Physics 403, Spring 2011
Problem Set 4

due Thursday, March 3

1. **Everything is hypergeometric (20 pts)**: Express the functions

$$(1+x)^c \ , \quad \frac{1}{x}\sin^{-1} x \ , \quad \frac{1}{x}\ln(1+x) \ ,$$

in terms of hypergeometric functions.

2. **Or confluent hypergeometric (20 pts)**: The confluent hypergeometric function $\Phi(\alpha,\gamma;z)$ satisfies the differential equation

$$y''(z) + \left(\frac{\gamma}{z} - 1\right) y'(z) - \frac{\alpha}{z} y(z) = 0 \ .$$

Many familiar functions can be written in terms of confluent hypergeometric functions. For example, Hassani describes how to write the Bessel functions in terms of $\Phi(\alpha,\gamma;z)$ on pp 423-4.

(a) The associated Laguerre polynomials $L_n^\nu(x)$ satisfy the differential equation

$$xy''(x) + (\nu+1-x)y'(x) + ny(x) = 0 \ .$$

Put this equation in confluent hypergeometric form and relate $\Phi(\alpha,\gamma;z)$ to $L_n^\nu(x)$.

(b) The Hermite polynomials $H_n(x)$ satisfy the differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0 \ .$$

Put this equation in confluent hypergeometric form and relate the solutions of the confluent hypergeometric equation to $H_n(x)$.

(c) Express the error function,

$$\text{erf}(x) = \int_0^x e^{-t^2} \ ,$$

in terms of a confluent hypergeometric function. In particular, verify that the error function can be expressed as $\text{erf}(x) = x\Phi(\alpha,\gamma;-x^2)$ and find α and γ.

3. **The Sum of the critical exponents (10 pts)**: Let

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0$$

be a Fuchsian second order differential equation with n regular singular points. Let $\lambda_{\pm,j}$ be the critical exponents of the j'th regular singular point. Evaluate $\sum_{j=1}^n(\lambda_{+,j} + \lambda_{-,j})$. [Hint: If you have no idea where to begin, see if you can figure out the answer for $n = 1$, 2, 3, and 4. Do you see a pattern?]