Physics 105 Problem Set 5 Solutions

Problem 5.1 *(10 Points)*
Rolling without slipping means that at any instant of time the point of the cylinder in contact with the surface has zero velocity with respect to the surface. If the CM has linear velocity, $v(t)$, then the velocity of the contact point is $v(t) - R\omega(t)$. If this is to be zero, $v = R\omega$ and so $a = R\alpha$ where a is the linear acceleration. In setting up this problem, we use the fact that any motion can be divided into a translational of the CM and rotation about the CM *even if the CM is accelerating*. This is discussed in K&K section 6.7. The width of the cylinder, l, does not enter into the problem.

(a)The equation of motion for accelerating down the plane is $Mg\sin(\theta) - f = Ma$. Normal to the plane, we have $N = Mg\cos(\theta)$. The torque around the CM is $\tau = fR = I\alpha$. For a cylinder, $I = MR^2/2$. Putting all of these together, we get:

$$a = \frac{2}{3}g\sin(\theta)\alpha = \frac{a}{R} = \frac{2g}{3R}\sin(\theta) \tag{1}$$

(b)Because there is no slipping, the total mechanical energy is conserved. Thus:

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + Mgh \tag{2}$$

where h is the height above some reference. We can go through this, solve for v,use the kinematic equation that $v_f^2 = 2ax$ and find the same velocity as above. Or, we can note that E is constant and so $dE/dt = 0$. This gives

$$\frac{d}{dt}(\frac{3}{4}Mv^2 + Mgh) = 3Mva/2 + Mgv\sin(\theta) = 0 \tag{3}$$

or $a = 2g\sin(\theta)/3$. Note that friction does not dissipate energy but transfers energy from translation to rotation.

(c)We can use the same method as above. Let f_i be the frictional force on each wheel with $i = 1, 2, 3, 4$. Then,

$$Mg\sin(\theta) - (f_1 + f_2 + f_3 + f_4) = Ma \tag{4}$$

where $f_iR = Ia/R$. We find:

$$a = \frac{g\sin(\theta)}{1 + 4I/MR^2} \tag{5}$$

To have a large acceleration, I/MR^2 should be small. For a heavy car (M large) and small light wheels (I small), the car appears to glide. In other words, it accelerates with $a = 2g\sin(\theta)/3$, just like a block on a frictionless plane.

When taking limits it is important to compare quantities with the same units. To say that I is small compared to one doesn't make much sense because I has units. I must be compared to something with units of MR^2.

Problem 5.2 *(10 Points)*
Let's divide up this problem into acceleration around the CM and translation of the CM. If the angular acceleration is clockwise (CW) the yo-yo moves to the right. Let θ be the angle the tension vector makes with respect to the floor so that the equation of motion in the x direction is $Ma = T\cos(\theta) - f$. The torque around the center of mass is given by $\tau = -Tr_1 + fr_2 = I\alpha = Ia/r_2$. Solving for a we find that

$$a = \frac{T(\cos(\theta) - r_1/r_2)}{M + I/r_2^2} \tag{6}$$

(a)For $\theta = 0$, the numerator is greater than zero and thus the yo-yo moves to the right.

(b)For $\theta_0 = \cos^{-1}(r_1/r_2)$, $a = 0$ and the yo-yo remains stationary if it was stationary to begin with or just translates with uniform velocity. Geometrically, this corresponds to when the tension vector is parallel to a line that runs from the point of contact with the floor to the tangent point on the inner drum of the yo-yo. Since $\vec{F}$ and $\vec{r}$ are parallel, there can be no angular acceleration about the point of contact.

(c)The tension will be a maximum when the frictional force is a maximum. From the forces in the y direction, we have $f_{max} = \mu N = \mu(mg - T_{max}\sin(\theta_0))$ where θ_0 is the angle found in part (b). For the x-direction $f_{max} - T_{max}\cos(\theta_0) = 0$. By combining these we get:

$$T_{max} = \frac{\mu mg}{\cos(\theta_0) + \mu\sin(\theta_0)} \tag{7}$$

Problem 5.3*(10 Points)*
The moment of inertia of a uniform density sphere about any axis through its center is $\frac{2}{5}MR^2$. There are a lot of ways to do the integral. Recall that we want to find I around an *axis*. In spherical coordinates;

$$I = \int \rho r_{perp}^2 dV \tag{8}$$

where $r_{perp} = r\sin(\theta)$ is the perpendicular distance from the axis about which we are computing the moment of inertia. Here, ρ is the density and is $\rho = 3M/4\pi R^3$.

$$I = \rho \int_{r=0}^{R} \int_{\phi=0}^{2\pi} \int_{\theta=-\pi/2}^{\pi/2} r^4 sin^3(\theta) dr\, d\theta\, d\phi \tag{9}$$

We do the ϕ and r integrals first because they are easy. We also plug for the density.

$$I = (3M/4\pi R^3)2\pi R^5/5 \int_{\theta=-\pi/2}^{\pi/2} \sin^3(\theta) d\theta = (3MR^2/10)(2 - 2/3) = 2MR^2/5 \qquad (10)$$

Problem 5.4 *(10 Points)*
(a) The net normal force of both rear wheels is N_r and on both front wheels it is N_f. One may choose any place as the origin but the CM and the bottom of the rear wheels are particularly convenient. We'll take the CM because we can also use it in part (b). From the balance of forces in the y direction, $N_f + N_r = Mg$. The torque from the front wheels is $(d - x)N_f$ and the torque from the rear wheels is xN_r and they bust balance because the car is not rotating about the CM. Thus $(d - x)N_f = xN_r$. From the force and torque balance equations we find $N_f = Mgx/d$ and $N_r = Mg(1 - x/d)$.

(b) We now want to know how fast the car can accelerate before the front wheels just leave the ground. In other words, what is the acceleration such that $N_r = Mg$ and $N_f = 0$? When the car is accelerating, there must be a net force, f^E, on the car at the point where the wheels contact the ground. That is, if the car is accelerating forward, the force producing this acceleration is the force of friction on the tires pushing forward. This force is the result of the engine putting a torque on the tires. The superscript is for "engine." When the car is accelerating on flat ground, $f^E = f^E_r + f^E_f = Ma$ where again we have used the "front" and "rear" subscripts. The force f_E is produced by friction.

We want to set up an inertial coordinate system to do the problem. Let's take the origin to be at the point where the rear tires touch the ground at the instant that the front tires leave the ground (*e.g.* K&K pg. 266). The angular momentum of the car around that point is $L = Mvh$ (always). The torque about that point is $\tau = dL/dt = Mgx$. Thus $Mah = Mgx$ when the front wheels leave. This means the acceleration is $a = gx/h$.

Lets look at this from another point of view. We know that any motion can be broken down into translation of the CM and rotation about the CM even if the CM is accelerating (K&K pg. 263). At the instant the front tires leave the ground, $f^E_r = Ma$ and $N_r = Mg$. The torque about the CM from these two forces is $Mah = Mgx$ leading to the same result as above.

If the car has just front wheel drive, then the torque rotating the car up goes to zero right after the front tires leave the ground and the angular acceleration stops. If the car has rear wheel drive, the angular acceleration is maintained.

Problem 5.5 *(10 Points)*
The physics of this problem is to understand the criterion for which the can tips over. In essence, the balls apply a net torque to the can and tip it over. We need to find the mass of

the can, m_c, so that this cannot happen. Just before the can tips over, it is supported on one point, namely the point on the lower right of the figure.

Before considering the tipping can, let's find the forces in the case that the can is just sitting there (an infinitely massive can). Draw a FBD for each ball and look at the forces acting on each. On the lower ball, we have the wall F_A, the table F_{N_a}, and the upper ball F_{21} acting on it. We find $F_{21}/\sqrt{2} = F_A$ and $F_{21}/\sqrt{2} + m_b g = F_N$ where m_b is the mass of a ball. Likewise, on the upper ball we find $F_{12}/\sqrt{2} = F_B$ and $F_{12}/\sqrt{2} = m_b g$ where F_B is the force of the wall on the ball. By combining these, we find $F_{12} = F_{21} = \sqrt{2} m_b g$, $F_{N_a} = 2m_b g$ and $F_A = F_B = m_b g$.

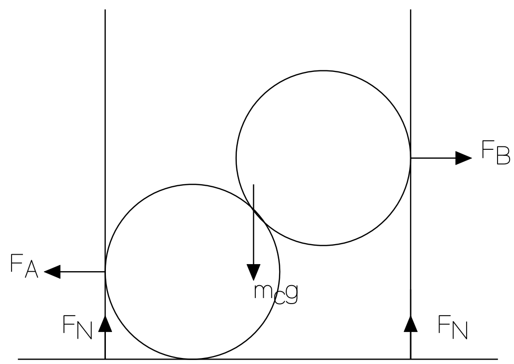

Figure 1: Forces on the can before it tips over.

The forces that act on the can are shown in Fig. 1. There is the weight of the can $m_c g$ acting downwards on the center of mass of the can, the two forces F_A and F_B exerted by the two balls and the normal reaction F_N from the floor which acts at all the points on the perimeter of the can in contact with the floor. In this cross-sectional diagram, F_N is shown to act at the lower left and right corners. The net torque due to all these forces should be equal to zero if the can is to remain in equilibrium.

Now consider what happens when the can tips, Fig. 2. Just slightly after it starts to rotate, the only point of contact is the lower right corner. We want to compute all the torques on the can about this point to see if it has a net clockwise (CW) torque. There are three contributions: (1) The torque from the can is $m_c gL/2 = m_c g(R + R/\sqrt{2})$ (force of gravity times the lever arm to the can's CM) in the CCW direction. (2) The force of the lower ball on the can F_A (opposite in direction to the force of the can on the ball) also wants to torque the can in the CCW direction. The value of the torque is $m_b gR$ where R is the radius of the ball. (3) The force of the upper ball on the can wants to torque the can in the CW direction. The lever arm

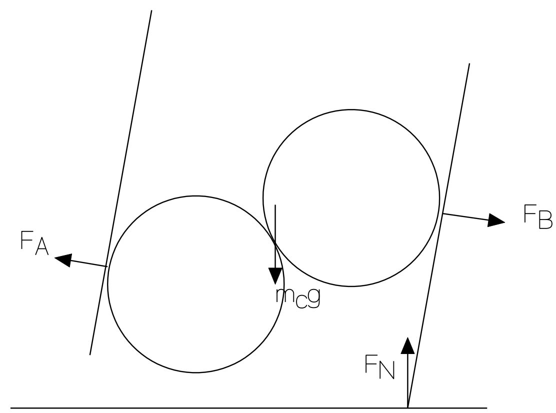

Figure 2: Forces on the can while it is tipping over.

is $R + 2R/\sqrt{2}$ so the torque is $m_b g(R + 2R/\sqrt{2})$ in the CW direction. For balance:

$$m_c g(R + R/\sqrt{2}) + m_b g R = m_b g(R + 2R/\sqrt{2}) \tag{11}$$

so that

$$m_c = \frac{2m_b}{1 + \sqrt{2}} \tag{12}$$

The key to this problem is that only the lower right corner is in contact with the ground while the can is tipping over. The normal force at this point does not contribute to the torque if the torque is measured about this point. The torque due to the remaining forces as shown above is positive and the can tips over. For larger masses of the can, the net torque from the forces mentioned would be negative and it would not tip over and remain standing on the table. This is an equilibrium situation and we know the net torque on the can has to be zero. The net negative torque from the three forces considered will be balanced by the torque due to the normal reaction at the other points on the perimeter of the can in contact with the floor in this case.