Physics 403, Spring 2011
Problem Set 7

due Thursday, April 7

1. **Uniqueness of the identity and inverse** [5 pts]: Let G be a group. Show that the identity element $e \in G$ and the inverse g^{-1} of an element $g \in G$ are unique.

2. **Conjugation and the group of automorphisms** [15 pts]: By an automorphism, one means an isomorphism of a group onto itself. Let $\mathrm{Aut}(G)$ be the set of automorphisms of a group G. Let a be an element of G. Let
$$\gamma_a : G \to G$$
be the map such that $\gamma_a(x) = axa^{-1}$.

   (a) Prove that $\mathrm{Aut}(G)$ is a subgroup of the group of permutations of G, denoted $\mathrm{Perm}(G)$ or $S_{|G|}$.

   (b) Show that $\gamma_a : G \to G$ is an automorphism of G.

   (c) Show that the set of all such maps γ_a for $a \in G$ is a subgroup of $\mathrm{Aut}(G)$. This set of maps is called the group of inner automorphisms.

3. **Fractional Linear Transformations** [15 pts]: Let $M = \mathbb{R} \cup \{\infty\}$, and define an action of $SL(2, \mathbb{R})$ on M by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax + c}{bx + d} \ .$$
Show that this is indeed a group action with a law of multipication identical to the matrix multiplication. Show that the action is transitive but not effective.

4. **A Normal Subgroup** [15 pts]: Let G be a finite group of order $2k$ for some positive integer k.

   (a) Prove that G has an element of period 2, i.e. show that there exists $x \in G$, $x \neq e$, such that $x = x^{-1}$.

   (b) Assume that k is odd. Let $a \in G$ have period 2 and let $T_a : G \to G$ be translation by a. Prove that T_a is an odd permutation.

   (c) Still assume that k is odd. Prove that G has a normal subgroup of order k.