Physics 305, Fall 2008
Problem Set 10

due Thursday, December 11

1. **Scattering off a Gaussian potential (25 pts):** Consider the scattering off the central force potential given by $V(r) = V_0 \exp(-r^2/\alpha^2)$. The incident particle has mass m and kinetic energy $E = \hbar^2 k^2/2m$.

   a. Using the Born approximation, calculate the scattering amplitude $f(\theta)$ and differential cross-section $d\sigma/d\Omega$. Sketch or make a plot of the differential cross-section as a function of the scattering angle for $k\alpha = 1$.

   b. What is the total cross-section σ? Show that the total cross-section in the limit of high incident energy scales as $1/E$. Calculate also the low energy limit of the total cross-section.

   c. Explain how the Born approximation is justified in the regime where $m\alpha^2 V_0 \ll \hbar^2$. Show that in this regime, $\sigma \ll \alpha^2$.

2. **A density matrix, maybe (20 pts):** Suppose we have a system with total angular momentum 1. Pick a basis corresponding to the three eigenvectors of the z-component of angular momentum, J_z, with eigenvalues $+1, 0 -1$, respectively. We are given an ensemble described by the density matrix:

$$\rho = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

   a. Is ρ a permissible density matrix? Give your reasoning. For the remainder of this problem, assume that it is permissible. Does it describe a pure or mixed state? Give your reasoning.

   b. Given the ensemble described by ρ, what is the average value of J_z?

   c. What is the spread (standard deviation) in measured values of J_z?

3. **Beam me up, Bob (25 pts):** Charlie has bought a machine on the flea market that makes EPR (Einstein-Podolsky-Rosen) pairs. The state of such a pair reads

$$|\Phi^{(-)}\rangle = \frac{1}{\sqrt{2}} \left(|\updownarrow\rangle_1 |\leftrightarrow\rangle_2 - |\leftrightarrow\rangle_1 |\updownarrow\rangle_2 \right) .$$

   Charlie sends particle 1 through a polarization filter that only transmits vertically polarized photons.

   a. What is the probability that the particle passes the filter? After the measurement on photon 1, he sends photon 2 through the same filter. Given that photon 1 passed, what is the probability that photon 2 will *not* pass the filter?

Besides the state $|\Phi^{(-)}\rangle$, Charlies also makes the following two-photon states

$$
\begin{aligned}
|\Phi^{(+)}\rangle &= \frac{1}{\sqrt{2}}\left(|\updownarrow\rangle_1|\leftrightarrow\rangle_2 + |\leftrightarrow\rangle_1|\updownarrow\rangle_2\right) \ , \\
|\Psi^{(-)}\rangle &= \frac{1}{\sqrt{2}}\left(|\updownarrow\rangle_1|\updownarrow\rangle_2 - |\leftrightarrow\rangle_1|\leftrightarrow\rangle_2\right) , \\
|\Psi^{(+)}\rangle &= \frac{1}{\sqrt{2}}\left(|\updownarrow\rangle_1|\updownarrow\rangle_2 + |\leftrightarrow\rangle_1|\leftrightarrow\rangle_2\right) \ .
\end{aligned}
$$

Make sure you understand why these four states form a complete orthonormal basis for the two photon Hilbert space. Charlie takes $|\Phi^{(-)}\rangle$ and sends photon 1 to Alice and photon 2 to Bob. Besides the photon Alice received from Charlie, she also possesses a mystery photon M in an unknown quantum state, that she wants to teleport to Bob. We write its state as

$$|M\rangle = a|\updownarrow\rangle + b|\leftrightarrow\rangle.$$

Alice doesn't know a or b. The teleportation protocol works as follows.

b. Write the state of the EPR pair 1 and 2 together with the mystery photon M in terms of a basis $\{|\Phi^{(\pm)}_{M1}\rangle, |\Psi^{(\pm)}_{M1}\rangle\}$.

c. Alice performs a measurement on the particles 1 and M, in which she projects the state on one of the four basis elements $|\Phi^{(+)}_{M1}\rangle$, $|\Phi^{(-)}_{M1}\rangle$, $|\Psi^{(+)}_{M1}\rangle$, or $|\Psi^{(-)}_{M1}\rangle$. In each case, what is the state of the photon 2, the one in Bob's possession?

d. Depending on which of the 4 states Alice finds, she sends a short message of 2 ordinary classical bits to Bob. Using this information, Bob performs a simple unitary transformation on his photon 2, such that its state is identical to that of photon M. Which transformation does Bob have to perform in each case?

e. The end result is that state $|M\rangle$ has been successfully transferred from Alice to Bob. Has any information been transmitted faster than the speed of light? Why not? Do you agree with the terminology "quantum teleportation" for this protocol?

(For more info, see C. H. Bennett and S. J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).)