Physics 305, Fall 2008
Problem Set 1

due Thursday, September 24

1. **Linear Algebra warm up (20 points):** Let $V = \mathbb{C}^n$ be a finite dimensional vector space over the complex numbers $\mathbb{C}$ of dimension n with the usual inner product. Let H be a Hermitian operator and U be a unitary operator defined over V.

   a. Demonstrate that the eigenvalues of H are real and that eigenvectors of H with distinct eigenvalues are orthogonal.

   b. Demonstrate that eigenvalues λ of U must have absolute value $|\lambda| = 1$ and that eigenvectors with distinct eigenvalues are orthogonal.

   c. **Extra Credit (5 points):** Demonstrate that the (properly normalized) eigenvectors of H provide an orthonormal basis of V. (Hint: First argue that H must have at least one eigenvector v. Consider the subspace $v_\perp$ orthogonal to v. Argue that H restricted to $v_\perp$ is Hermitian.)

2. **The Dirac delta function (10 points)**: Dirac delta functions are useful mathematical tools in quantum mechanics. (Technically, they are not functions but distributions.) The Dirac delta function $\delta(x)$ has the property that

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0) \ .$$

   a. Show that $\delta(ax) = \delta(x)/|a|$. [You may assume $a \neq 0$.]

   b. **Extra Credit (5 points):** Let x_i, $i = 1, \ldots, n$, be the zeroes of a function $f(x)$, i.e. $f(x_i) = 0$. Let $f'(x) = df/dx$ be the derivative of $f(x)$. Show that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|} \ .$$

   [You may assume $f'(x_i) \neq 0$.]

3. **"Supersymmetric" quantum mechanics (50 points):** Let the operators A and $A^\dagger$ be Hermitian conjugates of each other. Define the Hermitian operators

$$H_+ = \hbar A A^\dagger \qquad \text{and} \qquad H_- = \hbar A^\dagger A \ .$$

   Assume that the eigenvalues of $H_\pm$ are all distinct.

   a. Show that the eigenvalues of $H_\pm$ are non-negative.

   b. Given an eigenvector $|\psi_+\rangle$ of H_+ with eigenvalue $E \neq 0$, construct an eigenvector $|\psi_-\rangle$ of H_- with the same eigenvalue E.

Consider a Hamiltonian H for a one dimensional system corresponding to a particle of mass m placed in an attractive potential $V(x)$ with minimum at $x = 0$ ($V(x) \leq 0$ and $V(x)$ tends to zero as $|x| \to \infty$):

$$H = \frac{p^2}{2m} + V(x) \; .$$

We would like to express this Hamiltonian in the form

$$H = \hbar A A^\dagger + \alpha$$

for a real constant α where A and $A^\dagger$ are defined by

$$\begin{aligned} A &= \frac{i}{\sqrt{2m\hbar}} p + \sqrt{\frac{m}{2\hbar}} W(x) = \sqrt{\frac{\hbar}{2m}} \frac{d}{dx} + \sqrt{\frac{m}{2\hbar}} W(x) \\ A^\dagger &= \frac{-i}{\sqrt{2m\hbar}} p + \sqrt{\frac{m}{2\hbar}} W(x) = -\sqrt{\frac{\hbar}{2m}} \frac{d}{dx} + \sqrt{\frac{m}{2\hbar}} W(x) \; . \end{aligned}$$

$W(x)$ is called the superpotential.

c. Calculate $H_+ = \hbar A A^\dagger$ and $H_- = \hbar A^\dagger A$ as a function of $\frac{d^2}{dx^2}$, $W(x)$, and its derivative $W'(x)$.

d. Determine the relation between $W(x)$, $W'(x)$, $V(x)$ and α such that H can be written in the factorized form $H = \hbar A A^\dagger + \alpha$. The Hamiltonian $H_S = \hbar A^\dagger A + \alpha$ is called the supersymmetric partner of H.

e. Consider the states $|\psi\rangle$ and $|\tilde{\psi}\rangle$ with the corresponding properties $A^\dagger|\psi\rangle = 0$ and $A|\tilde{\psi}\rangle = 0$. Express $\psi(x)$ and $\tilde{\psi}(x)$ in terms of $W(x)$. Show that only one of these states can be normalizable.

f. Assume $\tilde{\psi}(x)$ is normalizable. Show that $\tilde{\psi}(x)$ is the ground state wave function of H_S with energy α.

The machinery can be used to diagonalize a Hamiltonian with the potential

$$V_\mu(x) = -\frac{\hbar^2 \kappa^2}{2m} \frac{\mu(\mu+1)}{\cosh^2 \kappa x} \; .$$

This problem is already quite long, and in the following, we will content ourselves with studying only the $\mu = 0$ and 1 cases.

g. Consider the free Hamiltonian $H_0 = p^2/2m$. Show that the choice $A_0 = ip/\sqrt{2m\hbar}$ factorizes H_0.

h. Determine the superpotential $W_1(x)$ that leads to the following factorization

$$H_0 = \hbar A_1 A_1^\dagger - \frac{\hbar^2 \kappa^2}{2m} \; .$$

i. Show that the Hamiltonian H_1 with potential

$$V_1(x) = -\frac{\hbar^2\kappa^2}{m}\frac{1}{\cosh^2\kappa x} \ .$$

can be obtained as the supersymmetric partner of H_0:

$$H_1 = \hbar A_1^\dagger A_1 - \frac{\hbar^2\kappa^2}{2m} \ .$$

j. Find the ground state wave function of H_1 and its corresponding energy.

k. Use your knowledge of the eigenstates of H_0 and part (b) to calculate the scattering states of H_1. What are the transmission and reflection coefficients of a plane wave scattering off of $V_1(x)$? What is the phase shift of the transmitted wave?