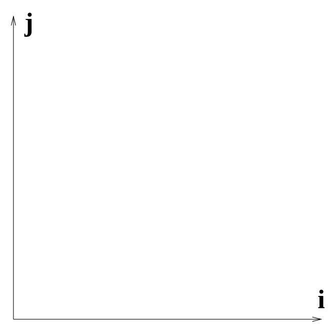
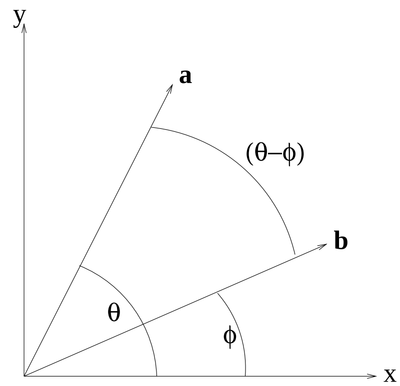
Physics 103H/105 Problem Set 1 Solutions

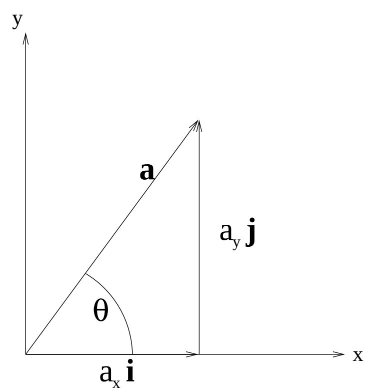
Problem 1. *(10 Points)*
$\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors in the x-y plane making angles θ and ϕ with the x-axis respectively.





$\hat{\mathbf{i}}$ is the unit vector in the x direction and $\hat{\mathbf{j}}$ is the unit vector in the y direction.

a) From vector addition we can write $\hat{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}$.



Using trigonometry, $a_x = \cos\theta$, $a_y = \sin\theta$, since the length of $\hat{\mathbf{a}}$ is unity.
Hence

$$\hat{\mathbf{a}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}},$$

as required.
If we repeat the same argument for $\hat{\mathbf{b}}$ and replace θ by ϕ, we get

$$\hat{\mathbf{b}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}.$$

To show that $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$, we use the dot product. The definition of the dot product for two arbitrary vectors $\mathbf{p}$ and $\mathbf{q}$ is

$$\mathbf{p}\cdot\mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\alpha, \tag{1}$$

where $|\mathbf{p}|$ denotes the magnitude of $\mathbf{p}$ (similarly for $\mathbf{q}$) and α is the angle between $\mathbf{p}$ and $\mathbf{q}$.

We can calculate the dot product in two different ways. Firstly we note that $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1$ and the angle between the vectors is $\theta - \phi$. This gives us

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos(\theta - \phi).$$

Secondly,

$$\begin{aligned} \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} &= (\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) \cdot (\cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}) \\ &= \cos\theta\cos\phi\,\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}}\cdot\hat{\mathbf{j}} + \cos\theta\sin\phi\,\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} + \sin\theta\cos\phi\,\hat{\mathbf{j}}\cdot\hat{\mathbf{i}}. \end{aligned}$$

Now, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors and have magnitude 1, and $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{i}} = 0$, since $\hat{\mathbf{i}} \perp \hat{\mathbf{j}}$, so if we equate the results from the two ways of calculating $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ we get

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi,$$

as required. (N.B. it doesn't matter if ϕ is bigger than θ since $\cos(\theta - \phi) = \cos(\phi - \theta)$).

Take the unit vector $\hat{\mathbf{a}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} = (\cos\theta, \sin\theta)$. We can rewrite this in the form of a 2×1 matrix (rows $\times$ columns)

$$\hat{\mathbf{a}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

b) Multiply $R(\alpha)$ by $\hat{\mathbf{a}}$.

$$\begin{aligned} R(\alpha)\,\hat{\mathbf{a}} &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \sin\alpha\cos\theta + \cos\alpha\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{pmatrix} \\ &= \hat{\mathbf{c}}. \end{aligned}$$

We have found that the matrix product of $R(\alpha)$ and $\hat{\mathbf{a}}$ is another column vector, $\hat{\mathbf{c}}$. From part a) we know that $\hat{\mathbf{c}}$ is another unit vector in the x-y plane, which makes angle $\alpha + \theta$ to the x-axis, so the effect of $R(\alpha)$ is to rotate $\hat{\mathbf{a}}$ anti-clockwise by an angle α.

Aside: Matrix Multiplication
If matrix multiplication is still a bit unclear, one way to think of it is

$$\begin{pmatrix} row\,1 \\ row\,2 \\ row\,3 \\ \ldots \\ row\,i \\ \ldots \end{pmatrix} \times \begin{pmatrix} column\,1 & column\,2 & column\,3 & \ldots & column\,j & \ldots \end{pmatrix}.$$

The entry in the i^{th} row and j^{th} column of the product matrix is the sum of the 1^{st} element in the i^{th} row by the first element in the j^{th} column with the product of the second elements in each respectively and so on. This means you can only take a matrix product between two matrices when the first has the same number of rows as the second has columns.

c) Let the product of $R(\alpha)$ and $R(\beta)$ be M, i.e. $M = R(\beta)R(\alpha)$. Note that we write $R(\beta)R(\alpha)$ rather than $R(\alpha)R(\beta)$ because we rotate by α before rotating by β. We wish to show that the combined operation of a rotation of α followed by a rotation of β is equal to M.

The product of the individual rotations is

$$\begin{aligned} R(\beta)R(\alpha) &= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -(\cos\beta\sin\alpha + \sin\beta\cos\alpha) \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & \cos\beta\cos\alpha - \sin\beta\sin\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \\ &= R(\alpha+\beta), \end{aligned}$$

where we have used the definition in part b) to get from the second last to the last line. Hence we have shown that

$$R(\alpha+\beta) = R(\beta)R(\alpha).$$

It is also true that $R(\alpha+\beta) = R(\alpha)R(\beta)$, so it doesn't matter whether we rotate by α then β or β then α, which follows our common sense. However we have to be careful with matrix products, because it isn't always true that $AB = BA$ for two matrices A and B. For example, consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

whose products are

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

If you go on with physics, you'll find that the non-commutivity of matrices plays a fundamental part in quantum mechanics, and gives a way of understanding Heisenberg's uncertainty principle.

Problem 2. *(10 Points)*
The position of the particle on the x-axis is given by

$$x(t) = 3\,\mathrm{m} \times \sin(2\,\mathrm{rad/s} \times t) + 1\,\mathrm{m},$$

with x in meters and t in seconds.

a) To calculate the velocity, note that it is the time derivative of the position,

$$\text{i.e.} \quad v(t) = \frac{d}{dt}x(t),$$

so

$$v(t) = 6\,\mathrm{m\,s^{-1}} \times \cos(2\,\mathrm{rad/s} \times t).$$

Hence

$$v(1) = 6\,\mathrm{m\,s^{-1}} \times \cos(2\,\mathrm{rad}).$$

Thus the velocity of the particle at $t = 1\,\mathrm{s}$ is v = -2.5 m s^{-1}.
(The sign of the velocity is negative since the particle is moving in the negative-x direction, the magnitude of 2.5 $\mathrm{m\,s^{-1}}$ is the speed).

b) To find where the particle is at the first time it is instantaneously at rest, take the equation for $v(t)$ and note that $v(t) = 0$ when $\cos(2t) = 0$. Hence $2t = \pi/2$, since $\pi/2$ is the first point where $\cos x = 0$ for $x > 0$, hence

$$t = \pi/4.$$

This corresponds to a position of

$$\begin{aligned} x(t) &= 3\,\mathrm{m} \times \sin(\pi/2) + 1\,\mathrm{m} \\ &= 4\,\mathrm{m} \\ y(t) &= -0.674\,\mathrm{m}. \end{aligned}$$

Alternatively, we could note that sine takes a maximum value of 1, and it is multiplied by an amplitude of 3 m, and initially $x(t)$ is increasing. Hence $x(t)$ will increase until it reaches 4 m, then it will turn around and come back. To reverse direction the particle must instantaneously be at rest, and we get the answer of 4 m.

c) To find the particle's acceleration we note that acceleration is the time derivative of the velocity,

$$\begin{aligned} a(t) &= \frac{d}{dt} v(t) \\ &= -12\,\mathrm{m\,s^{-2}} \times \sin(2\,\mathrm{rad/s} \times t). \end{aligned}$$

The time calculated in part b) is $t = \pi/4$, hence

$$a(\frac{\pi}{4}) = -12 \times \sin(\pi/2)\mathrm{m\,s^{-2}} = -12\mathrm{m\,s^{-2}}.$$

Thus the magnitude of the acceleration is 12 m s^{-2} and it is directed in the negative-x direction, so it is negative.

Problem 3. *(10 Points)*
a) In the first part of the problem, the situation is the same as the one in the question, and just like we found the velocity of the particle in O', we can also find the acceleration in O' by simply using its definition :

$$a' \equiv \frac{dv'}{dt} = \frac{d}{dt}(v - u) = \frac{dv}{dt} - \frac{du}{dt} = a.$$

since u is a constant.

Thus, under a Galilean transformation, the acceleration remains unchanged. This makes sense, because Newton's second law is the same in any inertial reference frame.

(b) Here, O' is not moving at a constant velocity with respect to O - it is moving with a constant acceleration.

O and O' coincide at $t = 0$ and O' is at rest in O at $t = 0$. Denoting the position and velocity of O' in O as $s(t)$ and $u(t)$, this means

$$s(0) = 0 \ ; \ u(0) = 0.$$

Also,

$$\frac{du}{dt} = \alpha.$$

where α is a constant. So,

$$u(t) = u(0) + \alpha t \ ,$$

and

$$s(t) = s(0) + u(0)t + \frac{1}{2}\alpha t^2$$

Thus,

$$s(t) = \frac{1}{2}\alpha t^2$$

(c) Denoting the position, velocity and acceleration of the particle in O' by x'_2, v'_2 and a'_2,

$$\begin{aligned} x'_2 &= x_2 - s = x_2 - \frac{1}{2}\alpha t^2 \ , \\ v'_2 &= \frac{dx'_2}{dt} = \frac{dx_2}{dt} - \alpha t \ , \\ a'_2 &= \frac{dv'_2}{dt} = a_2 - \alpha \ . \end{aligned} \tag{2}$$

Problem 4. *(10 Points)*
The velocity of the bead has two components. One arises from its motion along the spoke of the wheel and the other from the rotation of the wheel. When using polar coordinates to describe the motion of the bead, it is easiest to let the $\hat{\mathbf{r}}$ direction be along the spoke pointing away from the origin. Then $\hat{\theta}$ is then counterclockwise and the angular velocity is along the $\hat{\mathbf{z}}$ direction.
a)The velocity is most easily expressed in polar coordinates (r, θ) as

$$\mathbf{v} = u \ \hat{\mathbf{r}} + \omega r \ \hat{\theta}. \tag{3}$$

We also know that the velocity has the form

$$\mathbf{v} = \dot{r} \ \hat{\mathbf{r}} + r\dot{\omega} \ \hat{\theta}. \tag{4}$$

Comparing the two equations above, we see that

$$\begin{aligned} r &= ut \\ \theta &= \omega t + \pi/2 \end{aligned} \tag{5}$$

where we have used the fact that the bead was at r=0 and the spoke along the y-direction ($\theta = \pi/2$) at $t = 0$. Thus, in polar coordinates

$$\mathbf{v} = u\hat{\mathbf{r}} + ut\omega\hat{\theta}. \tag{6}$$

Inserting the values of u and ω, we have

$$\mathbf{v} = 2\hat{\mathbf{r}} + 2t\hat{\theta} \quad \text{m/s}. \tag{7}$$

To express the velocity in cartesian coordinates, we need to express the unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. These relations are

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\ \hat{\mathbf{i}} + \sin\theta\ \hat{\mathbf{j}} \\ \hat{\theta} &= \cos\theta\ \hat{\mathbf{j}} - \sin\theta\ \hat{\mathbf{i}} \end{aligned} \tag{8}$$

Thus the velocity in cartesian coordinates is

$$\mathbf{v} = (u\cos\theta - r\omega\sin\theta)\ \hat{\mathbf{i}} + (u\sin\theta + r\omega\cos\theta)\ \hat{\mathbf{j}} \tag{9}$$

Plugging in the values of r and θ from Eq. 5 and the values of u and ω, we obtain

$$\mathbf{v} = [u\cos(\omega t + \pi/2) - u\omega t\sin(\omega t + \pi/2)]\ \hat{\mathbf{i}} + [u\sin(\omega t + \pi/2) + u\omega t\cos(\omega t + \pi/2)]\ \hat{\mathbf{j}} \tag{10}$$

which gives us

$$\mathbf{v} = [-u\sin(\omega t) - u\omega t\cos(\omega t)]\ \hat{\mathbf{i}} + [u\cos(\omega t) - u\omega t\sin(\omega t)]\ \hat{\mathbf{j}} \tag{11}$$

Inserting the given values of u and ω, we get

$$\mathbf{v} = [-2\sin t - 2t\cos t]\hat{\mathbf{i}} + [2\cos t - 2t\sin t]\hat{\mathbf{j}}\text{m/s} \tag{12}$$

b) Equation 5 gives us the equations for r and θ. When the values of u and ω are inserted, we find:

$$\begin{aligned} r &= 2t \text{ m} \\ \theta &= t + \pi/2 \text{ rad} \end{aligned} \tag{13}$$

Since the cartesian coordinates x and y are related to r and θ by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned} \tag{14}$$

we find that

$$\begin{aligned} x &= ut\cos(\omega t + \pi/2) = -2t\sin t \quad \text{m} \\ y &= ut\sin(\omega t + \pi/2) = 2t\cos t \quad \text{m}. \end{aligned} \tag{15}$$

Problem 5. *(no Points)* We are given $l = 10\,\text{cm}$, $D = 50\,\text{cm}$ and $v = 10\,\text{cm}\,\text{s}^{-1}$. We wish to calculate the values of ω for which the stick will hit the wall "flat-on". The first thing we note is that the time for the stick to reach the well will be $t = D/v = 5\,\text{s}$.

If $\omega = 0$, then the stick does not spin, and hence clearly hits the wall flat on.

If $\omega \neq 0$ then there are two conditions that need to be met. One is that the stick perform an integer multiple of half a rotation in the time it takes to reach the wall, this is the condition that ensures that the stick is "flat-on" when it hits the wall.

The total angle swept out by the stick as it rotates is ωt, hence we have

$$\omega t = \frac{n}{2}(2\pi),$$

where n is an integer. Using our equation for t, our first condition is

$$\omega = \frac{n\pi v}{D} = \frac{n\pi}{5}\,\text{rad/s}. \tag{16}$$

The second condition we obtain from requiring that the stick cannot go through the wall (i.e. we have a real wall). If any part of the rod is moving to the left when the rod reaches the wall, that means that it must have been beyond the wall just before impact. This is non-physical, so we have to make sure that it doesn't occur. Let's consider one of the ends of the rod, since if any point goes through the wall, these will be the ones that do. The motion of these points has two contributions, linear and rotational motion. To make sure that the rod doesn't go through the wall, we need their sum to be greater than zero, that is

$$v - \frac{\omega l}{2} > 0.$$

Hence

$$v > \frac{\omega l}{2},$$

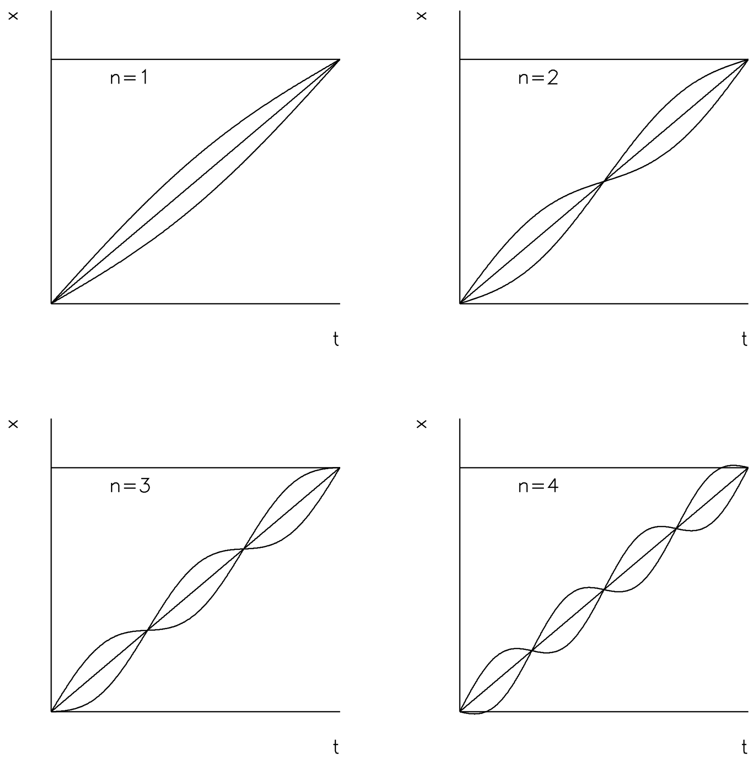
which implies

$$\omega < \frac{2v}{l} = 2\,\text{rad/s}. \tag{17}$$

Comparing (16) and (17), we need to satisfy both at the same time, which gives us solutions of

$$\omega_n = \frac{n\pi}{5}\,\text{rad/s}, n \in \{0, \pm 1, \pm 2, \pm 3\}. \tag{18}$$

The positions of the center of the rod and the ends for varying values of n are shown below – if we had $n = 4$, then one end of the rod would go through the wall before it hit the wall flat-on.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