Physics 305, Fall 2008
Problem Set 9

due Thursday, December 4

1. **Suppression of higher partial waves (30 pts):** Consider the scattering of a particle of mass m by a central potential $V(r)$ which is negligible for $r > d$.

   a. Show that
   $$\tan \delta_l(k) = \frac{k j_l'(kd) - \gamma_l(k) j_l(kd)}{k n_l'(kd) - \gamma_l(k) n_l(kd)} \ ,$$
   where $j_l'(x) = dj_l(x)/dx$, $n_l'(x) = dn_l(x)/dx$ and
   $$\gamma_l(k) = \left[\frac{dR_l^I(k,r)/dr}{R_l^I(k,r)}\right]_{r=d}$$
   is the value of the logarithmic derivative of the regular, interior solution $R_l^I(k,r)$, evaluated at $r = d$ of
   $$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2}V(r) + k^2\right] R_l(k,r) = 0 \ .$$

   b. Using the properties of the functions j_l and n_l, prove that in the small k limit
   $$\tan \delta_l(k) = \frac{(kd)^{2l+1}}{D_l} \frac{l - \hat{\gamma}_l d}{l + 1 + \hat{\gamma}_l d} + \dots \ ,$$
   where $D_l = (2l+1)!!(2l-1)!!$ for $l > 0$, $D_0 = 1$, and $\hat{\gamma} = \lim_{k\to 0} \gamma_l(k)$.

   c. Assuming first that $\hat{\gamma}_l d \neq -(l+1)$, prove that the partial wave amplitudes $f_l(k)$ exhibit the low-energy behavior
   $$f_l(k) \sim k^{2l} \qquad \text{where} \qquad f(k,\theta) = \sum_{l=0}^{\infty} f_l(k) P_l(\cos\theta) \ ,$$
   so that except for the s-wave ($l = 0$) contribution which in general tends to a non-zero constant, all partial cross sections σ_l ($l \geq 1$) vanish as k^{4l}. The scattering is therefore isotropic at very low energies. Prove also that the scattering amplitude f is given as $k \to 0$ by
   $$\lim_{k\to 0} f(k,\theta) = \lim_{k\to 0} \frac{\tan \delta_0(k)}{k} \ . \qquad (1)$$
   Typically, this quantity which has dimension of length is defined to be the negative of the scattering length, $-a$.

   d. We examine how the results obtained in part (c) must be modified when by accident
   $$\hat{\gamma}_l d = -(l+1) \ . \qquad (2)$$
   Show that for $l \geq 2$, the result (1) is unchanged. If (2) holds for $l = 0$, show that for small k
   $$f(k,\theta) \approx \frac{i}{k} \ .$$

If (2) holds for $l = 1$, show that

$$\lim_{k \to 0} f(k, \theta) = -a + b \cos\theta$$

where b is a constant.

2. **Low Energy Atomic Scattering (20 pts):** In problem set 4, we modeled a gas of cold bosonic atoms in a harmonic trap using the following mean-field "Hamiltonian":

$$H = N \left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 + \frac{(N-1)g}{2} |\psi|^2 \right) .$$

Here N is the number of atoms in the trap. We would like to derive H starting from the actual Hamiltonian H_{full} describing N identical bosons of mass m in an external spherical harmonic potential of frequency ω and interacting via a central force potential of the form $V(|\vec{r}_i - \vec{r}_j|)$.

   a. What is the actual Hamiltonian H_{full} describing this N-body system of atoms? Use the variational principle and the assumption that the N-particle wavefunction can be written as a product of single particle wavefunctions:

   $$\Psi(\{\vec{r}_n\}) = \prod_{n=1}^{N} \psi(\vec{r}_n) ,$$

   to derive the mean-field result H. What is g in terms of $V(r)$? (You may assume that the range of the potential is very small compared to the scales over which the probability density of the particles $|\psi(\vec{r})|^2$ varies.)

   b. Use the low energy Born approximation to relate g to the scattering length a.

3. **Neutrons scattering off a 1D crystal (25 pts):** Consider point-like s-wave scatterers with scattering length a. The "target" consists of ten such stationary scatterers (the nuclei), equally spaced along the z-axis, with spacing $b \gg a$. The neutron is incident along the z-axis.

   a. What is the differential cross-section as a function of scattering angle?

   b. What is the maximum value of the differential cross-section, and at what angle(s) does this maximum value occur?

   c. At what angles does the scattering vanish?

   d. For $b = 3$ Angstroms, what is the limit on the incident energy of the neutron (in eV) in order to be able to see a scattering maximum at nonzero angle?

4. **Elastic scattering (Extra Credit 40 pts):** Consider a beam of nonpolarized light consisting of photons of initial momentum $\hbar\vec{k}$. The photons elastically scatter off hydrogen in the ground state. Show that in the second order Born approximation and in the limit where the photons have a very long wavelength, the differential cross section has the angular dependence $1 + \cos^2\theta$ and that the total cross-section tends to zero as $k \to 0$.