)}:\left(\frac{1}{(2z)^2} + :\eta(0)\partial\xi(0): + O(z)\right) \\
&= 2z\left(1 + 2z\partial\phi(0) + 2z^2(\partial\phi(0))^2\right)\left(\frac{1}{(2z)^2} + :\eta\partial\xi(0):\right) + O(z^2) \\
&= \frac{1}{2z} + \partial\phi(0) + z\left[\partial\phi(0)^2 + 2:\eta\partial\xi(0):\right] + O(z^2) \ .
\end{aligned} \tag{178}$$

Thus we can make the identification of currents $:\beta\gamma: = \partial\phi$. But we can also make an identification of stress tensors:

$$T_{\beta\gamma} = -\frac{1}{2}:(\partial\phi)^2: - :\eta\partial\xi: - \partial^2\phi = T_\phi + T_{\eta\xi} \ , \tag{179}$$

where the total derivative $\partial^2\phi$ term comes from $j_{\beta\gamma}$. The piece $-:\eta\partial\xi:$ we recognize as a stress tensor for the bc system with $\lambda = 1$ implying $h_\eta = 1$ and $h_\xi = 0$. The stress tensor for the $\phi(z)$ field,

[6] If we use exponential operators $e^{2n\omega H}$, where $\omega^2 = \pm 1$ and n is integer, then the operators commute, but β and γ will have the wrong OPE.

however, has been modified from the form we discussed previously. It is now the "linear dilaton" CFT, and we can determine the weight of $e^{k\phi}$ using our by now standard procedure computing the singular terms in $T_\phi(z)e^{k\phi(0)}$. The answer is that $h_k = -k(1+k/2)$. For $k = \pm 1$, we obtain $h_+ = -3/2$ and $h_- = 1/2$. We can then verify that the scaling dimensions before and after the "bosonization" procedure are compatible,

$$h_\beta = h_- + 1 + h_\xi \ , \quad h_\gamma = h_+ + h_\eta \ . \tag{180}$$

We now apply this modified "bosonization" procedure to look at the R and NS ground state of the $\beta\gamma$ system. First consider the NS ground state. On the cylinder, we have

$$\beta_r|0\rangle_{\rm NS} = \gamma_r|0\rangle_{\rm NS} = 0 \ , \quad r = \frac{1}{2}, \frac{3}{2}, \ldots \tag{181}$$

We assume an operator such that the cylinder vacuum can be related to the z-plane vaccum via

$$|0\rangle_{\rm NS} = \mathcal{A}^{\rm NS}_{\beta\gamma}(0)|1\rangle \ . \tag{182}$$

By definition of the cylinder vacuum, we must find the following singularities in the OPEs

$$\gamma(z)\mathcal{A}^{\rm NS}_{\beta\gamma}(0)|1\rangle \ = \ \sum_{n\leq -\frac{1}{2}} \frac{\gamma_n}{z^{n-1/2}}\mathcal{A}^{\rm NS}_{\beta\gamma}(0)|1\rangle = O(z) \ , \tag{183}$$

$$\beta(z)\mathcal{A}^{\rm NS}_{\beta\gamma}(0)|1\rangle \ = \ \sum_{n\leq -\frac{1}{2}} \frac{\beta_n}{z^{n+3/2}}\mathcal{A}^{\rm NS}_{\beta\gamma}(0)|1\rangle = O(z^{-1}) \ . \tag{184}$$

These OPEs then suggest that

$$\mathcal{A}^{\rm NS}_{\beta\gamma} = e^{-\phi(z)} \ . \tag{185}$$

Indeed,

$$:e^{\phi(z)}\eta(z)::e^{-\phi(0)}: \ \sim \ O(z) \ , \tag{186}$$

$$:e^{-\phi(z)}\partial\xi(z)::e^{-\phi(0)}: \ \sim \ \frac{1}{z}\partial\xi(0) \ . \tag{187}$$

The R sector works analogously. On the cylinder, the vacuum is defined by

$$\beta_r|0\rangle_{\rm R} \ = \ 0 \ , \quad r = 0, 1, 2, \ldots \tag{188}$$

$$\gamma_r|0\rangle_{\rm R} \ = \ 0 \ , \quad r = 1, 2, 3, \ldots \ . \tag{189}$$

We let $|0\rangle_{\rm R} = \mathcal{A}^R_{\beta\gamma}(0)|1\rangle$. The fields then satisfy the OPEs

$$\gamma(z)\mathcal{A}^{\rm R}_{\beta\gamma}(0)|1\rangle \ = \ \sum_{n=-\infty}^{0} \frac{\gamma_n}{z^{n-1/2}}\mathcal{A}^{\rm R}_{\beta\gamma}(0)|1\rangle = O(z^{1/2}) \ , \tag{190}$$

$$\beta(z)\mathcal{A}^{\rm R}_{\beta\gamma}(0)|1\rangle \ = \ \sum_{n=-\infty}^{-1} \frac{\beta_n}{z^{n+3/2}}\mathcal{A}^{\rm R}_{\beta\gamma}(0)|1\rangle = O(z^{-1/2}) \ . \tag{191}$$

These OPEs then suggest the identification

$$\mathcal{A}^{\rm R}_{\beta\gamma} = e^{-\phi/2} \ , \tag{192}$$

as we may check

$$
\begin{aligned}
{:}e^{\phi(z)}\eta(z){:}{:}e^{-\phi(0)/2}{:} &= O(z^{1/2}) \;, && (193) \\
{:}e^{-\phi(z)}\partial\xi(z){:}{:}e^{-\phi(0)/2}{:} &= O(z^{-1/2}) \;. && (194)
\end{aligned}
$$

9 From Operators to States: Virasoro and Super Virasoro

In the previous section, we mapped out the relation between the vacuum state on the cylinder – a natural starting point for the quantization of the string – and the vacuum state on the plane – perhaps a more natural starting point in the context of conformal field theory. Here, we would like to consider excited states. In the context of conformal field theory, these excited states can be grouped together into representations of the Virasoro algebra, as we will now see.

For the stress-tensor, the transformation rule from the cylinder to the plane involves a Schwarzian derivative, as we saw previously

$$
z^2 T(z) = T(w) + \frac{c}{24} \;. \qquad (195)
$$

There is thus a choice of where it is most natural to define the modes. Conventionally, the modes are defined in the z-plane:

$$
L_n = \oint \frac{\mathrm{d}z}{2\pi i z} z^{n+2} T(z) \;. \qquad (196)
$$

Relative to modes T_n on the cylinder, there is then a shift:

$$
L_n = T_n + \frac{c}{24}\delta_{n,0} \;. \qquad (197)
$$

The modes L_n are conventionally called Virasoro generators.

Let us consider the action of the Virasoro generators L_n on the z-plane vacuum $|1\rangle$. By the relation (132), we have that

$$
L_m|1\rangle = 0 \quad \text{if} \quad m \geq -1 \;. \qquad (198)
$$

In particular $L_{\pm 1}$ and L_0 annihilate the z-plane vacuum. As $L_{\pm 1}$ and L_0 generate (half) of the $PSL(2,\mathbb{C})$ subgroup of the conformal group, the vacuum $|1\rangle$ is sometimes called the $PSL(2,\mathbb{C})$ (or $\mathfrak{sl}(2,\mathbb{C})$) invariant vacuum.

To check this claim about the invariance of $|1\rangle$, let us use our OPE technology to investigate the commutator algebra of the Virasoro generators. Their commutator algebra we can deduce from our OPE rules

$$
\begin{aligned}
\mathrm{Res}_{z_1\to z_2}\, z_1^{m+1}T(z_1) z_2^{n+1}T(z_2) &= \mathrm{Res}_{z_1\to z_2}\, z_1^{m+1} z_2^{n+1}\left(\frac{c}{2z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2)\right) \\
&= \frac{c}{12}(\partial^3 z_2^{m+1}) z_2^{n+1} + 2(\partial z_2^{m+1}) z_2^{n+1} T(z_2) + z_2^{m+n+2}\partial T(z_2) \\
&= \frac{c}{12}(m^3 - m) z_2^{m+n-1} - (\partial z_2^{m+1})(z_2^{n+1}) T(z_2) \\
&\quad - z_2^{m+1}(\partial z_2^{n+1}) T(z_2) + \partial(z_2^{m+n+2} T(z_2)) \\
&= \frac{c}{12}(m^3 - m) z_2^{m+n-1} + (m-n) z_2^{m+n+1} T(z_2) + \partial(\cdots) \qquad (199)
\end{aligned}
$$

From this residue calculation, we can read off the commutator of two Virasoro generators

$$[L_m, L_n] = \frac{c}{12}(m^3 - m)\delta_{m+n} + (m-n)L_{m+n} \ . \tag{200}$$

Indeed, when $m = -1$, 0, and 1, the central term vanishes, and the generators satisfy the usual $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra.

We can also consider the commutator of a Virasoro generator with a mode of a primary field

$$\begin{aligned} \mathrm{Res}_{z_1\to z_2}\, z_1^{m+1} T(z_1) z_2^{n+h-1} O(z_2) &\sim \mathrm{Res}_{z_1\to z_2}\, z_1^{m+1} z_2^{n+h-1}\left(\frac{h}{z_{12}^2}O(z_2) + \frac{1}{z_{12}}\partial O(z_2)\right) \\ &= h(\partial z_2^{m+1}) z_2^{n+h-1} O(z_2) + z_2^{m+n+h}\partial O(z_2) \\ &= h(m+1) z_2^{m+n+h-1} O(z_2) - (\partial z_2^{m+n+h}) O(z_2) + \partial(z_2^{m+n+h} O(z_2)) \\ &= h(m+1) z_2^{m+n+h-1} O(z_2) - (m+n+h) z_2^{m+n+h-1} O(z_2) + \partial(\cdots) \ , \end{aligned} \tag{201}$$

from which we can read off

$$[L_m, O_n] = [(h-1)m - n] O_{m+n} \ . \tag{202}$$

One special case is the dilation operator L_0 for which the action on a conformal primary state is

$$L_0|O\rangle = [L_0, O_{-h}]|1\rangle = hO_{-h}|1\rangle \ . \tag{203}$$

Thus the eigenvalues of L_0 are the conformal weights of the states. Another special case is L_{-1}, which acts like a derivative operator

$$L_{-1}\partial^n O(0)|1\rangle = n![L_{-1}, O_{-h-n}]|1\rangle = (n+1)! O_{-h-n-1}|1\rangle = \partial^{n+1} O(0)|1\rangle \ . \tag{204}$$

More generally, if we act on a conformal primary state, we obtain

$$L_m|O\rangle = [L_m, O_{-h}]|1\rangle = [(h-1)m + h] O_{m-h}|1\rangle \ . \tag{205}$$

This relation has an interesting consequence from (131): the relation implies that L_m annihilates a conformal primary state if $m > 0$. Under the action of L_m then $|O\rangle$ has an interpretation as a highest weight state. The descendants are obtained by acting with L_{-m} operators, $m \geq 1$, in all possible ways subject to the commutation relations (200). A descendant has the form

$$|d\rangle = L_{-m_1} L_{-m_2} \cdots L_{-m_k}|O\rangle \ . \tag{206}$$

The conformal weight of a descendant is obtained by acting with L_0

$$L_0|d\rangle = (h + m_1 + m_2 + \cdots + m_k)|O\rangle \ , \tag{207}$$

because $[L_0, L_{-m}] = mL_{-m}$. The integer $N = m_1 + m_2 + \ldots + m_k$ is called the level of the descendant. Such a representation of the Virasoro algebra is called a Verma module.

Assuming an inner product such that $L_m^\dagger = L_{-m}$, which should hold true for unitary CFTs, we can find some interesting constraints. One such constraint is the positivity of the inner product

$$\langle O| L_m L_{-m}|O\rangle \geq 0 \ . \tag{208}$$

Using the commutation relations (200), this inner product is equivalent to

$$\langle O|2mL_0 + \frac{c}{12}(m^3 - m)|O\rangle = \left[2mh + \frac{c}{12}(m^3 - m)\right] . \tag{209}$$

Restricting to $m = 1$, it follows that $h \geq 0$. According to this restriction, our ghost systems bc and $\beta\gamma$ are non-unitary since $h_c = -1$ and $h_\gamma = -1/2$. For m sufficiently large, it follows also that $c \geq 0$.

There are additional constraints one may obtain by looking at inner products involving more than two L_m. This line of reasoning leads to some very interesting physics, including the development of minimal models in CFT. But this line of development takes us too far afield from the subject of the course – string theory.

Instead, we should also mention the supersymmetric extension of the Virasoro algebra – the super Virasoro algebra. Given the supercurrent $J(z)$ (110), we can form a generalized holomorphic current $j^\eta(z) = \eta(z)J(z)$ by multiplying by an arbitrary holomorphic Grassman valued function $\eta(z)$. This process is similar to the $v(z)T(z)$ current we considered before. Such currents generate the local superconformal transformations, as we can verify using our OPE technology:

$$\frac{1}{i\epsilon}\delta X(0) = \mathrm{Res}_{z\to 0}\, \eta(z)J(z)X(0) \sim \eta\psi(0) , \tag{210}$$

$$\frac{1}{i\epsilon}\delta\psi(0) \sim \eta\partial X(0) , \tag{211}$$

$$\frac{1}{i\epsilon}\delta b(0) \sim \eta\partial\beta(0) , \quad \frac{1}{i\epsilon}\delta c(0) \sim \eta\gamma(0) , \tag{212}$$

$$\frac{1}{i\epsilon}\delta\beta(0) \sim \eta b(0) , \quad \frac{1}{i\epsilon}\delta\gamma(0) \sim \eta\partial c(0) . \tag{213}$$

To figure out the supersymmetric analog of the Virasoro algebra, starting with the formulae (109) and (110), consider the OPEs of the stress tensor $T(z)$ and supercurrent $J(z)$:

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) , \tag{214}$$

$$T(z)J(0) \sim \frac{3}{2z^3}J(0) + \frac{1}{z}\partial J(0) , \tag{215}$$

$$J(z)J(0) \sim \frac{2c}{3z^3} + \frac{2}{z}T(0) . \tag{216}$$

Decomposing $J(z)$ into modes

$$J(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}} , \tag{217}$$

we obtain the following commutation relations, additional to (200) which remains unchanged,

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0} , \tag{218}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r} . \tag{219}$$

As before $\nu = 0$ in the R sector and $\nu = 1/2$ in the NS sector. A highest weight state of a super-Virasoro primary would then be annihilated by G_r for all $r > 0$. A Verma module is created by acting with all possible combinations of G_r, $r < 0$, subject to the commutation relations.

10 Thermal Partition Function

We wish to calculate the partition function of the bosonic string at a temperature $T = 1/\beta$:

$$Z(\beta) = \mathrm{tr}\, e^{-\beta H} \ . \tag{220}$$

The open string has mass spectrum $M^2 = (N-1)/\alpha'$. The degeneracy comes from two contributions. One part of the degeneracy is the number of different ways of getting N with raising operators α^μ_{-n}, $n > 0$. In light cone gauge, we can further restrict a^μ_{-n} such that $\mu = 1, \ldots, 24$. The second contribution is from momentum vectors k such that $k^2 + M^2 = 0$. The partition function then takes the schematic form

$$Z(\beta) = \sum_{N=0}^{\infty} \int \frac{\mathrm{d}^{26}k}{(2\pi)^{26}} \delta\left(k^2 + \frac{N-1}{\alpha'}\right) p(N) e^{-\beta\sqrt{(N-1)/\alpha'}} \ , \tag{221}$$

where $p(N)$ is a degeneracy factor coming from the raising operators α^μ_{-n}.

We can write a generating function for $p(N)$ in the following way:

$$\sum_{N=0}^{\infty} p(N) q^N = \left(\prod_{n=0}^{\infty} \frac{1}{1-q^n}\right)^{24} \ . \tag{222}$$

If we focus on creation operators for a fixed μ, a general word will have N^μ_n α^μ_{-n} raising operators. The degeneracy $p(N^\mu)$ is then the number of ways of writing N^μ as a sum of positive integers while the degeneracy $p(N)$ is further the number of ways of expressing N as a sum over the N^μ.

Defining $q = e^{2\pi i \tau}$, the quantity on the right hand side of (222) is closely related to the Dedekind eta function

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - q^n) \ . \tag{223}$$

The Dedekind eta function $\eta(\tau)$ has simple transformation properties under the discrete group $SL(2, \mathbb{Z})$. This group can be generated by the elements $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$:

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau) \ , \tag{224}$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau) \ . \tag{225}$$

The T transformation rule is straightforward to understand. The S rule requires a fair amount of work to derive. The quantity $\eta(\tau)^{24}$ is sometimes called the modular discriminant $\Delta(\tau)$.

We would like to study the asymptotic behavior of $Z(\beta)$ at high temperatures. The answer to this question requires an understanding of the behavior of $p(N)$ at large N, a question that Hardy and Ramanujan first answered almost a hundred years ago. We have the following contour integral expression for $p(N)$:

$$\begin{aligned} p(N) &= \oint \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^{24} q^{-N} \frac{\mathrm{d}q}{2\pi i q} \\ &= \oint e^{2\pi i \tau} \eta(\tau)^{-24} q^{-N} \frac{\mathrm{d}q}{2\pi i q} \\ &= \oint e^{2\pi i \tau} \eta(-1/\tau)^{-24} \tau^{12} q^{-N} \frac{\mathrm{d}q}{2\pi i q} \\ &= \int_0^1 e^{-2\pi(N-1)i\tau - 2\pi/i\tau} (i\tau)^{12} \left(\prod_{n=1}^{\infty} \frac{1}{1-e^{2\pi/i\tau}}\right)^{24} \mathrm{d}\tau \\ &= \int_0^1 e^{f(\tau)} \left(\prod_{n=1}^{\infty} \frac{1}{1-e^{2\pi/i\tau}}\right)^{24} \mathrm{d}\tau. \end{aligned} \tag{226}$$

where we have defined the function of τ:

$$f(\tau) \equiv -2\pi(N-1)i\tau - \frac{2\pi}{i\tau} + 12\log(i\tau) \ . \tag{227}$$

In the large N limit, the integral has saddlepoints at $\tau = \pm i N^{-1/2}$. A saddlepoint evaluation then leads to the approximate result

$$p(N) \sim \frac{1}{2N^{27/4}} e^{4\pi\sqrt{N}} \ . \tag{228}$$

We see then that the partition function ceases to be well defined at high temperatures

$$4\pi\sqrt{N} - \beta\sqrt{\frac{N}{\alpha'}} > 0 \ , \tag{229}$$

or equivalently $\beta < 4\pi\sqrt{\alpha'}$. This divergence is associated with what is called a Hagedorn phase transition. The divergence typically does not mean that the theory ceases to be well defined above a certain temperature. Rather, it typically indicates that the nature of the fundamental degrees of freedom change.

A Bosonization and Cocycles

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