Physics 105 Problem Set 6 Solutions

Problem 6.41 *(10 Points)*
There are two keys to solving this problem. The first is to note that the ladder may start at any sufficiently high angle. It is easiest to start with the ladder leaning against the wall. The second key is to realize that it takes only one variable to describe the motion of the plank (which can be considered a rod for this problem). This variable is most conveniently chosen to be the angle θ given in the figure. Since the top of the plank is sliding down the wall (until the moment it loses contact with the wall), the x and y coordinates of the center of mass can be expressed in terms of θ as

$$x_{CM} = L\cos\theta \quad (1)$$

$$y_{CM} = L\sin\theta \quad (2)$$

The angle θ also describes rotational motion about the center of mass and thus we see that it is the only variable required to describe the motion of the system. Let us consider torques about the center of mass. There is a torque due to the normal reaction N_w from the wall and one due to the normal reaction N_f from the floor. The torque equation is

$$N_f L\cos\theta - N_w L\sin\theta = -\frac{1}{3}mL^2\ddot{\theta} \quad (3)$$

since the moment of inertia of a rod of length $2L$ about its center is $\frac{1}{3}mL^2$. The force N_w is also the one on the rod acting in the x direction, so

$$N_w = m\ddot{x}_{CM} = -mL(\cos\theta\dot{\theta}^2 + \sin\theta\ddot{\theta}) \quad \text{(FromEqn. 1)} \quad (4)$$

Similarly

$$N_f - mg = m\ddot{y}_{CM} = mL(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) \quad \text{(FromEqn. 2)} \quad (5)$$

or

$$N_f = mL(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2 + mg) \quad (6)$$

At the point that the plank loses contact with the wall, $N_w = 0$ and Eqn. 4 gives

$$\ddot{\theta} = -\cot\theta\dot{\theta}^2 \quad (7)$$

Putting this value into Eqn. 3 and using Eqn. 6 for N_f gives

$$\dot{\theta}^2 = \frac{3g}{4L}\sin\theta \quad (8)$$

at the point the plank loses contact with the wall. From energy conservation,

$$\frac{1}{2}m(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_{CM}\dot{\theta}^2 + mgy_{CM} = mgL \tag{9}$$

since mgL is the initial energy when the rod is leaning vertically against the wall. Wecan use Eqns. 1 and 2 to substitute for y_{CM}, $\dot{x}_{CM}$ and $\dot{y}_{CM}$. The result is

$$\frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{6}mL^2\dot{\theta}^2 + mgL\sin\theta = mgL \tag{10}$$

Using the value of $\dot{\theta}$ obtained in Eqn. 8, we get

$$\frac{3}{2}mgL\sin\theta = mgL \tag{11}$$

or

$$\sin\theta = \frac{2}{3} \tag{12}$$

at the point where the plank loses contact with the wall. The height is given by

$$\sin\theta = \frac{h}{2L} = \frac{2}{3} \tag{13}$$

so h is 2/3 the initial height $2L$.

Problem 7.3 *(10 Points)*

Let us set up polar coordinates with the origin at the point of suspension as shown in the figure. The total angular momentum of the system $\mathbf{L}$ can be split upinto the angular momentum of the center of mass ($\mathbf{L}_{CM}$) and the angular momentum about the center of mass ($\mathbf{L}'$).

$$\mathbf{L} = \mathbf{L}_{CM} + \mathbf{L}' \tag{14}$$

The center of mass goes around the z axis with constant angular velocity Ω at a distance $L\sin\beta + l$ from the axis of rotation. Thus

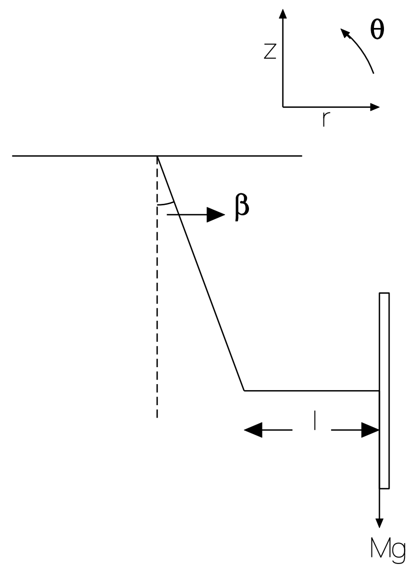
$$\mathbf{v}_{CM} = (L\sin\beta + l)\Omega\hat{\theta} \tag{15}$$

The radius vector to the center of mass $\mathbf{r}_{CM}$ is

$$\mathbf{r}_{CM} = (L\sin\beta + l)\hat{r} - L\cos\beta\hat{z} \tag{16}$$

Thus,

$$\mathbf{L}_{CM} = M\mathbf{r}_{CM} \times \mathbf{v}_{CM} = M\Omega(L\sin\beta + l)[(L\sin\beta + l)\hat{z} + L\cos\beta\hat{r}] \tag{17}$$



There are **two** contributions to $\mathbf{L}'$, one from the spinning of the wheel about its symmetry axis and the other from a spinning about the z axis. This second component is due to the fact that the wheel is precessing in the horizontal plane. We thus obtain

$$\mathbf{L}' = I_0 \omega_s \hat{r} + I_1 \Omega \hat{z} \tag{18}$$

where I_1 is the moment of inertia about an axis perpendicular to the symmetry axis. We know that

$$\frac{d\mathbf{L}}{dt} = \tau \tag{19}$$

where τ is the torque. About the point of suspension, only the weight contributes to the torque.

$$\tau = (L \sin\beta + l) M g \hat{\theta} \tag{20}$$

From Eqns. 17, 18 and 20

$$\frac{dL}{dt} = \Omega[ML \cos\beta (L \sin\beta + l)\Omega + I_0 \omega_s] = (L \sin\beta + l) M g \tag{21}$$

where we have also used the fact that $\frac{d\hat{z}}{dt} = 0$, $\frac{d\Omega}{dt} = 0$ and $\frac{d\hat{r}}{dt} = \Omega\hat{\theta}$. There is no component of the torque corresponding to the spin of the wheel about the z axis because the corresponding angular momentum is constant.

We next consider the forces. From the force equations along the r and z directions we have

$$T \sin\beta = M\Omega^2(L\sin\beta + l) \tag{22}$$

and

$$T\cos\beta = Mg \tag{23}$$

From Eqns. 22 and 23,

$$\tan\beta = \Omega^2 \frac{L\sin\beta + l}{g} \tag{24}$$

Using $\tan\beta \approx \sin\beta \approx \beta$

$$\beta = \frac{l}{\frac{g}{\Omega^2} - L} \tag{25}$$

From Eqns. 21 and 24

$$\Omega = \frac{Mgl}{I_0\omega_s} \tag{26}$$

From Eqns. 25 and 26,

$$\beta = \frac{l}{\frac{I_0^2\omega_s^2}{M^2gl^2} - L} \tag{27}$$

We could have instead chosen the origin to be at a point $L\cos\beta$ below where the rope is attached. The torque around that point now has two components, one from the wheel and the other from the tension in the line.

$$\tau = (L\sin\beta + l)Mg\hat{\theta} - L\sin\beta T\cos\beta\hat{\theta} \tag{28}$$

The total angular momentum is now

$$\mathbf{L} = M\Omega(L\sin\beta + l)^2\hat{z} + I_1\Omega\hat{z} + I\omega_s\hat{r} \tag{29}$$

and since $d\mathbf{L}/dt = \tau$ and $T\cos\beta = Mg$ we get:

$$I\omega_s\Omega\hat{\theta} = (L\sin\beta + l)Mg\hat{\theta} - MgL\sin\beta\hat{\theta} \tag{30}$$

Equation 24 gives us another relation between Ω and β based on the dynamics. We end up with the same result as before.

Lastly, we give a third approach to the problem that quantifies the approximations that are being made. These notes come from Prof. Groth.

Start with $F = ma$. In the vertical direction:

$$T\cos\beta = Mg \tag{31}$$

where T is the tension in the string, and in the horizontal direction:

$$T \sin\beta = Mv^2/(\ell + L\sin\beta)\,;\, v = (\ell + L\sin\beta)\Omega\,, \tag{32}$$

where v is the linear speed with which the disk precesses and Ω is the angular speed of precession, so

$$T \sin\beta = M\Omega^2(\ell + L\sin\beta)\,. \tag{33}$$

Compute torques and angular momentum about the fixed point which is the center of precession of the center of mass of the disk. We are only interested in the horizontal component of angular momentum due to the spin of the disk about the shaft. The vertical components of angular momentum, due to the rotation of the disk about a vertical axis throught its center of mass and the precession of the center of mass are constant for the postulated motion and so there should be no torque in the vertical direction. The torque in the horizontal plane (perpendicular to the axis of the disk) is due to the weight and the tension:

$$\tau = Mg(\ell + L\sin\beta) - T\cos\beta L\sin\beta = Mg\ell\,, \tag{34}$$

where we have replaced $T\cos\beta$ with Mg according to eq. (31). This is the rate of change of the (horizontal component of the) spin angular momentum, dS/dt, and the spin angular momentum is $S = I_0\omega_s$. Since the torque is perpendicular to the spin, the spin precesses at a rate given by $\Omega = (1/S)\cdot(dS/dt)$. Putting this altogether, we have,

$$Mg\ell = \Omega S = \Omega I_0\omega_s\,, \qquad \text{or} \qquad \Omega = \frac{Mg\ell}{I_0\omega_s}\,. \tag{35}$$

Note that the precession rate is determined only by the properties of the disk and its axle. The angle of the string, β, does not enter.

Now divide eq. (33) by eq. (31) and substitute for Ω from eq. (35):

$$\tan\beta = A(1 + B\sin\beta)\,, \qquad A = \frac{M^2 g\ell^3}{I_0^2\omega_s^2}\,, \qquad \text{and} \qquad B = \frac{L}{\ell}\,. \tag{36}$$

For a good gyroscope, ω_s will be large and A will be very small. In this case, β is small and we might try setting $\tan\beta = \sin\beta = \beta$. This gives

$$\beta = \frac{A}{1 - AB}\,, \tag{37}$$

Since A is very small, we can ignore AB in the denominator unless $B = L/\ell$ is pathologically large, giving

$$\beta = A = \frac{M^2 g\ell^3}{I_0^2\omega_s^2}\,. \tag{38}$$

B	Eq. (36)	Eq. (37)
10	0.0010101	0.0010101
100	0.0011111	0.0011111
999	0.1205763	1.0000000
1000	0.1258252	∞
1001	0.1310690	-1.0000000
10000	1.4706390	-0.1111111

Of course, one could have an arbitrarily long string which would cause β as given by eq. (37) to blow up or even become negative. In this case, one should return to eq. (36). In the event that $B \gg 1$ and $AB > 1$, we can ignore 1 compared to $B \sin\beta$. We find

$$\cos\beta = \frac{1}{AB} = \frac{I_0^2\omega_s^2}{M^2 g\ell^2 L} = \frac{g}{L\Omega^2}\,, \tag{39}$$

which is exactly the angle made by a conical pendulum of length L and frequency Ω. Recall that we are assuming the that $L \gg \ell$, so to first order, the disk is a point mass on the end of a string of length L. The fact that we have a precessing disk on the end of the string is of no importance except that it determines the frequency (Ω) at which we would like the pendulum to go around.

For intermediate cases, we should solve the transcendental eq. (36). On method is to plot the functions $\tan\beta/A$ and $1 + B\sin\beta$ versus β and see where they intersect. A few sketches will convince one that there's always a solution between 0 and $\pi/2$ no matter what the values of A and B (as long as they're positive!).

Also, one can just numerically solve the equation by making a guess and correcting the guess until both sides of the equation give the same answer. Here's a short table giving β for various values of B with $A = 0.001$ held fixed. Values are given in radians as solutions of eq. (36) and eq. (37):

Note that eq. (38) gives $\beta = 0.001$ in all cases above. I conclude that eq. (37) provides a useful correction to eq. (38) when B is not too large. However, when B is large, eq. (36) should be solved. Eq. (38) just goes bad as B gets large. Eq. (37) goes catastrophically bad as B gets large!

Problem 7.4 *(10 Points)*
This problem has all the same angular momentum components as the previous one but now the forces are different. The velocity of the center of mass of the millstone is

$$v = \Omega R \tag{40}$$

Since the millstone is rolling without slipping its angular velocity is

$$\omega = \Omega \frac{R}{b} \quad (\text{using } \omega b = \Omega R) \tag{41}$$

As in the previous problem $\mathbf{L} = \mathbf{L}_{CM} + \mathbf{L}'$.

$$\mathbf{L}_{CM} = M\Omega R^2 \hat{z} \tag{42}$$

$$\mathbf{L}' = -\frac{1}{2} M b^2 \omega \hat{r} + I_1 \Omega \hat{z} \tag{43}$$

where $\hat{z}$ is in the vertical direction and $\hat{r}$ points outward along the axle. The minus sign in $\mathbf{L}'$ arrises because the millstone is rotating clockwise. We are calculating the angular momenta are measured about the point where the horizontal axle is connected to the vertical shaft. Now,

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}'}{dt} = -\frac{1}{2} M b^2 \omega \Omega \hat{\theta} = \tau \tag{44}$$

where τ is the torque. Once again, $\frac{d\hat{z}}{dt} = 0$ and$\frac{d\hat{r}}{dt} = \Omega\hat{\theta}$. Let us look at the forces that give rise to the torque τ. These are the weight of the millstone, Mg, and the normal reaction N which act in opposite directions along the same vertical line. They thus have the same lever arm R but produce torques along opposite directions, Mg produces a torque along $\hat{\theta}$ and N produces a torque along $-\hat{\theta}$. Thus weobtain

$$MgR - NR = \frac{1}{2} M b^2 \omega \Omega \tag{45}$$

which gives us

$$N = M\left(g + \frac{1}{2} b \Omega^2\right) \tag{46}$$

where we have used Eqn. 29 to substitute for ω. The weight on the millstone and the normal reaction are in opposite directions but not equal. Why does the millstone not accelarate upwards? The force that keeps the center of mass stationary is a vertical force at the junction of the horizontal axleand the vertical shaft. We did not consider this force while calculating torques because we were calculating torques about its point of application.

Problem 7.5 *(10 Points)*
As the car rounds the curve, there is a torque about its CM due to the friction that acts to keep the car moving in a circle. If L is the height off the road of the CM, the torque is given by

$$\tau = Lf = M\frac{v^2}{l} L \tag{47}$$

where l is the radius of the curve. The torque is directed so as to cause the car to flip over away from the direction in which it is turning. On a flat stretch of road, the loading on the wheels is equal. When rounding, say, a righthand curve the normal force on the righthand wheel goes to zero just as the car starts to roll over away from the direction of the turn.

a) We mount the flywheel with its axis of rotation horizontal and perpendicular to the length of the car. The flywheel rotates, so the angular momentum vector points to the right side of the car. If the car is taking a right curve, the direction of the change of the angular momentum of the wheel is toward the rear or the car. There has to be a torque on the car to produce this change in angular momentum. The torque comes from the normal force on the wheels. The direction of the torque is such that one wants to decrease the normal force on the lefthand wheels, counteracting the tendency to roll and equalizing the wheel loading. For the car turning to the left, the direction of the change of angular momentum points forward. This leads to an decrease in the normal component of the righthand wheels. Note that in both cases it is the change in normal forces that counteracts the torque associated with the centripetal force.

b) The angular momentum of the flywheel due to rotation around it's axis is

$$L_w = \frac{1}{2} m R^2 \omega. \tag{48}$$

This vector is pointing to the right of the car and is rotating with angular velocity

$$\Omega = \frac{v}{l} \tag{49}$$

while the car turns. Hence, it's change in time is

$$\frac{dL_w}{dt} = L_w \Omega = \frac{1}{2} m R^2 \omega \frac{v}{l} \tag{50}$$

Setting this equal to the torque due to centripetal force we found above, we have

$$M \frac{v^2}{l} L = \frac{1}{2} m R^2 \omega \frac{v}{l} \tag{51}$$

hence the angular velocity of the wheel needed for equal loading of the wheels is

$$\omega = 2v \frac{ML}{mR^2}. \tag{52}$$

Problem 7.9 *(10 Points)*

The torque on the bike equals the change in the angular momentum of the wheels. The torque with respect to the center of mass has two components, the first one due to the centripetal force which is the frictional force between the wheels and the road and the second one from the normal force of the road on the bike. The frictional force that keeps the bike moving in a circle has a magnitude $f = M\frac{v^2}{R}$ and lever $2l\cos(\theta)$. The lever of the normal force is $2l\sin(\theta)$. Hence the total torque around the axis of rotation is

$$\tau = M\frac{v^2}{l}2l\cos(\theta) - Mg2l\sin(\theta) \tag{53}$$

pointing forward in the direction of the motion of the bike. The magnitude of the angular momentum of both wheels together is $L = 2ml^2\frac{v}{l}$ where $\frac{v}{l}$ is the angular velocity of wheels around their axes ofrotation. The angular momentum points perpendicular to the plane of the bike towards the center of the curve. The vertical component of the angular momentum $L\sin(\theta)$ points down and does not change with time. The horizontal component points towards the center of the curve and rotates with angular velocity $\frac{v}{R}$ around it. Hence it's change with time is

$$\frac{dL}{dt} = L\cos(\theta)\frac{v}{R} \tag{54}$$

and it points opposite to the motion of the bike. Setting this equal to the torque we have

$$M\frac{v^2}{R}2l\cos(\theta) - Mg2l\sin(\theta) = -2ml^2\frac{v}{l}\cos(\theta)\frac{v}{R} \tag{55}$$

so the angle the bike subtends with the vertical is

$$\tan(\theta) = \frac{v^2}{Rg}(1 + \frac{m}{M}). \tag{56}$$

K&K 7.6

The following solution is given so that you can expand your problem solving ability! K&K tell us that a coin of mass M and radius b rolls without slipping around a circular path of radius R at a speed v. They want to know the angle ϕ that the spin axis makes with respect to the horizontal.

The total angular momentum of the coin can be written $\vec{L} = \vec{L}_{\text{cm}} + \vec{L}_{\text{s}}$, where $\vec{L}_{\text{cm}}$ is the angular momentum of the center of mass (cm) around a fixed inertial coordinate system, and $\vec{L}_{\text{s}}$ is the spin angular momentum around the center of mass. Taking the origin of the inertial reference frame to be at the center of the cm's circular motion, we have ($\hat{z}$ points up):

$$\vec{L}_{\text{cm}} = \vec{r}_{\text{cm}} \times \vec{p}_{\text{cm}} = Mr_{\text{cm}}v_{\text{cm}}\ \hat{r} \times \hat{\theta} = M\Omega r_{\text{cm}}^2\ \hat{z}, \tag{57}$$

and

$$\vec{L}_{\rm s} = I_0\omega_{\rm s}\left[-\sin\phi\ \hat{z} - \cos\phi\ \hat{r}\right] + I'\Omega\ \hat{z}, \tag{58}$$

where $\Omega = v_{\rm cm}/r_{\rm cm} = v/R$ is the angular velocity of the cm around the origin and $\omega_{\rm s} = v/b$ is the spin angular velocity of the coin. Here, I_0 is the moment of inertia around an axis through the cm and perpendicular to the face, while I' is the moment of inertia around an axis through the cm and parallel to $\hat{z}$. Taking the time derivative of $\vec{L}$ we obtain

$$\frac{d\vec{L}}{dt} = -I_0\omega_{\rm s}\Omega\cos\phi\ \hat{\theta}. \tag{59}$$

There are three forces acting on the coin: gravity $M\vec{g}$, normal force $\vec{N}$, and friction $\vec{f}$. Summing the torques (around the origin) arising from these three forces gives

$$\sum\vec{\tau} = Mgr_{\rm cm}\ \hat{\theta} + fb\cos\phi\ \hat{\theta} - NR\ \hat{\theta}. \tag{60}$$

Summing the forces along $\hat{z}$ and $\hat{r}$ provides two additional equations we can use to eliminate f and N:

$$\sum F_z = N - Mg = 0 \Rightarrow N = Mg;\ \sum F_r = -f = -M\frac{v_{\rm cm}^2}{r_{\rm cm}} \Rightarrow f = Mv^2\frac{r_{\rm cm}}{R^2}, \tag{61}$$

where we have used $v_{\rm cm} = v\frac{r_{\rm cm}}{R}$. Substituting Eqs. 61 into Eq. 60 and noting that $r_{\rm cm} = R - b\sin\phi$, we obtain

$$\sum\vec{\tau} = \left[-Mgb\sin\phi + Mv^2b\frac{r_{\rm cm}}{R^2}\cos\phi\right]\ \hat{\theta}. \tag{62}$$

We are now ready to use the dynamical equation $\sum\vec{\tau} = \frac{d\vec{L}}{dt}$ to solve for ϕ:

$$-Mgb\sin\phi + Mv^2b\frac{r_{\rm cm}}{R^2}\cos\phi = -I_0\omega_{\rm s}\Omega\cos\phi, \tag{63}$$

and simplifying

$$\tan\phi = \frac{Mv^2b\frac{r_{\rm cm}}{R^2} + \frac{1}{2}Mb^2\left(\frac{v}{b}\right)\left(\frac{v}{R}\right)}{Mgb} = \frac{v^2}{gR}\left(\frac{1}{2} + \frac{r_{\rm cm}}{R}\right), \tag{64}$$

where we have used $I_0 = \frac{1}{2}Mb^2$, $\omega_{\rm s} = \frac{v}{b}$, and $\Omega = \frac{v}{R}$. Finally, substituting for $r_{\rm cm}$ gives

$$\tan\phi = \frac{v^2}{gR}\left(\frac{3}{2} - \frac{b}{R}\sin\phi\right). \tag{65}$$

If the friction force providing the centripetal acceleration is large enough, the angle ϕ need not be small and we cannot assume the small-angle approximation. However, the gyroscope approximation tells us that the precession is stable if $\Omega << \omega_{\rm s}$, or $b << R$. Taking this limit leads to the final relation

$$\tan\phi = \frac{3v^2}{2gR}. \tag{66}$$