Physics 403, Spring 2011
Problem Set 9

due Thursday, April 28

1. **Campbell-Baker-Hausdorff** [10 pts]: Let $X, Y \in \mathfrak{g}$ be elements of a Lie algebra. In class, we claimed that
$$e^X e^Y = e^{X+Y+\dots}$$
where the ellipsis in the above equation denotes terms that only involve commutators of X and Y and hence are also elements of $\mathfrak{g}$. Verify this claim to third order in X and Y, i.e. up to commutators of the form $[X, [X, Y]]$ and $[Y, [Y, X]]$.

2. **Fun with** $SU(2)$ [15 pts]:

   (a) Let $(z_0, z_1) \in \mathbb{C}^2$. Show that the operators
$$\begin{aligned} J_x &= \frac{1}{2}\left(z_0 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_0}\right) \\ J_y &= \frac{1}{2i}\left(z_0 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_0}\right) \\ J_z &= \frac{1}{2}\left(z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1}\right) \end{aligned}$$
   satisfy the right commutation relations to generate the Lie algebra for $\mathfrak{so}_3$.

   (b) Show that monomials of the form $z_0^\alpha z_1^\beta$ can be used to form finite dimensional irreps of this Lie algebra. What are the bounds on α and β for a given irrep? What is the relation between α and β and eigenvalues of J_z and $J^2 = J_x^2 + J_y^2 + J_z^2$?

3. **Coherent States of** $SU(2)$ [20 pts]: Consider the so-called "coherent state"
$$|\vec{z}, j\rangle = \sum_{m=-j}^{j} \frac{\sqrt{2j!}}{\sqrt{(j+m)!(j-m)!}} z_1^{j+m} z_2^{j-m} |j; m\rangle$$
where $|j; m\rangle$ is the usual basis of orthonormal angular momentum states from quantum mechanics and $\vec{z} = (z_1, z_2) \in \mathbb{C}^2$ is a vector that satisfies the constraint $|z_1|^2 + |z_2|^2 = 1$.

   (a) Verify that $|\vec{z}, j\rangle$ is properly normalized and compute the expectation values $\langle J_x\rangle$, $\langle J_y\rangle$, and $\langle J_z\rangle$ in the state $|\vec{z}, j\rangle$.

   (b) Compute the dispersion $\Delta J_z = \sqrt{\langle J_z\rangle^2 - \langle J_z^2\rangle}$.

   (c) Use angles on S^2 to parametrize the ratio $z_2/z_1 = \tan(\theta/2)e^{i\phi}$. What is $\langle \vec{J}\rangle$ in terms of θ and ϕ. Use rotational symmetry to argue what ΔJ_x and ΔJ_y must be. Calculate $(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2$. Physically, how do you interpret these results?