Physics 305, Fall 2008
Group Theory Remarks

1 Some mathematical preliminaries

We should begin with a couple of definitions.

Definition 1.1. A *group* is a set G with an operation $\cdot$ called multiplication

$$\begin{array}{ccccc} \cdot & : & G \times G & \to & G \\ & & (a,b) & \mapsto & a \cdot b \ , \end{array}$$

satisfying certain properties for all a, b, and $c \in G$:

- associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- the existence of an identity element $e \in G$ such that $e \cdot a = a \cdot e = e$.
- for every $a \in G$, the existence of an inverse element $a^{-1} \in G$ such that $a^{-1} \cdot a = a \cdot a^{-1} = e$

This definition tells us something about how a group acts on itself, but it tells us nothing about how a group acts on other things. In some sense, that's very strange, because the way most of us first start thinking about groups is how they act on physical objects. For example, we have the group of rotations spinning a compass needle or an arrow inside a sphere. Mathematically, however, we need to introduce a separate structure to understand how groups act on "other things", and that structure is called a representation. A representation of a group G is a prescription for how G acts on a vector space V. If V is d-dimensional, we map elements of G to $d \times d$ matrices that then act on V in the usual way. To make sure these $d \times d$ matrices have the structure of a group, we need to make sure each matrix has an inverse which in turn means the determinant of the matrix is not allowed to vanish. The set of such matrices is usually denoted $GL(V)$.

Definition 1.2. A *representation* is a map $\rho : G \to GL(V)$ from a group G to the general linear group $GL(V)$ acting on the vector space V. This map must satisfy the following properties

- $\rho(a \cdot b) = \rho(a)\rho(b)$
- $\rho(e) = \mathrm{Id}$
- $\rho(a^{-1}) = \rho(a)^{-1}$

In quantum mechanics, our vector space V is the Hilbert space of allowed states. Many times, the physical system has a symmetry. For example, if the Hamiltonian is time independent, then there is symmetry under $t \to t + \Delta t$. For a free particle, we have symmetry under spatial translations $\vec{x} \to \vec{x} + \Delta\vec{x}$. For central force potential problems, there is rotational symmetry. To each of these symmetries, we associate a group: the group of time translations, the group of spatial translations, and the group of rotations respectively. Finally, there will be a representation ρ of these groups that acts on any state $|\psi\rangle$ in our Hilbert space:

$$g : |\psi\rangle \to \rho(g)|\psi\rangle \ .$$

Given this prescription for how g acts on states, it is straightforward to see how g acts on operators. Note first that for any operator O, $O|\psi\rangle$ is again a state. Therefore

$$g : O|\psi\rangle \to \rho(g)O|\psi\rangle = \left(\rho(g)O\rho(g)^{-1}\right)\rho(g)|\psi\rangle \ .$$

Thus, we conclude that

$$g : O \to \rho(g)O\rho(g)^{-1} \ .$$

That our system is invariant under the symmetry group means that the Hamiltonian does not transform under g:

$$g : H \to \rho(g)H\rho(g)^{-1} = H \ .$$

We now argue that these representations must be unitary — that the corresponding matrices in $GL(V)$ are unitary. Take $\rho(g) \in GL(V)$ for a group element $g \in G$. Physically, if the system is symmetric under the action of G, then for any state, $|\psi\rangle$, $\langle H\rangle = \langle\psi|H\psi\rangle$ should be invariant under the action of g:

$$g : \langle\psi|H\psi\rangle \to \langle\rho(g)\psi|\rho(g)H\psi\rangle = \langle\psi|\rho(g)^\dagger\rho(g)H\psi\rangle \ .$$

Thus we conclude that $\rho(g)^\dagger\rho(g) = \mathrm{Id}$.

Given that our representations are unitary, there is an interesting mathematical theorem that becomes relevant.

Theorem 1.3. *If L is a Hermitian operator and $\alpha \in \mathbb{R}$, the operator $U = e^{i\alpha L}$ is unitary.*

Proof. We present a quasi-proof. Think about a basis in which L is diagonal. L has real eigenvalues by the spectral theorem. U is also diagonal with eigenvalues of the form $e^{i\lambda}$ with $\lambda \in \mathbb{R}$. Thus, in this basis, $U^\dagger = U^{-1}$. □

We've now hopefully convinced ourselves that in quantum mechanics, the $\rho(g)$ are unitary operators. Let's therefore write them as $U(g)$ to emphasize that fact. I would like to present an "infinitesimal" version of the statement $U(g)HU(g)^\dagger = H$. In the case of a continuous group where there are elements that are infinitesimally close to the identity operator (e.g. time or space translations or the rotation group mentioned above), we may write

$$U(g) = e^{i\epsilon L(g)} = 1 + i\epsilon L(g) + \ldots$$

where $\epsilon \ll 1$ is a small parameter and L is a Hermitian operator. In this case, we find

$$H = U(g)HU(g)^\dagger = (1 + i\epsilon L(g) + \ldots)H(1 - i\epsilon L(g) + \ldots)$$

from which we conclude the familiar statement

$$[H, L] = 0 \ .$$

For each symmetry group, we have conserved quantities via Noether's theorem. In quantum mechanics, these quantities are observables such as the angular momentum. These observables commute with the Hamiltonian.

2 Time and space translation

For a free particle, we can write a general wave function as

$$\psi(x) = \int \frac{dk}{\sqrt{2\pi}} \phi(k) e^{ikx} \ .$$

Let's introduce an operator T_a which shifts $x \to x + a$:

$$\begin{aligned} T_a\psi(x) = \psi(x+a) &= \int \frac{dk}{\sqrt{2\pi}} \phi(k) e^{ik(x+a)} \\ &= \int \frac{dk}{\sqrt{2\pi}} \phi(k) e^{ikx} e^{ika} \ . \end{aligned}$$

How do we pull the e^{ika} factor outside of the integral and deduce the form of T_a? Replacing $\hbar k$ with the momentum operator $\hat{p}$ should do the trick. Thus we set $e^{ika} = e^{i\hat{p}a/\hbar}$ and find

$$T_a(x)\psi(x) = e^{i\hat{p}a/\hbar} \int \frac{dk}{\sqrt{2\pi}} \phi(k) e^{ikx} \ .$$

From this expression we conclude that $T_a = e^{i\hat{p}a/\hbar}$. Because $\hat{p}$ is a Hermitian operator, T_a must be unitary, by our theorem. The elements T_a generate the continuous one dimensional group of spatial translations.

Next consider a Hamiltonian H which is independent of time. The general solution to Schrödinger's equation is

$$\psi(x,t) = \sum_n c_n \varphi_n(x) e^{-iE_n t/\hbar} \ ,$$

where $H\varphi_n(x) = E_n\varphi_n(x)$. Let's introduce an operator $U(\Delta t)$ which shifts $t \to t + \Delta t$:

$$\begin{aligned} U(\Delta t)\psi(x,t) = \psi(x, t+\Delta t) &= \sum_n c_n \varphi_n(x) e^{-iE_n(t+\Delta t)/\hbar} \\ &= \sum_n e^{-iE_n\Delta t/\hbar} c_n \varphi_n(x) e^{-iE_n t/\hbar} \ . \end{aligned}$$

We are left with a similar problem, how to pull the Δt dependent factor outside the sum. We replace E_n with the Hamiltonian, finding

$$U(\Delta t)\psi(x,t) = e^{-iH\Delta t/\hbar} \sum_n c_n \varphi_n(x) e^{-iE_n t/\hbar} \ .$$

Thus we conclude that time translations are generated by the unitary operators

$$U(\Delta t) = e^{-iH\Delta t/\hbar}$$

for arbitrary Δt.

3 Fun with SU(2) and SO(3)

SU(2) and SO(3) are both three dimensional continuous groups. SU(2) is the group of 2×2 unitary matrices with unit determinant. SO(3) is the group of 3×3 orthogonal matrices with unit determinant. SO(3) should be familiar. It's the ordinary group of rotations in three dimensions.

At this point, you should be slightly confused. SU(2) and SO(3), in addition to being groups, are also representations. SU(2) acts naturally on a two dimensional vector space $\mathbb{R}^2$ while SO(3) acts naturally on a three dimensional vector space $\mathbb{R}^3$. But there are also ways of mapping the elements of SU(2) and SO(3) to the $GL(V)$ of other vector spaces. We will get some inkling of how this works in the following in addition to discovering a surprising relationship between SU(2) and SO(3).

Consider the element $g \in \mathrm{SO}(3)$ which generates a rotation around the z-axis by an angle θ:

$$g = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Clearly, $g(x,y,z) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, z)$. Note that in general elements $g \in \mathrm{SO}(3)$ acting on (x,y,z) leave the length $x^2+y^2+z^2$ invariant.

Consider the following strange rewriting of the vector $v = (x,y,z)$ as a 2×2 matrix:

$$v = \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix} .$$

The determinant $\det v = -(x^2+y^2+z^2)$. Take any element $h \in \mathrm{SU}(2)$, and write hvh^{-1}. From the fact that $\det h = 1$ and $\det ab = \det a \det b$, if follows that this "adjoint" action of SU(2) on $\mathbb{R}^3$ leaves $x^2+y^2+z^2$ invariant.

We can do a little better here. Under the rotation about the z-axis $g \in \mathrm{SO}(3)$ considered above, note that $x+iy \to e^{-i\theta}(x+iy)$. Similarly $x-iy \to e^{i\theta}(x-iy)$. Now take

$$h = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} .$$

I claim

$$hvh^{-1} = \begin{pmatrix} z & e^{-i\theta}(x+iy) \\ e^{i\theta}(x-iy) & -z \end{pmatrix} .$$

It turns out that SU(2) and SO(3) are nearly the same group. One often says that SU(2) is the double cover of SO(3) and writes $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$. Consider the case $\theta = 2\pi$ above. Clearly $hvh^{-1} = v$ because $e^{2\pi i} = 1$. However, when we think of h acting on $w \in \mathbb{R}^2$, where $w = (x,y)$, this action $hw = -w$.

What is this 2 dimensional representation of SU(2)? It is the $l = 1/2$ representation under which the spin of the electron transforms.

"What's the deal?" you ask. "You tell me taking $\theta = 2\pi$ sends the electron wave function to minus the electron wave function? How can that be physical? I should end up where I started." It's quantum mechanics. We square everything at the end of the day. Observables will not see this minus one.