Physics 403, Spring 2011
Problem Set 8

due Thursday, April 21

1. **Subrepresentations** [10 pts]: In class, we showed that for an inner product space V and a unitary representation $\rho : G \to GL(V)$, if a subspace $W \subset V$ was stable under ρ, then the orthogonal complement $W^\perp$ was also stable under ρ. Thus ρ decomposed into two subrepresentations given by the restrictions of ρ to W and $W^\perp$ respectively. Here, we would like to remove the assumption that V be an inner product space and that ρ be unitary.

   (a) Let $p : V \to W$ be a linear operator such that $p(x) = x$ if $x \in W$. This operator is called a projection operator. We can define a p dependent orthogonal complement via $W_p^\perp = \ker(p)$. Construct
   $$\tilde{p} = \frac{1}{|G|} \sum_{g \in G} \rho(g) \cdot p \cdot \rho(g)^{-1} \ .$$
   Show that $\tilde{p} : V \to W$ is also a projection operator.

   (b) Show that $\rho(g) \cdot \tilde{p} = \tilde{p} \cdot \rho(g)\ \forall g \in G$.

   (c) Let $W^\perp = \ker(\tilde{p})$. Show that $W^\perp$ is stable with respect to the action of G.

2. **Tensor Products of Character** [10 pts]: Let $\rho : G \to GL(V)$ be a representation of the group G on the vector space V, and let χ be the corresponding character. Let $g \in G$ be an arbitrary group element.

   (a) Show that the characters χ_α of $\mathrm{Alt}^2(V)$ and χ_σ of $\mathrm{Sym}^2(V)$ can be written in terms of χ in the following way
   $$\begin{aligned} \chi_\alpha(g) &= \frac{1}{2}[\chi(g)^2 - \chi(g^2)] \ , \\ \chi_\sigma(g) &= \frac{1}{2}[\chi(g)^2 + \chi(g^2)] \ . \end{aligned}$$

   (b) Let $\rho' : G \to GL(V')$ be another representation and χ' the corresponding character. Find the characters of $\mathrm{Alt}^2(V \oplus V')$ and $\mathrm{Sym}^2(V \oplus V')$ in terms of χ and χ'.

3. **Tensor Product and S_3** [10 pts]: Let V be the irreducible representation of the permutation group of three elements, S_3, of dimension two. Use the character table of S_3 to decompose $V^{\otimes n}$ into irreducible representations.

4. **Character Tables** [20 pts]: Work out the character table for S_5.

5. **Ammonia** [20 pts]: The molecule ammonia NH_3 is symmetric with respect to the action of a finite group G that acts by rotations and reflections.

   (a) What is this group G? Write down the character table for the group.

(b) Consider the 12 dimensional (reducible) representation ρ obtained by letting G act on the coordinates of the hydrogen and nitrogen atoms. What is the character ϕ of this representation?

(c) Decompose ϕ into characters of the irreps in your character table.

(d) Remove the characters corresponding to translations and rotations to see which irreps correspond to vibrational modes.