Physics 305 Midterm Exam

11 am – 12:20 pm, October 23, 2008

This exam consists of **5** short answer questions, **3** problems, and a formula sheet. When we begin, check to see that this copy of the exam is complete. Use the same exam booklet for all problems, continuing to another booklet if necessary. **Print** your name on **each** booklet as you start it.

You must answer all five short answer questions and two out of three problems.

On the cover of your first booklet, write and sign the Honor Code pledge:
"I pledge my honor that I have not violated the Honor Code during this examination"

At the end of the exam, indicate clearly on the cover of your first exam booklet how many booklets you used and which two of the problems you selected.

Some useful test-taking hints:

- You may not be able to complete every problem. Keep moving – do what you know first.
- Make your answer clear by circling it.
- Use symbols rather than numbers wherever possible and CHECK UNITS.
- Whenever possible, check whether an answer or intermediate result makes sense before moving on.
- There is a list of formulas on a page you can tear out attached to the exam. Use it as a reminder of details. Don't try to do problems by searching through the sheet!
- If you get stuck on an early part of a problem, check the later parts — some may be independent and doable.
- If you get stuck on an early part of a problem, and a later part depends on it, **clearly** define a symbol for the unknown answer and use it in later parts. Note: this is an act of desperation – we often give multiple parts to guide you through a problem.
- **Show your work!**

The exam will last one hour and twenty minutes (11 am – 12:20 pm)

USE OF CALCULATORS IS NOT PERMITTED!!

Good Luck!

Short answers [5 pts each]. (Answer all five. A couple of sentences for each should be enough.)

1. Give a statement of the spectral theorem for Hermitian operators.
2. Provide a statement of the spin-statistics theorem.
3. What is the exchange force?
4. What is an Airy's function?
5. The term symbol for the ground state of carbon is ${}^3P_0$. What are the total L, J and S for the electrons in carbon?

Choose two of the following three questions to answer. Make sure it is clear which of the questions you have chosen.

1. 1D rubber band helium [30 pts]. We consider a toy, one dimensional model of the helium atom where the Coulombic potential is replaced with a Hooke's law potential. We fix the position of the nucleus of the "atom" to be $x = 0$ and let the electrons have positions x_1 and x_2. You may remember the following Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) - \frac{\lambda}{4}m\omega^2(x_1 - x_2)^2$$

from problem set 2. You may not have realized that the system is exactly solvable. Assume $\lambda > 0$.

a) [10 pts] Apply the change of variables $u = (x_1 + x_2)/2$ and $v = (x_1 - x_2)$ to the Hamiltonian. What simple quantum mechanical system do you now recognize? What are the eigen-energies of the Hamiltonian? What constraint must be imposed on λ for the system to be well behaved?

b) [15 pts] Assume $\lambda < 3/4$. Taking into account the spin of the electrons, consider the three lowest eigen-energies, E_0, E_1 and E_2. What are the values of these energies? How many eigenstates correspond to each of these eigen-energies (i.e. what are the degeneracies of the eigenstates corresponding to E_0, E_1, and E_2)? Where does the condition $\lambda < 3/4$ come from?

c) [5 pts] In problem set 2, you were asked to treat this system using time independent perturbation theory in the limit $\lambda \ll 1$. Use your exact results above to figure out what first order perturbation theory would give for the energies E_0, E_1 and E_2.

2. 2D photon gas [30 pts]. Consider a gas of **two** dimensional photons in a square box of area A and temperature T. Express your answers to the following questions in terms of A, T, k_B, $\hbar$, c, and $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

a) [10 pts] What is the density of states $\mathcal{D}(E)$ for the gas of photons, i.e. how many states are there with energy between E and $E + dE$?

b) [10 pts] Consider the photons with frequency between ω and $\omega + d\omega$. How much energy do these photons carry? For what value of ω is the energy a maximum? (For this last question, you will not find an explicit solution for ω. Try however to simplify the condition on ω as much as possible.) Does the maximum value occur for $\hbar\omega > k_B T$ or $\hbar\omega < k_B T$?

c) [10 pts] What is the total amount of energy in the gas of photons?

3. The variational principle and WKB [30 pts]. Here follow two completely separate questions. If you choose to solve this problem, you must complete both parts for full credit.

a) [20 pts] Consider the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \alpha x^4 \ .$$

Use a Gaussian trial wave-function $\psi(x) = Ae^{-x^2/b^2}$ and the variational principle to estimate the ground state energy for this Hamiltonian.

b) [10 pts] We model a metal as a gas of free electrons in a box. The potential for the electrons vanishes inside the box and has height V_0 at the edges of the box. The electrons have Fermi energy $E_F < V_0$ and zero temperature. The difference $W = V_0 - E_F$ is the energy required to remove an electron from the metal and is often called the work function. Now apply an electric field to the metal with strength $\mathcal{E}$. Use WKB to estimate the tunneling probability for the highest energy electrons to escape from the surface of the metal. (The result is called the Fowler-Nordheim formula.) (See Figure 1.)

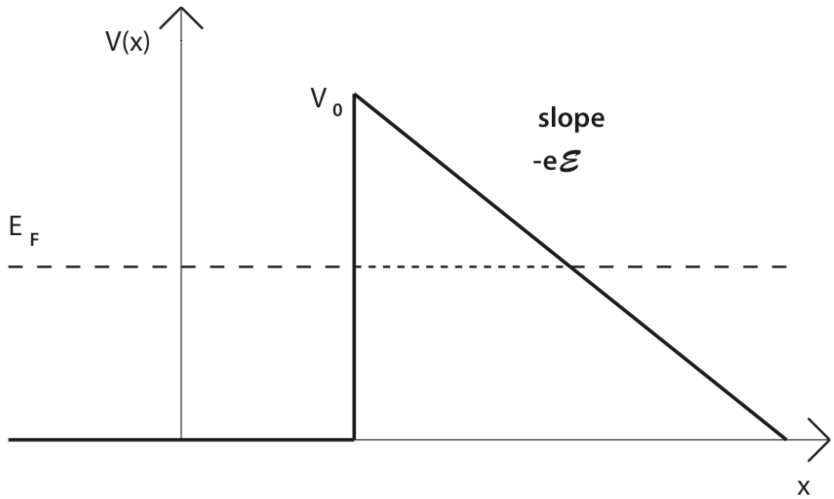

Figure 1: A picture for problem 3b.

Formula Sheet

Time independent perturbation theory: Given

$$H = H_0 + H_1 \qquad ; \qquad \psi_n = \psi_n^0 + \psi_n^1 + \dots \qquad ; \qquad E_n = E_n^0 + E_n^1 + E_n^2 + \dots$$

one has

$$E_n^1 = \langle \psi_n^0 | H_1 | \psi_n^0 \rangle \qquad ; \qquad |\psi_n^1\rangle = \sum_{m \neq n} \frac{\langle \psi_m^0 | H_1 | \psi_n^0 \rangle}{E_n^0 - E_m^0} \qquad ; \qquad E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H_1 | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

Central force potential: Given $\psi(r, \theta, \phi) = u(r) Y_m^l(\theta, \phi)/r$, one has

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} u(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} u(r) + V(r) u(r) = E u(r) \ .$$

Hydrogen atom:

- Bohr radius: $a = 4\pi\epsilon_0 \hbar^2 / me^2 = 0.529 \times 10^{-10}$ m.
- Fine structure constant: $\alpha = e^2 / 4\pi\epsilon_0 \hbar c \approx 1/137$.
- Bohr energies:

$$E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} = \frac{E_1}{n^2}$$

  where $E_1 = -13.6$ eV.
- Ground state wavefunction:

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

Fermion in a magnetic field:

$$H = -\vec{\mu} \cdot \vec{B}$$

where $\vec{\mu} = ge\vec{S}/2m$. For electrons, $g \approx 2$.

Fermi-Dirac and Bose-Einstein distributions:

$$f_{FD}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1} \qquad ; \qquad f_{BE}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1} \; .$$

WKB:

- Connection Formula (for a sloping wall):

$$\frac{C}{2\sqrt{|p|}} \exp\left[-\frac{1}{\hbar}\left|\int_a^x p\,dx\right|\right] \to \frac{C}{\sqrt{p}} \cos\left[\frac{1}{\hbar}\left|\int_a^x p\,dx\right| - \frac{\pi}{4}\right]$$

- Bohr-Sommerfeld quantization condition (for sloping walls):

$$\oint p\,dx = h\left(n + \frac{1}{2}\right)$$

- Tunneling probability:

$$T = \exp\left[-\frac{2}{\hbar}\int_a^b |p|\,dx\right]$$

Mathematical formulae:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad ; \qquad \int_0^{\infty} x^n e^{-x} dx = n!$$

$$I(a) \equiv \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \qquad ; \qquad (-1)^n \frac{d^n I}{da^n} = \int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx$$

$$\pi = 3.14159\ldots \qquad ; \qquad e = 2.71828\ldots$$