

Physics 305, Fall 2008  
Problem Set 3

due Thursday, October 8

1. **Degenerate Stars (40 pts):** If a star is in a certain mass and age range, it reaches a stage where the fusion of its nuclei nearly stops and it is supported against gravitational collapse by the zero-point kinetic energy of its electrons. If it has a high enough density, it will be fully ionized with all of its electrons moving freely, not bound to nuclei. Model such a star as being electrically neutral, with equal numbers of electrons, protons and neutrons (the nuclei present are mostly helium-4, carbon-12 and oxygen-16). Assume it is a sphere of uniform density and total mass  $M$  and radius  $R$ , and, for now, that the velocities of the electrons are well below  $c$ .
  - a. Neglecting the Coulomb interactions, calculate the total energy of the ground state of this model white dwarf star. This energy consists of the gravitational potential energy and the kinetic energy of the electrons. Approximate the electrons as a noninteracting and nonrelativistic Fermi gas. [Note: the kinetic energy of a Fermi gas of many particles is set by its density and does not depend on its shape, so you do not have to solve for the states of a particle in a spherical box here.] Why is the kinetic energy of the nuclei small compared to that of the electrons?
  - b. Determine the radius  $R$  that minimizes this total energy. What is the Fermi energy of the electrons?

Estimate the range of validity of the approximations you have made: We have assumed the Fermi energy is high enough that all the electrons are unbound from the nuclei (and the Coulomb interactions can then be neglected). The binding energy of the inner electrons to a carbon nucleus ( $Z=6$ ) is roughly 500 eV.

- c. For what mass range (give it in multiples of the Sun's mass, which is  $2 \times 10^{30}$  kg) is the Fermi energy of this cold star more than 500 eV? If  $M$  is such that the Fermi energy is well below 500 eV and many of the electrons are strongly bound to the nuclei, is a star of this  $M$  larger or smaller than your estimate of its radius from (b)?

In (a) and (b) you also assumed the electrons are nonrelativistic. This stops being true when the Fermi energy approaches the rest energy of the electron,  $m_e c^2$ .

- d. For what mass range (again, in multiples of the Sun's mass) is the Fermi energy of this cold star that you obtained in part (b) less than  $m_e c^2$ ? If  $M$  is such that the Fermi energy is near or above  $m_e c^2$ , so that many of the electrons are moving relativistically, is a star of this  $M$  larger or smaller than your estimate of the radius from (b)?

- e. If the electrons are moving highly relativistically, then their rest mass is a small part of their energy, so we can take the approximation of ignoring it. Repeat parts (a) and (b), approximating the electrons as a *massless* Fermi gas, and see if there is a range of  $M$  that gives a stable cold star with highly relativistic electrons.

[Most of problem (1) could be repeated for neutron stars, which consist of just neutrons that can also be approximated as a cold Fermi gas. If a star exceeds the mass that allows stability of its ground state, then once it cools sufficiently it will collapse gravitationally and become a black hole.]

2. **Trapped Atoms and BEC (30 points):** Cornell, Ketterle, and Wiemann shared the 2001 Nobel prize for producing Bose-Einstein condensates of  $^{23}\text{Na}$  (Ketterle) and  $^{87}\text{Rb}$  (Cornell and Wiemann) atoms. Treating the  $^{87}\text{Rb}$  and  $^{23}\text{Na}$  atoms as non-interacting, we will attempt to calculate the temperature at which this BEC occurs. We model the behavior of a dilute gas of  $N$  atoms in a spherical (three dimensional) harmonic trap with frequency  $\omega$  as a function of temperature  $T$ . At low temperatures, a macroscopic number of particles come to occupy the ground state. Work in the grand canonical ensemble. Assume that the atoms are all in the same spin state and that you have either  $^{87}\text{Rb}$  or  $^{23}\text{Na}$  in your trap.

- a. Explain why  $^{87}\text{Rb}$  and  $^{23}\text{Na}$  are bosons.
- b. What is the allowed range of chemical potential  $\mu$ ? Assuming there are a large number of atoms in the ground state of the trap, solve for  $\mu$  to leading order in  $1/N_0$ .
- c. Show that the number of particles in the trap can be written

$$N_0 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (n+2)(n+1)x^m e^{-\hbar\omega mn/\tau}$$

where we have defined a modified fugacity  $x = \exp((\mu - 3\hbar\omega/2)/\tau)$  and  $\tau = k_B T$ .

- d. In the limit  $\hbar\omega \ll k_B T$ , the sum over  $n$  may be approximated by an integral:

$$\sum_{n=1}^{\infty} (n+2)(n+1)e^{-\hbar\omega mn/\tau} \approx \int_0^{\infty} n^2 e^{-\hbar\omega mn/\tau} dn .$$

Defining  $N_e = N - N_0$  to be the number of atoms in excited states, use this approximation and the result from part (b) to calculate  $N_e$  near the temperature  $T_c$  at which the system Bose condenses. Extract  $T_c$  and deduce a formula for  $N_0/N$  as a function of temperature.

- e. Using the values  $N = 10^6$  and  $\omega = 200$  Hz, calculate the critical temperature  $T_c$  (in Kelvin) at which BEC occurs. (Hint:  $\sum_{n=1}^{\infty} n^{-3} \approx 1.20206$ ).