# Physics 105 Problem Set 8 Solutions

**Problem 8.1** (*10 Points*)

a) Dig a narrow tunnel straight through the center of the Earth. What is the force $F(x)$ as a function of distance $x$ from the center of the Earth? With gravity, a spherical shell produces no force on a mass inside of it. Hence only the mass of the Earth with a radius less than $x$ contributes to the force. For a gravitational force we have:

$$\vec{F}(x) = -\frac{GMm}{x^2}\,\hat{x} \tag{1}$$

However now $M$ does not equal $M_e$ the mass of the Earth. We can find M by multiplying the density of Earth $\rho$ by the volume $V$ of a sphere with radius $x$.

$$\rho = \frac{M_e}{\frac{4}{3}\pi R_e^3} \quad V = \tfrac{4}{3}\pi x^3 \tag{2}$$

$$M = \rho V = \frac{M_e x^3}{R_e^3} \tag{3}$$

Inserting this into our equation for force we get:

$$\vec{F}(x) = -\frac{GM_e m x}{R_e^3}\,\hat{x} \tag{4}$$

b) What is the component $F_x(x)$ of the force along the 1000 km tunnel as a function of position $x$ from the center of the tunnel? Here we follow the same prescription as above, except now we take the component of force along the direction of the tunnel. Let $F(r)$ be the force from part a) and let $\theta$ be the angle between the tunnel and a radial vector to position $x$. Here $r$ is the distance between the center of earth and position $x$. See Figure 1.

$$F_x(x) = F(r)\cos\theta = F(r)\frac{x}{r} = -\frac{GM_e m x}{R_e^3} \tag{5}$$

c) What is the period of the oscillation in each tunnel? Notice that the force has the form $F = -kx$, the same as a spring force. The frequency of oscillation for a spring is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM_e}{R_e^3}} \tag{6}$$

Thus for the period of oscillation $T$ we find,

$$T = \frac{2\pi}{\omega} = 5058\,\mathrm{s} = 1.4\,\mathrm{h} \tag{7}$$

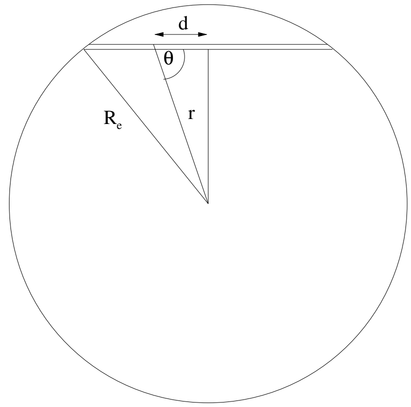


Figure 1: Tunnel Coordinates

**Problem 8.2,** (K&K 6.17, *10 Points*)
Here we have an inverted pendulum of mass $m$, and length $l$, with springs of spring constant $k$ attached to its midpoint and top end. And there's gravity. Define $\theta$ to be the angle from vertical in the clockwise direction and let $\Delta x_t, \Delta x_m$ be the linear displacements of the top and middle of the rod respectively. We will make the further assumption that the equlibrium length of the spring is much greater than $l$ the length of the rod and thus sthe springs remain essentially horizontal for small displacements. Taking vertical to be the equilibrium position and looking at the torque $\vec{\tau} = \vec{r} \times \vec{F}$ pointing into the page for small displacements have:

$$\tau = \frac{l}{2} mg \sin\theta - lk\Delta x_t \cos\theta - \frac{l}{2} k\Delta x_m \cos\theta \tag{8}$$

$$I\ddot{\theta} = \frac{l}{2} mg \sin\theta - l^2 k \sin\theta\cos\theta - \frac{l^2}{4} k \sin\theta\cos\theta \tag{9}$$

For a rod pivoting about an end point the moment of inertia $I = \frac{1}{3}ml^2$. For small angles $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Substituting this in and simplifying we find,

$$\ddot{\theta} + (\frac{15k}{4m} - \frac{3g}{2l})\theta = 0 \tag{10}$$

In this form we recognize the coefficient of $\theta$ as $\omega^2$, thus:

$$\omega = \sqrt{\frac{15k}{4m} - \frac{3g}{2l}} \tag{11}$$

**Problem 8.3,** (*10 Points*)
A particle of mass m moves in one dimension with a potential energy $U(x) = -ax^2 + bx^4$, where $a$ and $b$ are positive constants. What is the angular frequency of small oscillations about the equilibrium points of this system? First we find the equilibrium points $x_o$ by setting the derivative of the potential equal to zero.

$$\left.\frac{\partial U}{\partial x}\right|_{x=x_o} = 0 \tag{12}$$

$$0 = -2ax_o + 4bx_o^3 \tag{13}$$

$$x_o = 0, \pm\sqrt{\frac{a}{2b}} \tag{14}$$

We can take seconds derivatives or graph $U(x)$ to find that $x_o = 0$ is an unstable equilibrium and $x_o = \pm\sqrt{\frac{a}{2b}}$ are stable equilibria. To find the frequency we solve for the force $F = -\frac{\partial U}{\partial x}$.

$$m\ddot{x} = 2ax - 4bx^3 \tag{15}$$

Now we expand around the stable equilibrium points by letting $x \Rightarrow x_o + \Delta x$ and neglecting terms of order $\Delta x^2$ or higher.

$$m(\ddot{x_o} + \Delta\ddot{x}) = 2a(x_o + \Delta x) - 4b(x_o + \Delta x)^3 \tag{16}$$

$$m\Delta\ddot{x} = 2a(x_o + \Delta x) - 4b(x_o^3 + 3x_o^2\Delta x) \tag{17}$$

$$m\Delta\ddot{x} = (2a - 12bx_o^2)\Delta x \tag{18}$$

$$\Delta\ddot{x} + \frac{4a}{m}\Delta x = 0 \tag{19}$$

$$\omega = \sqrt{\frac{4a}{m}} \tag{20}$$

**Problem 8.4,** (*10 Points*)
This system has a natural frequency of 4 rad/s and is subject to a damping force such that $b/m = 10\ \mathrm{s}^{-1}$, or in other words $\omega_o = 4\,\mathrm{s}^{-1}$ and $\gamma = b/m = 10\,\mathrm{s}^{-1}$.
a) Show that the system is over-damped and find the general solution for the displacement. Our differential equations is,

$$\ddot{x} + \gamma x + \omega_o^2 x = 0 \tag{21}$$

Heavy damping occurs when

$$\frac{\gamma^2}{4} > \omega_o^2 \tag{22}$$

With our values of $\omega_o = 4\,\mathrm{s}^{-1}$ and $\gamma = 10\,\mathrm{s}^{-1}$ we get,

$$25 > 16 \tag{23}$$

so heavy damping occurs in this case. The general solution to this differential equation is derived in K&K Note 10.1 (Page 435). The solution has form,

$$x = Ae^{\alpha_+ t} + Be^{\alpha_- t} \tag{24}$$

where $A$ and $B$ are constants and

$$\alpha_\pm = -\frac{\gamma}{2} \pm \frac{\gamma}{2}\sqrt{1 - \frac{4\omega_o^2}{\gamma^2}} = -2, -8 \tag{25}$$

Thus our solution is

$$x = Ae^{-2t} + Be^{-8t} \tag{26}$$

b) If the mass is initially at $x = 0.5\,\mathrm{m}$, sketch the displacement as a function of time for initial velocities of -2 m/s and -6 m/s. We are given initial conditions that allow us to solve for the constants $A$ and $B$. The first conditions $x(t=0) = \frac{1}{2}$ and $\dot{x}(t=0) = -2$, lead to the equations,

$$\begin{aligned} \frac{1}{2} &= A + B & (27) \\ -2 &= -2A - 8B & (28) \end{aligned}$$

This solving these we find $A = \frac{1}{3}$ and $B = \frac{1}{6}$. The second conditions $x(t=0) = \frac{1}{2}$ and $\dot{x}(t=0) = -6$, lead to the equations,

$$\begin{aligned} \frac{1}{2} &= A + B & (29) \\ -2 &= -2A - 8B & (30) \end{aligned}$$

This solving these we find $A = -\frac{1}{3}$ and $B = \frac{5}{6}$. Now we have completely determined the motion of the particle in both cases. Plotting $x$ as a function of $t$ in Figure 2 we get,

$$\left.\begin{aligned} x(0) &= \tfrac{1}{2} \\ \dot{x}(0) &= -2 \end{aligned}\right\} \Longrightarrow x = \frac{1}{3}e^{-2t} + \frac{1}{6}e^{-8t}$$

$$\left.\begin{aligned} x(0) &= \tfrac{1}{2} \\ \dot{x}(0) &= -6 \end{aligned}\right\} \Longrightarrow x = -\frac{1}{3}e^{-2t} + \frac{5}{6}e^{-8t}$$

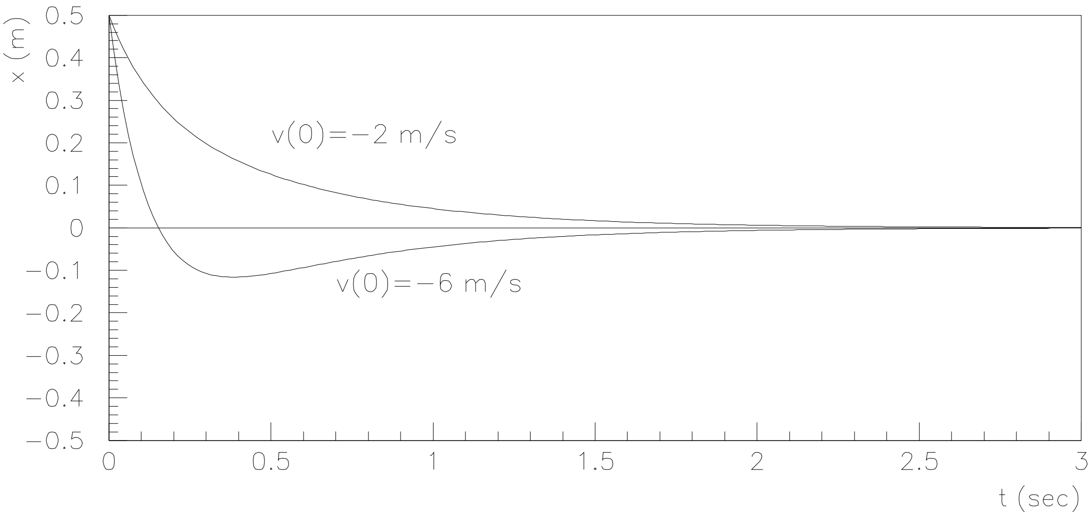


Figure 2: Heavily Damped Motion

**Problem 8.5,** (*10 Points*)
A seismograph is designed with a natural oscillation period of 2 s, whose amplitude decays with an exponential time constant of 20 s, thus,

$$e^{-\frac{\gamma}{2}t} = e^{-\frac{1}{20}t} \tag{31}$$

Or $\gamma = \frac{1}{10}\,\mathrm{s}^{-1}$ and $\omega_o = \frac{2\pi}{T} = \pi\,\mathrm{s}^{-1}$.
a) An earthquake provides a sinusoidal driving acceleration with a period of 0.1 s ($T = .1\,\mathrm{s}$) and an amplitude of $1\,\mathrm{cm/s}^2$, what is the amplitude of the steady-state response? We have $\omega = \frac{2\pi}{T} = 62.83\,\mathrm{s}^{-1}$ and $\frac{F_o}{m} = 1\,\mathrm{cm/s}^2$.

Here were are going to solve for the motion of a forced harmonic oscillator with damping. However before we dive in, a few words on differential equations. There are many ways to solve differential equations and for this class you are expected to be able to solve simple linear differential equations. There are several methods we can use to solve these types of equations. The usual method for solving differential equations involves guessing a solution of the right form and then showing that it satisfies the differential equation in question. Then by the uniqueness theorem we know that this is the only solution. Instead of having you memorize which type of solution to guess for each specific case, we will demonstrate that a complex exponential solution that will lead to the correct form of the solution without all the guesswork. This way you only need to remember one type of solution. I stress that these types of equations are very common in a wide range of scientific and technical subjects so it will be well worth your time to learn how to solve them. One method is presented in K&K Note 10.2 (Page 437) where they convert the equation into complex form, solve it by guessing a complex exponential solution, and then take the real part at the end. I will demonstrate another method that is a bit more direct. Our differential equation is,

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = \frac{F_o}{m}\cos\omega t \tag{32}$$

We could have also said that the driving acceleration was $\frac{F_o}{m}\sin\omega t$ and we would arrive at the same result. Depending on our choice, we next use one of the following identities to rewrite the $\cos\omega t$ or $\sin\omega t$ terms.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{33}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{34}$$

Our equation can then be written

$$\ddot{x} + \gamma x + \omega_o^2 x = \frac{F_o}{m} \cdot \frac{e^{i\omega t} + e^{-i\omega t}}{2}. \tag{35}$$

In class and in precept we have instead generalized $x$ to a complex variable $z$ and then taken the real part in the end. Here we show a different approach. The above suggests a solution of the form

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} \tag{36}$$

Where $A$ and $B$ are *complex* constants. Taking derivatives we find,

$$\dot{x} = Ai\omega e^{i\omega t} - Bi\omega e^{-i\omega t} \tag{37}$$

$$\ddot{x} = -A\omega^2 e^{i\omega t} - B\omega^2 e^{-i\omega t} \tag{38}$$

Plugging this into the differential equation we get,

$$-A\omega^2 e^{i\omega t} - B\omega^2 e^{-i\omega t} + \gamma(Ai\omega e^{i\omega t} - Bi\omega e^{-i\omega t}) + \omega_o^2(Ae^{i\omega t} + Be^{-i\omega t}) = \frac{F_o}{2m}(e^{i\omega t} + e^{-i\omega t}) \tag{39}$$

To proceed we need to realize something that may not be initially obvious. On each side of this equation we have two different functions of time, namely $e^{i\omega t}$ and $e^{-i\omega t}$. For this equation to hold for all time $t$ we need the constants multiplying the respective time functions to be equal on both sides of the equation. We can see this more clearly by factoring the equation in the following manner,

$$(-A\omega^2 + A\gamma i\omega + A\omega_o^2)e^{i\omega t} + (-B\omega^2 - B\gamma i\omega + B\omega_o^2)e^{-i\omega t} = \frac{F_o}{2m}e^{i\omega t} + \frac{F_o}{2m}e^{-i\omega t} \tag{40}$$

Setting the coefficients of $e^{i\omega t}$ equal and solving for A we get,

$$-A\omega^2 + \gamma Ai\omega + A\omega_o^2 = \frac{F_o}{2m} \tag{41}$$

$$A = \frac{F_o}{2m} \cdot \frac{1}{\omega_o^2 - \omega^2 + i\gamma\omega} \tag{42}$$

Setting the coefficients of $e^{-i\omega t}$ equal and solving for B we get,

$$-B\omega^2 - B\gamma i\omega + B\omega_o^2 = \frac{F_o}{2m} \tag{43}$$

$$B = \frac{F_o}{2m} \cdot \frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega} \tag{44}$$

Now notice that $B = A^*$ where $A^*$ is the complex conjugate of $A$. The solution of $x(t)$ then has the form,

$$x(t) = Ae^{i\omega t} + A^*e^{-i\omega t} \tag{45}$$

Here the two terms $Ae^{i\omega t}$ and $A^*e^{-i\omega t}$ are just complex conjugates of each other. When you add a complex number to its complex conjugate you just get twice the real part. We can see this by taking an arbitrary complex number of the form $z = x + iy$ where this $x$ and $y$ are *real* numbers. Then $z^* = x - iy$ and we have for their sum,

$$z + z^* = (x + iy) + (x - iy) = 2x \tag{46}$$

Thus we have that (different x here),

$$x(t) = 2\mathrm{Re}(Ae^{i\omega t}) \tag{47}$$

Where $\mathrm{Re}(z)$ is the real part of complex number $z$. Further simplifying A we get,

$$A = \frac{F_o}{2m} \cdot \frac{1}{\omega_o^2 - \omega^2 + i\gamma\omega} \tag{48}$$

$$A = \frac{F_o}{2m} \cdot \frac{1}{\omega_o^2 - \omega^2 + i\gamma\omega}\left(\frac{\omega_o^2 - \omega^2 - i\gamma\omega}{\omega_o^2 - \omega^2 - i\gamma\omega}\right) \tag{49}$$

$$A = \frac{F_o}{2m} \cdot \frac{\omega_o^2 - \omega^2 - i\gamma\omega}{(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2} \tag{50}$$

We can converting a complex number $z$ from a cartesian form $(x+iy)$ to a polar form $(Re^{i\phi})$ where $R = |z| = \sqrt{zz^*} = x^2 + y^2$ and $\phi = \tan^{-1}\frac{y}{x}$ are both *real* numbers. Doing this for A we have,

$$\begin{aligned} R &= \frac{F_o}{2m}\left[\left(\frac{\omega_o^2-\omega^2}{(\omega_o^2-\omega^2)^2+(\gamma\omega)^2}\right)^2+\left(\frac{\gamma\omega}{(\omega_o^2-\omega^2)^2+(\gamma\omega)^2}\right)^2\right]^{\frac{1}{2}} && (51)\\ R &= \frac{F_o}{2m}\left[\frac{1}{(\omega_o^2-\omega^2)^2+(\omega\gamma)^2}\right]^{\frac{1}{2}} && (52)\\ \phi &= \tan^{-1}\left(\frac{\gamma\omega}{\omega_o^2-\omega^2}\right) && (53)\end{aligned}$$

From above we had,

$$\begin{aligned} x(t) &= 2\mathrm{Re}(Ae^{i\omega t}) && (54)\\ x(t) &= 2\mathrm{Re}(Re^{i\phi}e^{i\omega t}) && (55)\\ x(t) &= 2\mathrm{Re}(Re^{i(\omega t+\phi)}) && (56)\\ x(t) &= 2\mathrm{Re}(R[\cos(\omega t+\phi)+i\sin(\omega t+\phi)]) && (57)\\ x(t) &= 2R\cos(\omega t+\phi) && (58)\\ x(t) &= \frac{F_o}{m}\left[\frac{1}{(\omega_o^2-\omega^2)^2+(\omega\gamma)^2}\right]^{\frac{1}{2}}\cos(\omega t+\phi) && (59)\end{aligned}$$

The amplitude we want is simply the coefficient of $\cos(\omega t+\phi)$.

$$Amplitude = \frac{F_o}{m}\left[\frac{1}{(\omega_o^2-\omega^2)^2+(\omega\gamma)^2}\right]^{\frac{1}{2}} = 2.539 * 10^{-4}\,\mathrm{cm} \qquad (60)$$

b) How far off (in percentage) is result for the amplitude in the approximation $\omega \gg \omega_o$? In this limit,

$$R_\omega = \frac{F_o}{m}\left[\frac{1}{(\omega)^4+(\omega\gamma)^2}\right]^{\frac{1}{2}} = 2.533\times 10^{-4}\,\mathrm{cm} \qquad (61)$$

$$\left|\frac{R_\omega - R}{R_\omega}\right| = 0.25\% \qquad (62)$$

We thank Michael Leung for preparing this document.