Physics 105 Problem Set 3 Solutions

Problem 1. *(10 Points)*

a) A strong human cyclist, weighing about 100 kg (including bicycle), can bicycle up a 3 percent grade at about 30 km/h. What is her or his power output in watts? in horsepower?

If θ is the angle the slope makes with the flat ground, $\tan\theta = 0.03$. Thus $\sin\theta \approx 0.03$. 1 hp = 746 W.

$$P = mgv\sin(\theta) \approx 250 \text{ W} \approx 0.34 \text{ hp} \quad (1)$$

b) How does this compare to a typical human office worker climbing stairs in an office building at a rate of one floor every 20 s?

Let us assume that the mass of a typical human worker m is 65 kg. The height of an office floor is $y = 15$ feet $= 4.5$ m. $v_y = 4.5/20 = 0.225$ m/s.

$$P = mgv_y \approx 143 \text{ W} \approx 0.19 \text{ hp} \quad (2)$$

c) How many kilocalories (which are known as "calories" in nutritional information) would you have to eat every day to sustain each of these (relatively high) levels of activity for 3 hours per day? $W = P\Delta t$ where $\Delta t = 3\ h = 10800\ s$. As there are 4184 J per 1 kcal, the work in case (a) is 650 kcal and in case (b) 370 kcal.

d) A person with an active lifestyle eats 3 to 5 thousand kilocalories per day. Do you think that the production of mechanical work is the primary use of food calories? It takes only 600 kcal or so to maintain these levels of daily activity, yet we take in 3000-5000 kcal per day. Most of our energy goes **not** to mechanical work but to our metabolism (primarily maintaining our high body temperature). If al you did was sit all day, you'd still need about 2000 kcal per day. If your average power input is 2000 kcal/day, your average power output must also be 2000 kcal/day or about 100 W. Thus, if 50 students are in Jadwin 303 on a Wednesday night and it feels hot in the room, it's because there is the equivalent of a $50 \times 100 = 5$ kW heater in there!

Problem 2. *(10 Points)*

(a) The work done by the force is

$$W = \int_{path} \vec{F} \cdot \vec{ds} = \int_{0,0,0}^{x_1,0,0} F_x\, dx + \int_{x_1,0,0}^{x_1,y_1,0} F_y\, dy$$
$$= \int_{0,0,0}^{x_1,0,0} 6xy\, dx + \int_{x_1,0,0}^{x_1,y_1,0} (3x^2 - 3y^2)\, dy$$
$$= 3x_1^2 y_1 - y_1^3 \quad (3)$$

(b) Likewise:

$$W = \int_{path} \vec{F} \cdot \vec{ds} = \int_{0,0,0}^{0,y_1,0} F_y\, dy + \int_{0,y_1,0}^{x_1,y_1,0} F_x\, dx$$
$$= \int_{0,0,0}^{0,y_1,0} (3x^2 - 3y^2)\, dy + \int_{0,y_1,0}^{x_1,y_1,0} 6xy\, dy$$
$$= -y_1^3 + 3x_1^2 y_1 \quad (4)$$

Aside. Since the work done is the same along these two paths, you might suspect that **F** is a conservative force. If you have done some vector calculus, you might know that the curl of **F** is zero if and only if **F** is conservative. You can check that the curl of **F** is zero in this example.

Problem 3. *(10 Points)*

When the elevator is a distance r from the center, it is held in circular motion by the centripetal force provided by the motor:

$$\vec{F} = -\frac{mv_{circ}^2}{r}\hat{r} = -\frac{m(2\pi)^2 r}{T^2}\hat{r} \quad (5)$$

where v_{circ} is the speed of the elevator around the circle. Note that the force is directed toward the center. The circular velocity is related to the period by $v_{circ} = 2\pi r/T$. The work done is the integral over distance of this expression:

$$W = -\int_{r=R}^{r=0} \frac{m(2\pi)^2 r}{T^2} dr = 2m(R\pi/T)^2 \quad (6)$$

The work is positive as we would expect because the elevator is moving in the direction that the motor is pulling.

Problem 4. *(10 Points)*
a)$\vec{F} = -\frac{dU}{dx}\hat{x} = (4x^3 - 16x)\hat{x}$

b)$x = 0, \pm 2$ m

c)An equilibrium at x_0 is stable if U(x) is a local minimum. It is unstable is U(x) is a local maximum. In other words: stable implies $d^2U/dx^2 > 0$ and unstable if $d^2U/dx^2 < 0$. For our U(x), $d^2U/dx^2 = 16$ for $x_0 = 0$ and so this point is stable. At $x_0 = \pm 2$, $d^2U/dx^2 = -32$ and so this point is unstable. Expanding U(x) in a Taylor series about $x = x_0$ gives:

$$U(x) = U(x_0) + \frac{dU}{dx}|_{x=x_0}(x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2}|_{x=x_0}(x - x_0)^2 + \text{higher order terms} \tag{7}$$

At the minimum of the potential, dU/dx is zero and the potential, to second order, approximates a parabola. thus, the functional form is

$$U(x) = U(x_0) + \frac{1}{2}k_{eff}(x - x_0)^2 \tag{8}$$

We recall that Hooke's law is just that $U = k(x - x_0)^2/2$ and so we can identify the effective spring constant of a minimum in the potential with the second derivative of the potential.

Note that that the higher order terms in the Taylor expansion are negligible for small displacements, which is why we can stop at $(x - x_0)^2$. (It's not because the 3! in the cubic term is larger than 2!). An exception to the observation that "any potential is a harmonic oscillator near its minimum" is $U(x) = (x - x_0)^4$.

Problem 5. *(10 Points)*

The rope is pulled in with a constant velocity v, so that the radial position of the block changes linearly with time,

$$r(t) = r_0 - vt \tag{9}$$

and

$$\dot{r} = -v. \tag{10}$$

At the same time, the block is rotating with an angular velocity $\omega(t)$. While v is constant in time, ω is not necessarily. (Do you have an intuition for whether ω will increase or decrease as we pull the string in?) The net motion of the block is an inward spiral. In polar coordinates, we can separate these two components of its motion and write

$$\vec{\boldsymbol{v}}(t) = \dot{r}\hat{\boldsymbol{r}}(t) + r(t)\omega(t)\hat{\boldsymbol{\theta}}(t) = -v\hat{\mathbf{r}}(t) + r(t)\omega(t)\hat{\boldsymbol{\theta}}(t). \tag{11}$$

One more derivative of this expression will tell us the acceleration vector $\vec{\mathbf{a}}$ of the block. We must be careful taking time derivatives in polar coordinates because our coordinate system is rotating around with angular velocity $\omega(t)$. In other words,

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \omega(t)\,\hat{\boldsymbol{\theta}}(t), \text{ and } \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\omega(t)\,\hat{\boldsymbol{r}}(t) \tag{12}$$

We find,

$$\begin{aligned} \vec{\boldsymbol{a}}(t) = \frac{d}{dt}\vec{\boldsymbol{v}}(t) &= \frac{d}{dt}\left[-v\hat{\boldsymbol{r}}(t)\right] + \frac{d}{dt}\left[r(t)\omega(t)\hat{\boldsymbol{\theta}}(t)\right] \\ &= -v\frac{d\hat{\boldsymbol{r}}}{dt} + (-v\omega(t) + (r_0 - vt)\dot{\omega}(t))\hat{\boldsymbol{\theta}}(t) + r(t)\omega(t)\frac{d\hat{\boldsymbol{\theta}}}{dt} \\ &= -r(t)\omega(t)^2\hat{\boldsymbol{r}}(t) + (r(t)\dot{\omega}(t) - 2v\omega(t))\hat{\boldsymbol{\theta}}(t). \end{aligned} \tag{13}$$

We see that the radial term is just the centripetal acceleration $r\omega^2$. Since the block is moving in at a constant velocity, it adds nothing extra to the acceleration.

We are now ready to apply $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. The force is directed only radially inward and has magnitude $F(t)$. In other words,

$$\vec{\boldsymbol{F}} = -F(t)\hat{\boldsymbol{r}}(t) = m\vec{\boldsymbol{a}} = -r(t)\omega(t)^2\hat{\boldsymbol{r}}(t) + (r(t)\dot{\omega}(t) - 2v\omega(t))\hat{\boldsymbol{\theta}}(t), \tag{14}$$

from which we can read off two equations. First, since there is no force in the $\hat{\theta}$-direction, we get

$$r(t)\dot{\omega}(t) - 2v\omega(t) = 0, \tag{15}$$

or

$$\dot{\omega}(t) = \frac{2v}{r_0 - vt}\omega(t). \tag{16}$$

This is a first order differential equation, which we can solve by separating the variables and integrating. If we are careful about the limits of integration, we don't have to worry about the constant term. We get

$$\int_{\omega_0}^{\omega} \frac{d\omega'}{\omega'} = \int_0^t \frac{2v}{r_0 - vt'}dt', \tag{17}$$

or

$$\ln\left(\omega'\right)_{\omega_0}^{\omega} = -2\ln\left(r_0 - vt'\right)_0^t. \tag{18}$$

This evaluates to

$$\ln\left(\frac{\omega}{\omega_0}\right) = 2\ln\left(\frac{r_0}{r_0 - vt}\right) = \ln\left(\frac{r_0}{r_0 - vt}\right)^2. \tag{19}$$

Finally, exponentiating both sides, we conclude that

$$\boxed{\omega(t) = \left(\frac{r_0}{r_0 - vt}\right)^2 \omega_0.} \tag{20}$$

In the $\hat{\mathbf{r}}$-direction, we have less work to do. The equation we get is

$$F(t) = mr\omega^2, \tag{21}$$

which we recognize as exactly the force we need to provide the centripetal acceleration of the block. Just plugging in, we find that

$$\boxed{F(t) = mr_0\omega_0^2\left(\frac{r_0}{r_0 - vt}\right)^3.} \tag{22}$$