Physics 403, Spring 2011
Problem Set 2

due Thursday, February 17

1. **A Differential Equation for the Legendre Polynomials (15 pts)**: In this problem, we will derive a second order linear differential equation satisfied by the Legendre polynomials
$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x^2)^n] \ .$$

   (a) Show that
$$\int_{-1}^{1} P_m(x)[(1-x^2)P_n'(x)]'\, dx = 0 \qquad \text{for } m \neq n \ . \tag{1}$$
   Conclude that
$$[(1-x^2)P_n'(x)]' = \lambda_n P_n(x)$$
   for λ_n a constant. [Prime $'$ here and in the following questions denotes d/dx.]

   (b) Perform the integral (1) for $m = n$ to find a relation for λ_n in terms of the leading coefficient of the Legendre polynomial $P_n(x) = k_n x^n + \ldots$. What is k_n?

2. **A Generating Function for the Legendre Polynomials (20 pts)**: In this problem, we will derive a generating function for the Legendre polynomials.

   (a) Verify that the Legendre polynomials satisfy the recurrence relation
$$P_n' - 2xP_{n-1}' + P_{n-2}' - P_{n-1} = 0 \ .$$

   (b) Consider the generating function
$$g(x,t) = \sum_{n=0}^{\infty} t^n P_n(x) \ .$$
   Use the recurrence relation above to show that the generating function satisfies the first order differential equation
$$(1-2xt+t^2)\frac{d}{dx}g(x,t) = tg(x,t) \ .$$
   Determine the constant of integration by using the fact that $P_n(1) = 1$.

   (c) Use the generating function to express the potential for a point particle
$$\frac{q}{|\mathbf{r}-\mathbf{r}'|}$$
   in terms of Legendre polynomials.

3. **The Dirac delta function (15 pts)**: Dirac delta functions are useful mathematical tools in quantum mechanics. (Technically, they are not functions but distributions.) The Dirac delta function $\delta(x)$ has the property that

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0) \; .$$

   (a) Show that $\delta(ax) = \delta(x)/|a|$. [You may assume $a \neq 0$.]

   (b) Let x_i, $i = 1, \ldots, n$, be the zeroes of a function $f(x)$, i.e. $f(x_i) = 0$. Let $f'(x) = df/dx$ be the derivative of $f(x)$. Show that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|} \; .$$

   [You may assume $f'(x_i) \neq 0$.]

4. **A Convolution (10 pts)**: Consider two functions $f(x)$ and $g(x)$ defined on the interval $0 \leq x < 2\pi$. These functions have the Fourier series representations

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx} \; ; \quad g(x) = \sum_{n=-\infty}^{\infty} g_n e^{inx} \; .$$

   Consider the convolution

$$h(x) = \int_{0}^{2\pi} g(x - y) f(y)\, dy \; .$$

   Express the Fourier series for $h(x)$ in terms of f_n and g_n.

5. **Two infinite sums (10 pts)**: Find the Fourier series expansion of $f(\theta) = \theta^2$ for $|\theta| < \pi$. Use the expansion to sum the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \; .$$