Physics 305, Fall 2008
Problem Set 6

due Thursday, November 12

1. **A charged particle in a magnetic field (30 points):** Consider an electron of charge $-e$ and mass m in a constant magnetic field $\mathbf{B} = B\hat{z}$. For simplicity, we will assume that the electron is constrained to move in the xy-plane, and we will ignore its spin.

   a. Using a separation of variables ansatz,

   $$\Psi(x, y) = \chi(x)\psi(y) \ ,$$

   find the eigen-energies for the electron in a gauge where $\mathbf{A} = By\hat{x}$.

   b. Now consider the case where the electron is moving inside a square with periodic boundary conditions and area $A = L^2$. Assume that

   $$\frac{AeB}{h} \gg 1 \ .$$

   How many quantum states are there with a given energy? What do you think is the reason for this assumption about the size of A?

   c. Increase the number of electrons inside the rectangle to N. Ignore electron-electron interaction. Describe qualitatively what happens to the Fermi sea as we increase the magnetic field from zero.

2. **Spin precession and "spin echo" (40 points):** The following problem is a warm-up for the time-dependent perturbation theory we will consider after fall break. The Hamiltonian of a $s = 1/2$ spin in a time-dependent magnetic field $\vec{B}(t)$ is

   $$H(t) = \mu\vec{B}(t) \cdot \vec{\sigma} \ ,$$

   where $\mu > 0$ is the spin's magnetic moment and $\vec{\sigma}$ are the Pauli matrices for the spin. Let the spin be initially oriented along the positive z-axis at time $t = 0$. We will subject it to a certain sequence of fields that are a caricature of some things that can be done in MRI imaging.

   a. The $\pi/2$ pulse: First apply a field of magnitude B_x along the positive x-axis from time $t = 0$ to $t = t_1$. What must the product $B_x t_1$ be in order for this application to cause a 90 degree rotation of the spin?

   b. What is the resulting orientation of the spin at time t_1?

   c. Next apply a field of magnitude B_z along the positive z-axis, and leave it on for a longer time, so the spin may rotate many times. What is the resulting time dependence of $\mu\langle\vec{\sigma}\rangle$, the expectation value of the spin's magnetic moment at time t (it is a vector)? For a proton spin in a 1 Tesla field, what is the precession frequency?

In solids (e.g. bone, etc.), variation in the material's magnetic properties cause the field $\vec{B}$ to vary slightly over the positions of the various nuclear spins. Variation in the magnetic field in turn causes the spins' precession rates to differ, and the rotating magnetic moments quickly get out of phase, eliminating the coherent macroscopic precessing magnetization that would otherwise be present. The following "spin echo" technique can bring back the coherence:

After B_z is on from time t_1 to $t_1 + t_2$, turn it off and apply B_x as in part (a), but now for time $2t_1$, so the spins rotate by 180 degrees about the x-axis (a "π-pulse"). Then apply B_z again for the same time duration t_2. Note t_2 is long, so each spin turns many times about the z-axis during the applications of B_z, and the product $B_z t_2$ is not chosen to give any particular number of rotations of the spin. However, we assume the two applications of field along the z-axis are identical in magnitude and duration.

d. At the end of these 4 applications of the field (B_x for t_1, B_z for t_2, B_x for $2t_1$, and then B_z for t_2 again), what is the orientation of the spin? Show that it does not depend on B_z, so all the spins will be aligned at this time, even though B_z varies over the material. This reappearance of a coherent rotating magnetization is called the "spin echo".

Have a happy fall break!