Physics 403, Spring 2011
Problem Set 5

due Thursday, March 24

1. **A Quantum Paradox** [15 pts]: Consider a particle in an infinite potential well on the interval $x \in [0, a]$. Physical considerations impose the boundary conditions $\psi(0) = \psi(a) = 0$. Assume the Hamiltonian operator $H = -d^2/dx^2$ (setting $\hbar = 1$ and $m = 1$) is self-adjoint.

   (a) Solve the eigenvalue equation $H\psi = E\psi$ associated to the boundary conditions $\psi(0) = \psi(a) = 0$. Deduce the spectrum and an orthonormal basis of eigenfunctions.

   (b) Consider the wave function given by $\psi(x) = \mu x(a - x)$ where μ is a normalization factor so that $\langle\psi|\psi\rangle = 1$. Show that $\langle H\psi|H\psi\rangle > 0$. Isn't it the case however that $H^2\psi = 0$ and hence $\langle\psi|H^2\psi\rangle = 0$? What's going on?

   (c) Decompose H^2 using the eigenfunction basis computed in (a). Use this "matrix" representation of H^2 to compute $\langle\psi|H^2\psi\rangle$.

2. **The Inverse of the Kinetic Energy Operator** [15 pts]: Consider again the operator $H = -d^2/dx^2$ on the domain $D_H = \{f, f'' \in \mathcal{L}^2(0, 1) : f(0) = f(1) = 0\}$. We would like to construct $(H - \omega^2)^{-1}$.

   (a) Find an orthonormal basis of eigenvectors for H. Use this basis to construct $(H - \omega^2)^{-1}$.

   (b) Construct solutions f_1 and f_2 of the equation $(H - \omega^2)f(x) = 0$ such that $f_1(0) = 0$ and $f_2(1) = 0$. Use these homogeneous solutions to find a general solution $f(x)$ to the inhomogeneous equation $(H - \omega^2)f(x) = g(x)$ that satisfies the boundary conditions $f(0) = f(1) = 0$.

   (c) Show that the inverse operators constructed in parts (a) and (b) are the same.

3. **Critical Mass** [15 pts]: An infinite slab of fissile material has thickness L. The neutron density $n(x, t)$ in the material obeys the equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} + \lambda n + \mu \ ,$$

   where $n(x, t)$ is zero at the surface of the slab at $x = 0$ and L. Here D is the neutron diffusion constant, the term λn describes the creation of new neutrons by induced fission, and the constant μ is the rate of production per unit volume of neutrons by spontaneous fission.

   (a) Expand $n(x, t)$ as a series

$$n(x, t) = \sum_m a_m(t)\varphi_m(x) \ ,$$

   where $\varphi_m(x)$ are a complete set of functions you think suitable for solving the problem.

(b) Find an explicit expression for the coefficients $a_m(t)$ in terms of their initial values $a_m(0)$.

(c) Determine the critical thickness L_{crit} above which the slab will explode.