Physics 403, Spring 2011
Problem Set 3

due Thursday, February 24

1. **A Recursion Relation for the Heun Equation (20 pts)**: In class we discussed the Heun equation, a second order linear ordinary differential equation with four regular singular points
$$y''(z) + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-z_0}\right] y'(z) + \frac{\alpha\beta z - Q}{z(z-1)(z-z_0)} y(z) = 0 \ .$$
The parameters satisfy the constraint $\alpha + \beta + 1 = \gamma + \delta + \epsilon$.

   (a) Where are the regular singular points? What critical exponents are associated with each singular point?

   (b) A solution near $z = 0$ has the power series representation
$$y(z) = \sum_{n=0}^{\infty} c_n z^n \ .$$
   Give a three term recursion relation for c_{n+2}, c_{n+1}, and c_n.

2. **A Black Hole Wave Equation at Small Momentum (10 pts)**: In class we considered the wave equation
$$u^3 \partial_u \left[u^{-3} f(u) \partial_u \psi(u)\right] + \frac{\omega^2}{f(u)} \psi(u) - q^2 \psi(u) = 0 \ ,$$
where $f(u) = 1 - u^4$ with "in-going" boundary conditions at $u = 1$: $\psi \sim (1-u)^{-i\omega/4}$. Find a solution to this equation for $\omega = 0$ and small q of the form
$$\psi(u) = \psi_0(u) + q^2 \psi_1(u) + O(q^4) \ .$$

3. **Exactly Solvable Variant of a Black Hole Wave Equation (30 pts)**: Consider the following variant of the wave equation we studied in class
$$u \partial_u \left[u^{-1} f(u) \partial_u \psi(u)\right] + \frac{\omega^2}{f(u)} \psi(u) - q^2 \psi(u) = 0 \ ,$$
where again $f(u) = 1 - u^4$. While the equation we considered in class is obeyed by a massless spin zero particle, the above equation is satisfied by a massless spin one particle in a black hole space time. Set the momentum $q = 0$ in what follows.

   (a) What are the singular points of this differential equation? What are their corresponding critical exponents?

   (b) Define a new coordinate $z = \frac{1}{2}(1 - 1/u^2)$. Write the differential equation in terms of z. Where are the singular points now? Introduce a rescaled wave function $y(z)$, where $\psi(z) = g(z) y(z)$ and $g(z)$ is a simple function that depends on the critical points and their scaling exponents, such that the differential equation for $y(z)$ takes a canonical form.

(c) If we impose "in-going" boundary conditions at $u = 1$, for what values of ω does the solution $\psi(u)$ satisfy Dirichlet boundary conditions at $u = 0$, i.e. $\psi(0) = 0$. These values of ω are often called quasinormal modes. How come they are complex?