Physics 105 Problem Set 4 Solutions

Problem 3.1 *(10 Points)*
3.1 *The density of a thin rod of length l varies with the distance x from one end as $\rho = \rho_0 x^2/l^2$. Find the position of the center of mass.*

The center of mass is given by $x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i}$ for a system of point particles. If we keep breaking up the particles then m becomes dm and $x_{cm} = \frac{\int xdm}{\int dm}$

We are given $\rho(x) = \rho_0 x^2/l^2$. By definition, $\rho = \frac{dm}{dV} = \frac{dm}{Adx} \Rightarrow dm = A\rho dx$, where A is the cross-sectional area. So,

$$\int xdm = \int x\rho(x)Adx = \int_0^l xA\rho_0 x^2/l^2 dx = A\rho_0/l^2 \int_0^l x^3 dx = A\rho_0 l^2/4 \tag{1}$$

and

$$\int dm = \int_0^l \rho(x)Adx = \int_0^l A\rho_0/l^2 x^2 = A\rho_0 l/3 \tag{2}$$

so,

$$x_{cm} = \frac{A\rho_0 l^2/4}{A\rho_0 l/3} = 3l/4 \tag{3}$$

Problem 3.4 *(10 Points)*
3.4 *An instrument carrying projectile accidentally explodes at the top of its trajectory. The horizontal distance between the launch point and the point of explosion is L. The projectile breaks into two pieces which fly apart horizontally. The larger piece has three times the mass of the smaller piece. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. How far away does the larger piece land? Neglect air resistance and effects due to the earth's curvature.*

Between any of the collisions in this problem, the only force acting on the masses is gravity. We also know that the trajectory of a particle influenced only by gravity is independent of mass, and solely determined by initial velocity. This follows from F = ma = mg $\Rightarrow$ a = g. If there were no exposion, the original mass would have followed a mirror image trajectory of its flight up all the way to a point 2L from the start. Or, equivalently, if it had bounced off a wall at the top of its trajectory, or if we followed the second half in a mirror, it would land back where it started. This means that the correct speed to get back to the launch is precisely the speed the mass has when it's at the top. But since the horizontal velocity has not changed from the start, this is the same horizontal velocity the mass always had, the only one in the problem so far. Call it v. So we conclude that the small mass must have velocity $-v$ right after the collison to get back to the launch pad. That is, $v_1 = -v$. We need to determine the velocity of the other half, v_2, by conservation of momentum:

$$mv = \frac{m}{4}v_1 + \frac{3m}{4}v_2 = -\frac{m}{4}v + \frac{3m}{4}v_2 \Rightarrow v_2 = 5v/3 \tag{4}$$

Now, $\Delta x = vt$. For the particle, when it was whole, L=vt $\Rightarrow$ t=L/v. So $\Delta x = \frac{5v}{3}\frac{L}{v} = \frac{5L}{3}$. So the larger piece lands L+5L/3 = 8L/3 from the launch point.

Problem 3.11 *(10 Points)*
3.4 *Material is blown into cart A from cart B at a rate b. The material leaves the chute vertically downward, so that it has the same horizontal velocity as cart B, u. At the moment of interest, cart A has mass M and velocity v. Find dv/dt, the instantaneous acceleration of A.*

First, note that since the matter leaving B has the same velocity as B, B's velocity cannot change. This is easy to see in B's rest frame. That is, $du/dt = 0$.
Now, by conservation of momentum for the system of BOTH cars plus sand:

$$\begin{aligned} 0 &= \frac{dP_{tot}}{dt} = \frac{d(Mv)}{dt} + \frac{d(M_B u)}{dt} = \frac{dM}{dt}v + M\frac{dv}{dt} + \frac{dM_B}{dt}u + M_B\frac{du}{dt} \\ &= bv + Ma - bu = 0 \\ &\Rightarrow a = b(u-v)/M \end{aligned} \tag{5}$$

Another way to solve this problem is by a very neat and underhanded trick physicists call dimensional analysis. Basically, it works by taking all the data we are given and constructing a quantity with the dimensions of the answer. This works remarkably well, but has the drawback that it cannot produce the dimensionless constant in the answer. In this case, we are looking for an acceleration $[LT^{-2}]$ and are given b in $[MT^{-1}]$, M in [M], and some velocities in $[ML^{-1}]$. The only possibility is that $[LT^{-2}] = [MT^{-1}][LT^{-1}][M^{-1}]$. Now we only have to decide which velocity to use, since we are give two, u and v. This can be solved by a special case. Here we see that if u=v, so the carts have the same speed, then in the rest frame of the carts, there is no way that A could accelerate. Therefore the velocity we need is (u-v). So we get: a = b(u-v)/M, just like before.

Problem 4.23 *(10 Points)*
4.23 *A small ball of mass m is placed on top of a superball of mass M, and the two balls are dropped to the floor from height h. How high does the small ball rise after the collision? Assume that collisions with the superball are elastic, and that $m \ll M$.*

If we think about what's happening step by step, first the superball hits the floor and bounces off, then the superball hits the small ball and the small ball is propelled away. Before the collision both balls have fallen from a height h, so by conservation of energy, $v_i = \sqrt{2gh}$. For the first collision, between the superball and the earth, the center of mass frame is the earth's frame, which is the lab frame. So when the superball bounces off the floor, its velocity reverses direction, but its speed is still $\sqrt{2gh}$. Now the small ball hits the superball, but since the superball is so much more massive, this is equivalent to the small ball hitting a wall. In the rest frame of the superball, which is equivalent to the center of mass frame of the two ball system, the small ball's speed is $2\sqrt{2gh}$, so it bounces off with the same speed in the opposite direction. Then, back in the lab frame, the small ball has velocity $3\sqrt{2gh}$ going up. By conservation of energy, this sends it to a height of $\frac{1}{2}\frac{v^2}{g} = 9h$.

Problem 4.29 *10 (Points)*
4.29 *A " superball" of mass m bounces back and forth between two surfaces with speed v_0. Gravity is neglected and the collisions are perfectly elastic.*
a. Find the average force F on each wall.
b. If one surface is slowly moved toward the other with speed $V << v$, the bounce rate will increase due to the shorter distance between collisions, and because the ball's speed increases when it bounces from the moving surface. Find F in terms of the separation of the surfaces, x.
c. Show that the work needed to push the surface from l to x equals the gain in the kinetic energy of the ball.

a. The force on the wall equals to the momentum transfered from the ball to the wall per unit time

$$F = \frac{\Delta p}{\Delta t}. \tag{6}$$

In one collition the ball changes its velocity from $+v_0$ to $-v_0$, hence the change in the momentum of the ball is $\Delta p = 2mv_0$. The ball bounces the wall again in $\Delta t = 2l/v_0$ because it has to traverse the distance l to the other wall and back. Combining the above formulae we find that the force on the wall is

$$F = m\frac{v_0^2}{l}. \tag{7}$$

b. In the rest frame of the moving wall, the ball does not change the magnitude of velocity after the bounce. It comes with relative velocity $V + v$ and it leaves with the same relative velocity. Adding to this the velocity V of the wall we see that the ball bounces back with speed $2V + v$. Hence the change in the speed of the ball is $\Delta v = 2V$. In the time $\Delta t = 2x/v$ between bounces the wall moves by $\Delta x = -\Delta t V = \frac{2Vx}{v}$. Hence the change of the speed with the decrease of the wall separation is

$$\frac{\Delta v}{\Delta x} = -\frac{v}{x}. \tag{8}$$

Because the wall is moving slowly relative the the ball, there are many bounces and we can approximate the speed by a continuum function of x and get a differential equation

$$\frac{dv}{v} = -\frac{dx}{x}, \tag{9}$$

which has the solution

$$v = v_0 \frac{l}{x}. \tag{10}$$

Substituting the new velocity v for v_0 and wall separation x for l into the result from part a. we find that the force on the moving wall is

$$F = m\frac{v_0^2 l^2}{x^3}. \tag{11}$$

c. The work done to push the wall is

$$W = -\int_l^x F dx = -\int m\frac{v_0^2 l^2}{x^3} = \frac{1}{2} m v_0^2 l^2 (\frac{1}{x^2} - \frac{1}{l^2}) \tag{12}$$
$$\tag{13}$$

where the minus sign takes into account that the decreasing separation between the walls corresponds to positive work because it is in the direction of the force F. The ball had initial velocity v_0 and final velocity $v_0 \frac{l}{x}$ so the increase in its kinetic energy is

$$\Delta K = \frac{1}{2} m v_0^2 (\frac{l^2}{x^2} - 1), \tag{14}$$

which is the same as the work done to push the wall.

Problem 6 *(10 points)*
6 *The slingshot maneuver. A spacecraft of mass m approaches a planet of mass M ($M \gg m$, of course) with velocity v, antiparallel to the planet's velocity v_p. The spacecraft comes pretty close to the planet, and it is observed that after the encounter it is moving at 90 degrees to its*

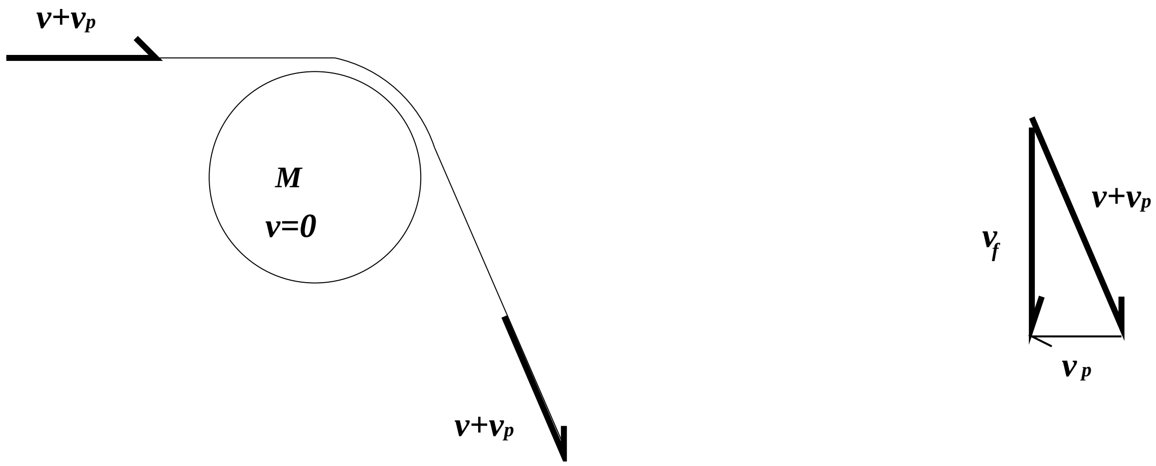

Figure 1: *Motion of the spacecraft in the rest frame of the planet.*

original line of motioin. Calculate the energy gained by the spacecraft from the planet.

A simple way to solve this problem is to study the motion of the spacecraft in the rest frame of the planet, Figure 1.

In this frame, the spacecraft is approaching the planet with relative velocity

$$\mathbf{v} = \mathbf{v} - \mathbf{v_p}, \tag{15}$$

which follows from the transformation law for velocities between frames of reference. Since $\mathbf{v}$ has opposite orientation to $\mathbf{v_p}$, the magnitudes of velocities add $v = v + v_p$. The spacecraft comes close to the planet and flies away in a different direction with the same velocity by conservation of energy. In the original frame of reference, the final velocity of the spacecraft is perpendicular to the initial velocity in the frame where the planet is moving. We transform the velocity of the spacecraft from the planet's frame to the original, Figure 1 using the transformation law Equation 22. with opposite sign in front of $\mathbf{v_p}$ because we are transforming back. The Pythagoras law gives that the final velocity of the spacecraft is

$$v_f = \sqrt{(v + v_p)^2 - v_p^2} = \sqrt{v^2 + 2vv_p}. \tag{16}$$

Hence the energy gained by the rocket in the slingshot maneuver is

$$\delta E = \frac{1}{2} m (v_f^2 - v^2) = mvv_p \tag{17}$$