Supersymmetry (CMMS40)

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Contents

1 Introduction **1**

2 Coleman-Mandula Theorem **4**

3 Spinors and Clifford Algebras **9**
3.1 Clifford Algebras 10
3.2 Majorana spinors 13
3.3 Spinor Inner Product 15
3.4 Fierz re-arrangement 17

4 Elementary Consequences of Supersymmetry **19**
4.1 Two Component Notation 23
4.2 Witten Index 25

5 4d Wess-Zumino Model **26**
5.1 Interactions 31

6 From Super Maxwell to Super Yang-Mills with Matter **34**
6.1 Maxwell Field 35
6.2 Super Yang-Mills 37
6.3 Super Yang-Mills with Matter 42

7 Scale Dependence of Super Yang-Mills with Matter **45**
7.1 The Theta Parameter and Non-Renormalization 50
7.2 Anomalies 53
7.3 Higgs vs. Coulomb vs. Confinement 55

8 The Low Energy Behavior of SQCD **57**
8.1 The Classical Moduli Space 57

A Sources **62**

1 Introduction

Why study supersymmetry? There are two sets of arguments. The first is experimental: For many years people have held out hope that next generation particle accelerators (or indeed other high energy particle experiments such as cosmic ray detectors, dark matter detectors, neutrino detectors, precision electroweak experiments, etc.) would see evidence for supersymmetry in the world around us. The second is theoretical: the presence of a symmetry in physics often helps in solving a problem, and supersymmetry is no exception.

The experimental set of arguments concerns what has come to be known as the Standard Model of Particle Physics. This relativistic quantum field theory describes essentially everything that we have observed in nature that is not gravitational. It postulates that the world around us is made of particles. In particular, the building blocks are fermionic spin 1/2 particles – electrons, muons, taus, neutrinos, and quarks – which interact by exchanging bosonic vector particles – gluons, W and Z bosons, and photons. The Standard Model is a gauge theory, which means it has a local continuous symmetry described by a Lie Group, in this case $SU(3)\times SU(2)\times U(1)$. Only an unbroken U(1) is observed at low energies, the U(1) associated with the photon of electricity and magnetism. A last critical ingredient is thus to explain the symmetry breaking pattern. The fact that we don't observe an SU(3) at low energies is associated with the imperfectly understood physics of confinement in quantum chromodynamics (QCD). The breaking of $SU(2)\times U(1)$ to a diagonal U(1) on the other hand is associated with a last critical ingredient of the Standard Model: the Higgs particle, a spin zero bosonic particle.

Despite its successes, there are a few key unsatisfactory aspects of the Standard Model:

- *Hierarchy Problem:* From a modern standpoint, the Standard Model is an effective field theory – something that accurately describes the physics at the relatively low energies available in today's particle accelerators. The mass of the heaviest observed fundamental particle, the top quark at 172 GeV, gives an order of magnitude estimate of the energy scales at and below which the Standard Model can be trusted to give accurate results. In contrast, we have no reason to expect the Standard Model to be accurate if extrapolated to very high energy scales, for example the Planck scale $E_P = \hbar^{1/2}G_N^{-1/2}c^{5/2} \sim 10^{19}$ GeV at which quantum gravitational affects are expected to become important. A symptom of the Standard Model's limitations are divergences that appear in loop corrections to the mass of the Higgs. A naive but standard way of regulating these divergences is to cut-off the integration at an energy scale where we expect new physics. If that scale is really E_P, then mass corrections will be huge, and the Higgs mass should be of the same order of magnitude as E_P. Through the Higgs mechanism, the other Standard Model particles will get huge masses as well. Of course, we do not observe such huge masses, and so, without some fine tuning that will arrange for cancellation between the various diagrams, there must be new physics at some lower scale. Supersymmetry provides for precisely such new physics, introducing a new class of particles that can run in loops and partially cancel these large corrections to the Higgs mass.

  The current experimental situation is not promising for supersymmetry. The LHC has observed the Higgs to have a mass of 125 GeV, but has not observed any supersymmetric partners. 125 GeV is relatively low, but the new physics is coming in at a relatively high scale, naively at least several hundred GeV, where the loop cancellations will not be particularly effective.

- *Unification:* The existence of three gauge groups $SU(3)\times SU(2)\times U(1)$ has long seemed inelegant to theorists. How much nicer would it be if the Standard Model could be embedded in a theory with a single gauge group, for example $SU(5)$, $SO(10)$, or even E_8. Giving some support to this idea is that the gauge couplings for the three gauge groups run with energy scale and become all roughly equal at 10^{15} GeV. Supersymmetry has the remarkable property of making the three couplings much closer together at this unification scale. In addition, it provides an accurate prediction for the Weinberg angle, i.e. the way in which the U(1) of the photon sits inside the original $SU(2) \times U(1)$ gauge group.

- *Dark Matter:* Roughly 70% of the matter in the universe is not particles in the Standard Model. Astrophysicists have come to this conclusion from a variety of observations, for example from looking at rotation of individual galaxies, rotation of clusters of galaxies, and the cosmic microwave background radiation. The new class of particles introduced by supersymmetry provide a host of dark matter candidates, the most serious of which is often called the LSP, the lightest super partner.

The second set of arguments for supersymmetry is that it helps solve various problems. Maybe the real question we are interested in does not involve supersymmetry, but if we add supersymmetry, we can often find solutions and then hopefully learn something about what to expect in answer to the original question.

- *String Theory:* Large energy (or equivalently short distance) divergences are a generic problem in quantum field theory. The hierarchy problem provides a case study, but the issue is larger. Intuitively, the problem is that point-like particles of relativistic quantum field theories are singular objects. The self energy of a charged point particle is infinite, for example. A theory of strings is somewhat less singular. An issue with non-supersymmetric versions of string theory however, is that we have not been able to find stable vacuum states. Supersymmetry cures this problem and indeed also provides a possible framework in which to unify the Standard Model (open strings) with gravity (closed strings).

- *Confinement:* You can make yourself a million dollars in the Clay Mathematics Prize Competition if you successfully explain why Yang-Mills theory (i.e. QCD without the quarks) develops a mass gap at low energies. Add supersymmetry, and the problem becomes much simpler. The vacuum structure of a very large variety of supersymmetric gauge theories has at this point been successfully analyzed, giving us some insight into the original problem of confinement in QCD.

- *Partition Functions and Localization:* The basic problem of quantum field theory is to compute the path integral (or partition function). In supersymmetric theories, this path integral can sometimes be computed exactly on special manifolds, for example spheres. Indeed, one can go further and compute correlation functions of certain supersymmetric operators as well.

Problem 1.1. *Using only the quantities $\hbar$, G_N, and c, construct quantities that have the units of length, mass, and time. Compute the corresponding Planck length, Planck mass, and Planck time, using SI units.*

Problem 1.2. *Another proposed source of extra physics is extra dimensions. Assume that we live not in a four dimensional world but a $(4+p)$-dimensional one where the extra dimensions are all extremely small circles of length ℓ.*

a) *Noting that the dimensionality of G_N is different in $(4+p)$ dimensions, what is the new expression for the Planck energy E_P in terms of $\hbar$, c, and G_N?*

b) *Find a relationship between G_N and the observed 4d value G_N^{4d}. Given the observed 4d value for G_N^{4d}, how small must ℓ be in order to have $E_P = 1$ TeV? Are there some values of p that you can rule out?*

2 Coleman-Mandula Theorem

Symmetry plays a critical role in quantum field theory, and we often distinguish several different types. There are gauge symmetries – the $SU(3) \times SU(2) \times U(1)$ of the standard model for instance. There are global symmetries; consider the approximate SU(2) flavor symmetry of the up and down quarks. There are discrete symmetries, for example charge conjugation C, parity P, and time T reversal. Most important of all, perhaps, are the space-time symmetries of special relativity, also known as the Poincaré group. After all, relativistic quantum field theories were developed out of an intent to wed quantum mechanics and special relativity.

Given the prominence of the Poincaré group in relativistic quantum field theory, one is led to ask whether this group might in certain contexts be a subgroup of some larger group. The contexts in which the Poincaré group can be enlarged turn out to be surprisingly limited. There is in fact a theorem, proven in 1967 by Coleman and Mandula, that the Poincaré group can be combined with internal, continuous symmetries, such as the $SU(3)$ of the standard model, in only a trivial way, as a direct product. In other words, if one takes an element g from the Poincaré group and an element h from a continuous internal symmetry group, then $gh = hg$.

There are various interesting exceptions to the theorem. The proof involves the scattering or S matrix, and if the theory contains only massless particles, for which the S matrix is a somewhat problematic concept, the Poincaré group can be enlarged to the conformal symmetry group. Furthermore, discrete symmetries are not included, nor are ones that are spontaneously broken. The biggest and most interesting loop hole, however, is supersymmetry. In this section, we will review some elementary facts about the Poincaré group and continuous internal symmetry groups, and then discuss how supersymmetry evades the Coleman-Mandula theorem.

The Poincaré group is a Lie group that is generated by space-time translations along with Lorentz transformations (which in turn consist of rotations and boosts). The infinitesimal version (or Lie algebra version) of this group action, under which the theory is invariant, can be written

$$x^\mu \to x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu \ , \tag{2.1}$$

where the quantity $\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$ is taken to be small.

In special relativity, the space-time proper distance $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ between two points must be invariant under these transformations, which in turn places a constraint on $\omega^\mu{}_\nu$:

$$\begin{aligned}
\Delta s^2 &\to \eta_{\mu\nu}(\Delta x^\mu + \omega^\mu{}_\lambda \Delta x^\lambda)(\Delta x^\nu + \omega^\nu{}_\rho \Delta x^\rho) \\
&= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu + \eta_{\mu\nu}\omega^\mu{}_\lambda \Delta x^\lambda \Delta x^\nu + \eta_{\mu\nu}\omega^\nu{}_\rho \Delta x^\mu \Delta x^\rho + \ldots \\
&= \Delta s^2 + (\omega_{\mu\nu} + \omega_{\nu\mu})\Delta x^\mu \Delta x^\nu + \ldots \;. \qquad (2.2)
\end{aligned}$$

In other words, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is antisymmetric under exchange of its indices.[1]

While elements of the Poincaré group compose to give new elements in the group, the infinitesimal version of this statement is that the commutator of two infinitesimal elements (i.e. elements of the corresponding Lie algebra) yields a new infinitesimal element. We consider infinitesimal elements δ_1 and δ_2 and compute

$$[\delta_1, \delta_2]x^\mu \equiv \delta_1\delta_2 x^\mu - \delta_2\delta_1 x^\mu \;. \qquad (2.3)$$

To compute $\delta_2\delta_1 x^\mu$, it is perhaps clearer to start with the arrow notation

$$\begin{aligned}
x^\mu &\to x^\mu + a_1^\mu + \omega_{1\nu}^\mu x^\nu \\
&\to x^\mu + a_1^\mu + \omega_{1\nu}^\mu x^\nu + a_2^\mu + \omega_{2\nu}^\mu (x^\nu + a_1^\nu + \omega_{1\lambda}^\nu x^\lambda) \;,
\end{aligned}$$

from which it follows that

$$\delta_2\delta_1 x^\mu = \omega_2{}^\mu{}_\nu a_1^\nu + \omega_2{}^\mu{}_\lambda \omega_1{}^\lambda{}_\nu x^\nu \;. \qquad (2.4)$$

The commutator then must be

$$[\delta_1, \delta_2]x^\mu = (\omega_1{}^\mu{}_\lambda a_2^\lambda - \omega_2{}^\mu{}_\lambda a_1^\lambda) + (\omega_1{}^\mu{}_\lambda \omega_2{}^\lambda{}_\nu - \omega_2{}^\mu{}_\lambda \omega_1{}^\lambda{}_\nu) x^\nu \;. \qquad (2.5)$$

The new infinitesimal Poincaré transformation is

$$a^\mu = \omega_1{}^\mu{}_\lambda a_2^\lambda - \omega_2{}^\mu{}_\lambda a_1^\lambda \;, \quad \omega^\mu{}_\nu = \omega_1{}^\mu{}_\lambda \omega_2{}^\lambda{}_\nu - \omega_2{}^\mu{}_\lambda \omega_1{}^\lambda{}_\nu \;. \qquad (2.6)$$

Note that $\omega_{(\mu\nu)} = \frac{1}{2}(\omega_{\mu\nu} + \omega_{\nu\mu}) = 0$, consistent with the requirement that Δs^2 is invariant.

We would like to be able to act not just on space-time points x^μ with the Poincaré group but on quantum fields as well. To that end, we introduce the linear operators P_μ and $M_{\mu\nu}$ which act on the coordinates such that

$$\delta x^\mu = i a^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda} M_{\nu\lambda}(x^\mu) \;. \qquad (2.7)$$

[1] We will use a Minkowski metric with mostly plus signature:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \;.$$

The factor of 1/2 is introduced because of the anti-symmetry so that, for example, $\omega_{12} = -\omega_{21}$ is only counted once. The factors of i allow the generators to be Hermitian rather than anti-Hermitian operators. However, we would like to be able to act on more general representations of the Poincaré group, in particular fields which are also functions of x^μ, $\Phi_I(x^\mu)$. Here I is some generalized index allowing for an arbitrary representation. The commutator (2.5) can be written more abstractly as

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \, , \\ [P_\mu, M_{\nu\lambda}] &= i\eta_{\mu\nu}P_\lambda - i\eta_{\mu\lambda}P_\nu \, , \\ [M_{\mu\nu}, M_{\lambda\rho}] &= i\eta_{\mu\lambda}M_{\nu\rho} - i\eta_{\nu\lambda}M_{\mu\rho} - i\eta_{\mu\rho}M_{\nu\lambda} + i\eta_{\nu\rho}M_{\mu\lambda} \, . \end{aligned} \tag{2.8}$$

Problem 2.1. *Reproduce the result (2.5) using P_μ and $M_{\nu\lambda}$ and in particular (2.7) and the commutator algebra (2.8).*

In general, we would like to be able to represent the action of P_μ and $M_{\mu\nu}$ not just on x^μ but on a quantum field $\Phi_I(x^\mu)$ which transforms under a representation of Poincaré and is additionally a function of a space-time point. Here I is some generalized index. An infinitesimal group element of Poincaré g consisting of the data $(a_\mu, \omega_{\mu\nu})$ and acting on $\Phi_I(x^\mu)$ thus has two pieces, one g^{IJ} acting by matrix multiplication on the generalized index of the field I and the second acting on x^μ,

$$\delta\Phi_M(x^\mu) = g^{IJ}\Phi_I(x^\mu) + \Phi_I(x^\mu + \delta x^\mu) - \Phi_I(x^\mu) \, . \tag{2.9}$$

By a Taylor series, we can write the second two terms, to leading order, as a derivative

$$\Phi_I(x^\mu + \delta x^\mu) - \Phi_I(x^\mu) = (a^\mu + \omega^\mu{}_\nu x^\nu)\partial_\mu \Phi_I(x^\mu) \, . \tag{2.10}$$

Now it turns out that g^{IJ} simplifies as well and depends only on the Lorentz part of the Poincaré group. Because of the nontrivial commutator $[P_\mu, M_{\nu\lambda}]$, the Poincaré group is not a direct but a semi-direct product of translations and Lorentz transformations. Translations by themselves are straightforward to understand. They form an abelian and non-compact subgroup of the full group. Their irreducible representations are always one dimensional, and the corresponding matrices just constants. In fact, as far as I'm aware, for fields of physical interest, these constants always vanish. For example, for tensor fields, shifting the location of the origin of spacetime clearly should not affect the structure of the tangent and cotangent bundles, leaving the space-time indices on some general tensor field $T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}$ invariant.

The nontrivial data in g^{IJ} is then a representation of the Lorentz algebra only, and $P_\mu = -i\partial_\mu$ reduces to a derivative acting on the fields, controlling how the shift in x^μ in turn affects the field Φ_I. Smooth functions can be expanded in terms of a Taylor series:

$$\begin{aligned} f(x+a) &= f(x) + a^\mu \partial_\mu f(x) + \ldots \\ &= f(x) + i a^\mu P_\mu f(x) + \ldots \end{aligned} \tag{2.11}$$

Finite translations can be obtained as an exponential of P_μ:

$$\begin{aligned} f(x+a) &= e^{ia^\mu P_\mu} f(x) \\ &= f(x) + a^\mu \partial_\mu f(x) + \frac{1}{2} a^\mu a^\nu \partial_\mu \partial_\nu f(x) + \ldots \end{aligned} \tag{2.12}$$

The Lorentz group is non-abelian and admits more interesting representations. The Standard Model that we discussed briefly in the first section contains a Higgs field $H(x)$ in the trivial representation, vector fields such as the photon $A_\mu(x)$, and many spinor fields, such as the electron $\psi_\alpha(x)$. In general, a nontrivial representation of the Lorentz group implies that the field carries some kind of index, for example μ and α for the vector and spinor fields respectively. Different representations imply that there are different choices of matrices which satisfy the commutation relations (2.8) of the Poincaré group.

Problem 2.2. *For a vector representation, one takes*

$$(M_{\mu\nu})^\lambda{}_\rho = i\eta_{\mu\rho}\delta^\lambda_\nu - i\delta^\lambda_\mu\eta_{\nu\rho} \ . \tag{2.13}$$

(Notice that the indices μ and ν take a dual role, labeling both the Lorentz generator and its matrix components.) For the spinor representation, one takes instead

$$(M_{\mu\nu})_\alpha{}^\beta = -\frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)_\alpha{}^\beta \ , \tag{2.14}$$

where $(\gamma_\mu)_\alpha{}^\beta$ are the Dirac γ-matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Verify that these two representations of the Lorentz group obey the commutation relations (2.8).

Quantum field theories often possess additional symmetries, most notably gauge symmetries. Associated with the gauged Lie group, there is a Lie algebra with commutation relations of the form

$$[T_a, T_b] = i f_{ab}{}^c T_c \ , \tag{2.15}$$

where the T_a are Hermitian generators, and $f_{ab}{}^c$ are the structure constants. The fields transform in representations of this algebra and carry associated indices. For example, the quarks ψ^a_α in the standard model in addition to a spinor index α carry an index a indicating that they transform in a fundamental representation of $SU(3)$.

The component P_t is both an energy and also a generator of infinitesimal translations in time. Because P_t exists as a well defined, time independent quantity, we expect that the total energy is conserved. Often a good first step in approaching a physics problem is to work out a complete set of conserved charges. In the context of our commutator algebra of P_μ, $M_{\mu\nu}$ and T_a, the set of conserved charges is the set which commutes with P_t. In the context of the Poincaré group, we expect the full four momentum P_μ to be conserved, along with angular momenta corresponding to M_{xy}, M_{yz}, and M_{zx}. The boosts M_{ti} on the other hand do not commute with P_t. Having written down the full set, as is typical in quantum mechanics one has to worry about whether the generators mutually commute as well. Otherwise, the operators will not all be simultaneously diagonalizable. In the context of spatial rotations, for example, one typically chooses $J_z = M_{xy}$ and the Casimir operator $J^2 = M_{xy}^2 + M_{yz}^2 + M_{zx}^2$.

From Noether's theorem, we expect that continuous symmetries are associated with conserved charges and more generally conserved currents. It should follow from Noether's theorem that $[P_t, T_a] = 0$. The content of the Coleman-Mandula theorem is much stronger, that the generators T_a commute with all of the generators of the Poincaré group:

$$[T_a, P_\mu] = 0 = [T_a, M_{\mu\nu}] \ . \tag{2.16}$$

Thus the T_a are not only conserved but transform under the trivial representation of the Poincaré group.

Theorem. (Coleman-Mandula) *In any spacetime dimension greater than two, the only interacting quantum field theories have Lie algebra symmetries which are a direct product of the Poincaré algebra with an internal symmetry.*

Supersymmetry requires the violation of a key assumption of the Coleman-Mandula theorem – that the symmetry algebra should be a Lie algebra. Recall that a Lie algebra is the tangent space at the identity element of a Lie group, with an infinitesimal group transformation of the form

$$g = 1 + i\epsilon A \tag{2.17}$$

where A is an element of the Lie algebra (e.g. P_μ, $M_{\mu\nu}$ or T_a from before) and ϵ is an infinitesimal parameter. The algebra is closed under an antisymmetric bilinear operation called the Lie bracket

$$[A, B] = -[B, A] \tag{2.18}$$

which is subject to the Jacobi identity

$$[A, [B, C]] + [B, [C, A] + [C, [A, B]] = 0 \ . \tag{2.19}$$

Supersymmetry replaces the Lie algebra with a superalgebra. A superalgebra should already be a familiar notion to you if you have worked with a quantum field theory that contains both fermions and bosons. By the spin statistics theorem, bosons carry representations of the Lorentz group with integer spin and their field operators must commute outside of the light cone. On the other hand, fermions carry half-integer spin representations and anti-commute outside the light cone. A standard Lie algebra can be constructed from bosonic generators and commutators $[,]$, but once we involve fermions, it is very natural to throw anti-commutators $\{,\}$ into the mix along with the rule that the product of two fermions is a boson, the product of a fermion and a boson is a fermion, and the product of two bosons is again a boson.

We can formalize this notion of a superalgebra as a $\mathbb{Z}_2$ graded Lie algebra where fermions have odd grading and bosons have even grading. It is also convenient to write a generalized commutator $[,\}$ where the decision to anti-commute or commute is based on what is inside the brackets:

$$[B, B\} = [B, B'] \quad \sim \quad B'' \ , \tag{2.20}$$

$$[B, F\} = [B, F] \quad \sim \quad F' \ , \tag{2.21}$$

$$[F, F'\} = \{F, F'\} \quad \sim \quad B \ . \tag{2.22}$$

B here is for boson and F for fermion, and the notation is schematic. There is furthermore a generalized Jacobi identity

$$(-1)^{ac}[A, [B, C\}\} + (-1)^{ba}[B, [C, A\}\} + (-1)^{cb}[C, [A, B\}\} = 0 \ . \tag{2.23}$$

where $a, b, c \in \mathbb{Z}_2$ are the gradings of A, B, and C respectively.

In this course, the even generators will be the P_μ and $M_{\mu\nu}$ generators of the Poincaré group. The odd generators, or supersymmetries Q_α, are then in a sense the square roots of the Poincaré generators, schematically

$$\{Q, Q\} = P + M \ . \tag{2.24}$$

There is thus a symmetry that is "deeper" than Poincaré and is surely therefore worthy of study.

One more comment needs to be made. While we have found a way to nontrivially enlarge the Poincaré algebra, the supercharges Q_α still generally commute with other internal continuous symmetry generators $[Q_\alpha, T_a] = 0$. This refined version of the Coleman-Mandula theorem is due to Haag, Sohnius, and Lopuszanszki and was proven in 1975.

Problem 2.3. *There is a loop hole in the Coleman-Mandula Theorem associated with theories where all the particles are massless, essentially because of difficulties in defining an S-matrix for massless particles. In this case, the Poincaré algebra is extended to the conformal algebra:*

$$\begin{gathered}[D, P_\mu] = iP_\mu \ , \quad [D, K_\mu] = -iK_\mu \ , \quad [D, M_{\mu\nu}] = 0 \ , \quad [K_\mu, K_\nu] = 0 \ , \\ [M_{\mu\nu}, K_\rho] = i(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu) \ , \quad [P_\mu, K_\nu] = -2i(\eta_{\mu\nu}D + M_{\mu\nu}) \ .\end{gathered}$$

where K_μ generate so-called special conformal transformations and D is the dilatation operator.

a) Compute the commutator of P^2 with K_μ and D. What happens to a massive particle state $|p\rangle$ (where $P^2|p\rangle = m^2|p\rangle$, $m^2 \neq 0$) under the infinitesimal special conformal transformation K_μ?

b) If $\mu, \nu = 0, \ldots, d-1$, then define $J_{\mu\nu} = M_{\mu\nu}$ along with $J_{\mu,d} = \frac{1}{2}(P_\mu - K_\mu)$, $J_{\mu,d+1} = \frac{1}{2}(P_\mu + K_\mu)$, and $J_{d,d+1} = D$, along with the constraint that $J_{ab} = -J_{ba}$ is antisymmetric. Show that the commutators of these generators are the same as for a (d+2)-dimensional orthogonal group, with metric signature $(2, d)$, i.e. $SO(2, d)$.

3 Spinors and Clifford Algebras

We must first master the formalism necessary to describe spinors and fermions. We can attribute much of this formalism to Dirac, who had the insight that the Dirac equation should be a kind of square root of the Klein-Gordon equation:

$$(\gamma^\mu \partial_\mu - m)\psi = 0 \ . \tag{3.1}$$

Acting on the left with $(\gamma^\mu \partial_\mu + m)$, one finds

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - m^2)\psi = 0 \ . \tag{3.2}$$

This second equation is equivalent to the Klein-Gordon equation,

$$(\partial^2 - m^2)\psi = 0 \ , \tag{3.3}$$

provided

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \ . \tag{3.4}$$

From this innocuous looking anti-commutation relation follows an intricate structure that depends sensitively on the space-time dimension – the Clifford algebra and its representations.[2] The matrix γ^μ has a vector index that we can lower and raise using the metric $\eta_{\mu\nu}$ and its inverse.

3.1 Clifford Algebras

Introducing fermions ψ requires also introducing a set of γ-matrices. The choice of ψ and the associated γ^μ furnish a representation of the Clifford algebra. Generically, we take the representation to be over the complex numbers. For now, we suppress the spinor indices $\alpha, \beta, \ldots$ on ψ and γ^μ. Thus when we write

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \tag{3.5}$$

there is an implicit identity matrix in spinor space $\mathrm{id}^\alpha{}_\beta$ on the right hand side.

Let us begin with the even dimensional case $d = 2k + 2$. We group the gamma matrices into $k + 1$ pairs of anti-commuting raising and lowering operators

$$\gamma^{0\pm} = \frac{1}{2}(\pm\gamma^0 + \gamma^1) \ , \tag{3.6}$$

$$\gamma^{a\pm} = \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1}) \ , \quad a = 1, \ldots, k \ . \tag{3.7}$$

Problem 3.1. *Show that these linear combinations satisfy the relations*

$$\begin{aligned} \{\gamma^{a+}, \gamma^{b-}\} &= \delta^{ab} \ , \\ \{\gamma^{a+}, \gamma^{b+}\} &= \{\gamma^{a-}, \gamma^{b-}\} = 0 \ . \end{aligned} \tag{3.8}$$

In particular, note that $(\gamma^{a+})^2 = 0 = (\gamma^{a-})^2$. By repeatedly acting with the $k + 1$ γ^{a-} on a spinor, we can eventually reach a lowest weight state ζ such that

$$\gamma^{a-}\zeta = 0 \text{ for all } a \ . \tag{3.9}$$

Starting from ζ and acting with the raising operators γ^{a+}, at most once each, we can obtain all of the $2^{k+1} = 2^{d/2}$ states in the representation. The states can be labeled $\mathbf{s} = (s_0, s_1, \ldots, s_k)$, where each of the $s_a = \pm\frac{1}{2}$:

$$\zeta^{(\mathbf{s})} \equiv (\gamma^{k+})^{s_k+1/2} \cdots (\gamma^{0+})^{s_0+1/2}\zeta \ . \tag{3.10}$$

[2] Clifford became a student at KCL in 1860, at the tender age of 15. He later was elected a fellow at Trinity College, Cambridge in 1868. After surviving a shipwreck along the Sicilian coast during a voyage to observe the solar eclipse of December 1870, he started work as a professor mathematics and mechanics at UCL. He suffered a pair of nervous breakdowns, perhaps due to overwork, and succumbed to tuburcolosis in 1879, at the age of 33. In the *Ethics of Belief*, he wrote “It is wrong always, everywhere, and for anyone, to believe anything upon insufficient evidence.”

The lowest weight state ζ corresponds to all $s_a = -\frac{1}{2}$.

Taking the $\zeta^{(\mathbf{s})}$ as a basis, we derive the matrix elements of γ^μ from the definitions and the anti-commutation relations. Starting in $d = 2$, we find

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3.11}$$

The 2×2 matrices are chosen to obey the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ but additionally such that the states in the representation take the simple form where

$$\zeta^{(-1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \zeta^{(+1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{3.12}$$

Note that the 2×2 matrices that appear are related to two of the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.13}$$

In particular, $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$. This relation is not surprising since these matrices give a three dimensional Euclidean representation of the Clifford algebra

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} . \tag{3.14}$$

Increasing d by two doubles the size of the γ-matrices. Given a representation Γ^μ in $2k$ dimensions, we can construct a representation γ^μ in $2k+2$ dimensions using the prescription,

$$\begin{aligned} \gamma^\mu &= \Gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\Gamma^\mu \otimes \sigma^3 , \quad \mu = 0, \ldots, d-3 , \\ \gamma^{d-2} &= \mathrm{id} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{id} \otimes \sigma^1 , \quad \gamma^{d-1} = \mathrm{id} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathrm{id} \otimes \sigma^2 . \end{aligned} \tag{3.15}$$

The 2×2 matrices that we add act on the index s_k, which newly appears in going from $2k$ to $2k + 2$ dimensions. (In what follows, we will set $d = 2k + 2$.)

The basis choice is not unique. There are many ways of constructing this $2^{d/2}$ dimensional representation of a Clifford algebra. We claim, however, that they are all equivalent up to an appropriate unitary transformation, $\gamma^\mu \to U\gamma^\mu U^{-1}$. In four dimensions, for instance, the construction above leads to the γ-matrices

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} , \\ \gamma^2 &= \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} , \quad \gamma^3 = \begin{pmatrix} 0 & -i\,\mathrm{id} \\ i\,\mathrm{id} & 0 \end{pmatrix} . \end{aligned} \tag{3.16}$$

A different, more popular choice of basis in four dimensions, often found in field theory text books, is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\bar{\sigma}^\mu & 0 \end{pmatrix} , \tag{3.17}$$

where $\sigma^\mu = (-\,\mathrm{id}, \sigma^i)$ and $\bar{\sigma}^\mu = (-\,\mathrm{id}, -\sigma^i)$.

Problem 3.2. *Demonstrate a unitary transformation U that relates the representations (3.16) and (3.17). See if you can choose a basis such that the 4d gamma matrices are purely real, a so-called "really real" representation.*

The representation $\zeta^{(\mathbf{s})}$ of the Clifford algebra is also a representation – the so-called Dirac spinor representation – of the Lorentz group. In an earlier exercise, we demonstrated that the Lorentz generators can be written as

$$M^{\mu\nu} = -\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \ . \tag{3.18}$$

The generators $M^{2a,2a+1}$ commute and can be simultaneously diagonalized to give the weights of the representation. (Each of the $M^{2a,2a+1}$ operators functions like a J_z angular momentum operator in quantum mechanics.) In terms of our raising and lowering operators, we have

$$S_a \equiv i^{\delta_{a,0}} M^{2a,2a+1} = \gamma^{a+}\gamma^{a-} - \frac{1}{2} \ . \tag{3.19}$$

In this way $\zeta^{(\mathbf{s})}$ is a simultaneous eigenstate of the S_a with eigenvalues s_a. The spinors $\zeta^{(\mathbf{s})}$ thus form the 2^{k+1} dimensional spinor representation of the Lorentz algebra $so(2k+1,1)$.

While the representation $\zeta^{(\mathbf{s})}$ is irreducible as a representation of the Clifford algebra, it is in general not irreducible as a representation of the Lorentz group. Because the Lorentz generator $M^{\mu\nu}$ is quadratic in the γ-matrices, it can never flip an odd number of spins when acting on $\zeta^{(\mathbf{s})}$. Thus the states with even and odd numbers of $-1/2$ spins will not mix under the action of $M^{\mu\nu}$, and the Dirac representation falls apart into two smaller representations in an even number of space-time dimensions.

In fact, we can construct a matrix, the analog of "gamma five" in four dimensions, to help perform this decomposition. The matrix detects the "chirality" of the state, i.e. the parity of the number of down spins, and commutes with $M^{\mu\nu}$. This matrix is a product of all the other gamma matrices:

$$\gamma \equiv i^{-k}\gamma^0\gamma^1\cdots\gamma^{d-1} \ . \tag{3.20}$$

Problem 3.3. *Show that in terms of the S_a operators, we can write*

$$\gamma = 2^{k+1} S_0 S_1 \cdots S_k \ . \tag{3.21}$$

As a result, it is clear that γ is diagonal in our $\zeta^{(\mathbf{s})}$ basis, taking the eigenvalue $+1$ when there are an even number of $-\frac{1}{2}$ spins and -1 for an odd number of $-\frac{1}{2}$ spins. The states with eigenvalue $+1$ form a Weyl representation of the Lorentz algebra, while the states with eigenvalue -1 form a second, inequivalent Weyl representation. The eigenvalue of γ is often called the *chirality* of the representation.

The matrix γ performs a second key function by allowing us to construct representations of the Clifford algebra in odd dimensions. We simply use γ as the d^{th} gamma matrix, as it satisfies the requisite anti-commutation relations with the other gamma matrices to give $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. One interesting fact is that we could just as well take $-\gamma$ as the d^{th} gamma matrix, which gives a second inequivalent representation of the Clifford algebra in odd dimensions. (Note we cannot change the overall sign by conjugation $\gamma \to U\gamma U^{-1}$.)

3.2 Majorana spinors

A subtle feature of representations of the Clifford algebra is the possibility of imposing a Majorana condition. A Majorana representation is a "real" (as opposed to complex representation), and thus has half the dimensionality of Dirac representation. The subtlety comes from the fact that we need to consider a more general reality condition than $\zeta^* = \zeta$. We need to allow for the fact that ζ^* is some linear operator B acting on ζ:

$$\zeta^* = B\zeta \ . \tag{3.22}$$

Indeed, under a unitary transformation $\zeta \to U\zeta$, and hence $B \to U^* B U^{-1}$. Thus even if we can find a particular basis where B is the identity, after acting by U, we will in general find some nontrivial B. Taking the conjugate of the definition (3.22) yields an additional consistency condition, $\zeta = B^*\zeta^* = B^*B\zeta$, implying $B^*B = \text{id}$.

As we saw earlier in an exercise, an infinitesimal Lorentz transformation is generated by something quadratic in the gamma matrices, $M_{\mu\nu} = -\frac{i}{2}\gamma_{\mu\nu}$. For this reality condition to make sense, we need it to be compatible with the Lorentz transformations:

$$\begin{aligned} (\delta\zeta)^* &= (B\delta\zeta) \ , & (3.23)\\ -\frac{i}{2}\omega^{\mu\nu}(M_{\mu\nu}\zeta)^* &= \frac{i}{2}\omega^{\mu\nu} B M_{\mu\nu}\zeta \ , & (3.24)\\ (M_{\mu\nu}\zeta)^* &= -BM_{\mu\nu}\zeta \ . & (3.25) \end{aligned}$$

On the right hand side, we can expand $BM_{\mu\nu}\zeta = BM_{\mu\nu}B^{-1}B\zeta$ while on the left $(M_{\mu\nu}\zeta)^* = M^*_{\mu\nu}\zeta^*$. Thus, the matrix B had better act as

$$BM_{\mu\nu}B^{-1} = -M^*_{\mu\nu} \tag{3.26}$$

on the Lorentz generators. On the individual gamma matrices, we are then allowed a sign ambiguity, $B\gamma_\mu B^{-1} = \pm\gamma^*_\mu$.

We will leave the general story as a problem and focus on three low dimensional cases of interest, $d = 2$, 3 and 4. In $d = 2$, the gamma matrices $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$ (3.11) are manifestly real. As a result, we can take $B = \text{id}$. The "gamma five" matrix $\gamma = \gamma^0\gamma^1 = \sigma^3$ is real and diagonal. While the original Dirac representation is two complex (or four real dimensional), we can reduce this representation into different types of smaller representations. There are Weyl representations with one complex (or two real) components. There are Majorana representations with two real components. Finally, because B is the identity in a basis where γ is diagonal, we can have Majorana-Weyl spinors with one real component.

In $d = 3$, the gamma matrices $\gamma^0 = i\sigma^2$, $\gamma^1 = \sigma^1$ and $\gamma^3 = \sigma^3$ are again all manifestly real, allowing for a Majorana representation with $B = \text{id}$. In odd dimensions, there are no Weyl representations.

In $d = 4$, for the basis (3.17), we can write $B = \gamma^2\gamma$. We know that B has the correct properties to impose a Majorana condition because

$$B\gamma^\mu B^{-1} = (\gamma^\mu)^* \ . \tag{3.27}$$

In the basis (3.17), "gamma five" is diagonal

$$\gamma = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix} \tag{3.28}$$

while B is not. Moreover, γ and B do not commute, implying that they cannot be simultaneously diagonalized. In other words, we cannot impose both a Majorana and a Weyl condition at the same time. We can have Majorana spinors or we can have Weyl spinors, but not both at the same time in four dimensions.

There is an elegant general story which we leave as a problem. Curiously, the representation theoretic structure has a periodicity modulo eight as a function of dimension. This periodicity turns out to be a rather deep feature of the Clifford algebra, with relations to other areas of mathematics, such as Bott periodicity.

Problem 3.4. *In $d = 2k+2$ dimensions, the matrices $\gamma^{\mu *}$ and $-\gamma^{\mu *}$ satisfy the same Clifford algebra as γ^μ and so must be related to γ^μ by a unitary similarity transformation. We would like to determine explicitly the form of this similarity transformation for the basis (3.15) and study its properties. Consider two candidate matrices*

$$B_1 = \gamma^3 \gamma^5 \cdots \gamma^{d-1} \ , \quad B_2 = \gamma B_1 \ . \tag{3.29}$$

a) Show that

$$B_1 \gamma^\mu B_1^{-1} = (-1)^k \gamma^{\mu *} \ , \quad B_2 \gamma^\mu B_2^{-1} = (-1)^{k+1} \gamma^{\mu *} \ , \tag{3.30}$$

and also that

$$B_i M_{\mu\nu} B_i^{-1} = -M^*_{\mu\nu} \tag{3.31}$$

for $i = 1$ and 2. As a result, the spinors ζ and $B_i^{-1}\zeta^*$ must transform in the same way under the Lorentz group.

b) Show that

$$B_1 \gamma B_1^{-1} = B_2 \gamma B_2^{-1} = (-1)^k \gamma^* \ . \tag{3.32}$$

As a result, both B matrices will change the eigenvalue of γ when k is odd and not when it is even. When ($d = 2 \mod 4$) each Weyl representation is its own conjugate, while when ($d = 0 \mod 4$), each Weyl representation is conjugate to the other.

c) That ζ and $B_i^{-1}\zeta^*$ transform the same way under the Lorentz group allow us to impose the Majorana reality condition $\zeta^* = B\zeta$, provided $B^* B = \mathrm{id}$ as discussed above. Show that a Majorana condition is possible using B_1 only if $k = 0$ or 3 (mod 4) and using B_2 only if $k = 0$ or 1 (mod 4).

d) Extending to odd dimensions, show that a Majorana condition is possible only when $k = 0$ or 3 (mod 4).

e) Make sure that you understand the contents of Figure 1.

d	Majorana	Weyl	Majorana-Weyl	min. rep.
2	yes	self	yes	1
3	yes	-	-	2
4	yes	complex	-	4
5	-	-	-	8
6	-	self	-	8
7	-	-	-	16
8	yes	complex	-	16
9	yes	-	-	16
10=2+8	yes	self	yes	16
11=3+8	yes	-	-	32
12=4+8	yes	complex	-	64

Figure 1: Properties of spinor representations in various dimension. A dash indicates the condition cannot be imposed. "self" means the Weyl representation is self-conjugate under complex conjugation while "complex" indicates complex conjugation gives the other Weyl representation. The dimension of the smallest representation is given in the last column. The conditions – Majorana, Weyl, Majorana-Weyl – that can be imposed on the representations repeat as a function of the dimension modulo 8.

f) *How do the details of the previous arguments change if we use a metric with mostly minus signature?*

Having completed the exercise above, one may ask if there are any other possible B's to consider which may satisfy the consistency conditions. If so, then $BM_{\mu\nu}B^{-1} = B'M_{\mu\nu}B'^{-1}$, which implies there is a linear operator $B^{-1}B'$ which commutes with all of the Lorentz generators. By Schur's Lemma, anything that commutes with all elements of an irreducible representation must be a multiple of the identity. In odd dimensions, where the Dirac representation is irreducible, there can be nothing else. In even dimensions, the Dirac representation splits into two Weyl representations, and $B^{-1}B'$ is not necessarily the identity. It is instead a linear combination of the identity with γ, the two operators which act like multiples of the identity when restricted to the Weyl representations. Indeed, in the exercise above, we had B_1 and B_2 in the even dimensional case, which differed by a factor of γ.

3.3 Spinor Inner Product

In addition to the annoying complexity that the reality condition for spinors should be generalized to $\psi^* = B\psi$, a second irritating feature about spinors is that $\psi^\dagger\psi$ is not a Lorentz scalar, as we now verify. Under an infinitesimal Lorentz transformation, we showed earlier that

$$\delta\psi_\alpha = \frac{1}{4}\omega^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\beta\psi_\beta \ . \tag{3.33}$$

It follows that the Hermitian conjugate spinor transforms as

$$\delta\psi^\dagger = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_\nu^\dagger\gamma_\mu^\dagger \tag{3.34}$$

From the iterative construction (3.15) of the gamma matrices, it is clear that

$$\gamma_0^\dagger = -\gamma_0 \ , \quad \gamma_i^\dagger = \gamma_i \ . \tag{3.35}$$

Problem 3.5. *Verify (3.35) for both odd and even dimensions.*

In fact, under a unitary similarity transformation $\gamma^\mu \to U\gamma^\mu U^{-1}$, this property is preserved, and we expect it to hold in general. As a result, we can write the Hermitian conjugation relation as

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \ , \tag{3.36}$$

and the transformation rule (3.34) can be written

$$\delta\psi^\dagger = -\frac{1}{4}\psi^\dagger \omega^{\mu\nu}\gamma_0\gamma_\nu\gamma_\mu\gamma_0 = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\mu\nu}\gamma_0 \ . \tag{3.37}$$

(While we have suppressed the spinor index, the structure $(\gamma_0\gamma_{\mu\nu}\gamma_0)_\alpha{}^\beta$ means it is most natural to write the Hermitian conjugate spinor with an upper index, $(\psi^\dagger)^\alpha$.) The sign and the additional factors of γ_0 will not cancel, and $\zeta^\dagger\psi$ has a nontrivial transformation under the Lorentz group. It is in other words not a scalar quantity. In the γ_0 sickness lies the cure, and we can define a modified conjugate spinor

$$\bar\psi \equiv \psi^\dagger \gamma_0 \ . \tag{3.38}$$

In this case, the infinitesimal Lorentz transformation becomes

$$\delta\bar\psi = -\frac{1}{4}\bar\psi\omega^{\mu\nu}\gamma_{\mu\nu} \tag{3.39}$$

and hence

$$\begin{aligned} \delta(\bar\zeta\psi) &= \delta\bar\zeta\psi + \bar\zeta\delta\psi \\ &= -\frac{1}{4}\bar\zeta\omega^{\mu\nu}\gamma_{\mu\nu}\psi + \frac{1}{4}\bar\zeta\omega^{\mu\nu}\gamma_{\mu\nu}\psi \\ &= 0 \ . \end{aligned} \tag{3.40}$$

Thus $\bar\zeta\psi$ is a Lorentz scalar.

From two spinors, we can construct other Lorentz covariant objects as well, such as vectors and anti-symmetric tensors:

$$\bar\zeta\gamma_\mu\psi \ , \quad \bar\zeta\gamma_{\mu\nu}\psi \ , \ldots \tag{3.41}$$

Problem 3.6. *Show that $v_\mu = \bar\zeta\gamma_\mu\psi$ is a vector, i.e. show that $\delta v_\mu = -\omega_\mu{}^\nu v_\nu$ under the transformation (3.33).*

We should next consider how this modified definition of spinor conjugation, $\bar\psi = \psi^\dagger\gamma_0$, interfaces with the Majorana condition $\psi^* = B\psi$:

$$\bar\psi = \psi^\dagger\gamma_0 = \psi^{T*}\gamma_0 = \psi^T B^T \gamma_0 \tag{3.42}$$

The combination $C \equiv B^T \gamma_0$ is often referred to as the charge conjugation matrix, and for Majorana spinors (in fact Dirac spinors as well), we can write the Lorentz covariant objects as $\zeta^T C\psi$, $\zeta^T C\gamma_\mu \psi$, $\zeta^T C\gamma_{\mu\nu}\psi$, etc. Before, we had the relation $\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$. The equivalent condition that guarantees compatibility with the Lorentz group for Majorana spinors is

$$-\gamma_\mu^T = C\gamma_\mu C^{-1} \ . \tag{3.43}$$

Restoring indices, we can think of $C^{\alpha\beta}$ as a metric on spinor indices, such that $\bar{\psi}\zeta = \psi^T C\zeta = \psi_\alpha C^{\alpha\beta}\zeta_\beta$. The inverse metric is then $C^{-1}_{\alpha\beta}$ with lower indices, and we can raise indices via $\psi^\alpha = \psi_\beta C^{\beta\alpha}$.

Before closing this section, we should discuss some elementary spinor manipulations that will be useful later on in demonstrating supersymmetry. First, spinor fields are Grassman valued, which means they anticommute:

$$\psi_\alpha \zeta_\beta = -\zeta_\beta \psi_\alpha \ . \tag{3.44}$$

There is a sign ambiguity then in how to define complex conjugation. We make the choice

$$(\psi_\alpha \zeta_\beta)^* = \zeta_\beta^* \psi_\alpha^* \ , \tag{3.45}$$

analogous to the way Hermitian conjugation acts on matrices.

We will often need to perform various manipulations with Majorana spinors, the simplest of which is perhaps

$$(\bar{\zeta}\psi)^* = (\zeta_\alpha C^{\alpha\beta}\psi_\beta)^* = \psi_\beta C^{\alpha\beta}\zeta_\alpha = -\zeta_\alpha C^{\alpha\beta}\psi_\beta = -\bar{\zeta}\psi \tag{3.46}$$

where we have made use of the fact that we can work in a basis where ζ, ψ, and C are real, yielding the curious result that $\bar{\zeta}\psi$ is pure imaginary.[3]

Problem 3.7. *The Majorana Flip Relations. Show that in $d = 2$, 3 and 4,*

$$\lambda^T C\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\chi = (-1)^p \chi^T C\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda \ . \tag{3.47}$$

In these dimensions, which allow for Majorana spinors, if we impose that λ and χ are Majorana, then we can replace $\lambda^T C$ with $\bar{\lambda}$ and similarly for χ. How are these rules modified in $d = 2$ and 4 to incorporate a γ matrix?

3.4 Fierz re-arrangement

Consider the following list of gamma matrices and antisymmetric products of gamma matrices:

$$\gamma_\Gamma \in \{\mathrm{id}, \gamma, \gamma_\mu, \gamma_\mu\gamma, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma, \ldots\} \tag{3.48}$$

[3] One way to change this property is to modify the definition of a barred spinor to include a factor of i (see for example Freedman and van Proeyen). Another is to work in a mostly minus convention for the metric (see for example Peskin and Schroeder).

where $\Gamma = \mu\nu\lambda\cdots$ is a generalized index and $\gamma_{\mu\nu\lambda\cdots}$ is an antisymmetric product over the given indices with weight one, e.g.

$$\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{3.49}$$

in the two index case. We would like to think about these matrices as vectors in a matrix valued inner product space, with inner product $\langle M_1, M_2\rangle = \mathrm{tr}(M_1^\dagger M_2)$. Because of the relation $\gamma \sim \gamma_0\gamma_1\cdots\gamma_{d-1}$, these matrices are not all linearly independent. In fact they stop being linearly independent once the number of indices is larger than $d/2$.

Problem 3.8. *Convince yourself that the counting works out, that there are precisely enough linearly independent matrices in the list (3.48) to span a vector space that has dimension $2^{\lfloor \frac{d}{2}\rfloor} \times 2^{\lfloor \frac{d}{2}\rfloor}$, i.e. the size of a gamma matrix.*

Provided we restrict the number of indices, the list of vectors is actually orthogonal with respect to our inner product. A key observation required is that a single gamma matrix is traceless:

$$\begin{aligned} 2\eta_{\mu\nu}\,\mathrm{tr}(\gamma_\lambda) &= \mathrm{tr}(\{\gamma_\mu,\gamma_\nu\}\gamma_\lambda) \\ &= \mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\nu\gamma_\mu\gamma_\lambda) \\ &= \mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\mu\gamma_\lambda\gamma_\nu) \\ &= \mathrm{tr}(\gamma_\mu\{\gamma_\nu,\gamma_\lambda\}) \\ &= 2\eta_{\nu\lambda}\,\mathrm{tr}(\gamma_\mu)\ . \end{aligned} \tag{3.50}$$

Choosing $\mu = \nu \neq \lambda$ then implies $\mathrm{tr}(\gamma_\lambda) = 0$.

Problem 3.9. *Generalize this result to show that $\mathrm{tr}(\gamma_{\mu_1\cdots\mu_n}) = 0$, provided $0 < n < d$. From this tracelessness, argue that the list of vectors $\gamma_\Gamma \in \{\mathrm{id}, \gamma_\mu, \gamma_\mu\gamma, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma, \ldots\}$ is orthogonal, provided we restrict the indices such that they are linearly independent.*

A completeness relation for our basis set (3.48) is then

$$\delta_\alpha^\beta\delta_\gamma^\delta = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'}(\gamma_\Gamma)_\gamma{}^\beta(\gamma_{\Gamma'})_\alpha{}^\delta \tag{3.51}$$

for some constants to be determined $c_{\Gamma\Gamma'}$ where Γ and Γ' are generalized indices that range over the list of independent elements in the list (3.48). To determine the $c_{\Gamma\Gamma'}$, we multiply both sides by $(\gamma_{\Gamma''})_\beta{}^\gamma$, and sum over β and γ:

$$(\gamma_{\Gamma''})_\alpha{}^\delta = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'}\,\mathrm{tr}(\gamma_\Gamma\gamma_{\Gamma''})(\gamma_{\Gamma'})_\alpha{}^\delta\ , \tag{3.52}$$

By orthogonality, $\mathrm{tr}(\gamma_\Gamma\gamma_{\Gamma''}) = 0$ unless $\Gamma = \Gamma''$ and the double sum reduces to a single sum

$$(\gamma_{\Gamma''})_\alpha{}^\delta = \sum_{\Gamma'} c_{\Gamma''\Gamma'}\,\mathrm{tr}(\gamma_{\Gamma''}^2)(\gamma_{\Gamma'})_\alpha{}^\delta\ . \tag{3.53}$$

For this equality to hold, we must have that $c_{\Gamma''\Gamma'} = 0$ unless $\Gamma' = \Gamma''$. In the case of equality, we have further that

$$c_{\Gamma\Gamma} = \frac{1}{\mathrm{tr}(\gamma_\Gamma^2)} = \pm_\Gamma \frac{1}{2^{\lfloor \frac{d}{2} \rfloor}} \ , \tag{3.54}$$

where $\pm_\Gamma$ arises because $\gamma_\Gamma^2 = \pm 1$ and the power of 2 occurs as a dimension of the representation of the Clifford algebra $\mathrm{tr}(\mathrm{id}) = 2^{\lfloor \frac{d}{2} \rfloor}$.

We have gone through this abstract argument because we will frequently be in a situation in the future where we want to be able to shuffle spinor bilinears around, a manipulation of the form $(\bar\lambda\psi)(\bar\zeta\chi) \to (\bar\lambda\chi)(\bar\zeta\psi)$. Consider a slightly more general situation where we have only three spinors, two of which are contracted together. We will use our decomposition of the identity $\delta_\alpha^\beta \delta_\gamma^\delta$ in terms of the generalized gamma matrix (3.48):

$$\begin{aligned}
(\bar\lambda\psi)\chi_\alpha &= \bar\lambda^\gamma \psi_\delta \chi_\beta \delta_\alpha^\beta \delta_\gamma^\delta & (3.55)\\
&= -\sum_\Gamma c_{\Gamma\Gamma} \bar\lambda^\gamma (\gamma_\Gamma)_\gamma{}^\beta \chi_\beta (\gamma_\Gamma)_\alpha{}^\delta \psi_\delta & \\
&= -\frac{1}{2^{\lfloor \frac{d}{2} \rfloor}} \sum_\Gamma \pm_\Gamma (\bar\lambda \gamma_\Gamma \chi)(\gamma_\Gamma \psi)_\alpha \ . & (3.56)
\end{aligned}$$

This swapping of ψ and χ in the contraction is called a Fierz re-arrangement identity.

Problem 3.10. *Show that in three dimensions, the Fierz re-arrangement identity is*

$$(\bar\lambda\psi)\chi_\alpha = -\frac{1}{2}(\bar\lambda\chi)\psi_\alpha - \frac{1}{2}(\bar\lambda\gamma_\mu\chi)(\gamma^\mu\psi)_\alpha \ . \tag{3.57}$$

Furthermore, show that in the special case $\lambda = \chi$ and the spinors are Majorana, this identity reduces to

$$(\bar\lambda\psi)\lambda_\alpha = -\frac{1}{2}(\bar\lambda\lambda)\psi_\alpha \ . \tag{3.58}$$

Problem 3.11. *There is yet another type of spinor representation, symplectic Majorana fermions. They can be useful for writing down actions with extended supersymmetry. These spinors ψ_α^i carry an extra index $i = 1$ or 2, and satisfy the following reality property:*

$$\bar\psi^i = \epsilon^{ij} \psi_j^T \tilde C \ . \tag{3.59}$$

The tensor $\epsilon^{12} = -\epsilon^{21} = 1$ and is zero otherwise. Construct $\tilde C$ using the B_1 and B_2 matrices from problem 3.4. In what dimensions are symplectic Majorana fermions allowed? In what dimensions can fermions be simultaneously symplectic Majorana and Weyl. (Note the language may be slightly confusing. Symplectic Majorana fermions are not also Majorana.)

4 Elementary Consequences of Supersymmetry

A generic supersymmetry algebra can be written as follows:

$$\begin{aligned}
\{Q_\alpha, \bar Q^\beta\} &= 2(\gamma^\mu)_\alpha{}^\beta P_\mu \ , \\
[Q_\alpha, P_\mu] &= 0 \ , \\
[M_{\mu\nu}, Q_\alpha] &= \frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta \ ,
\end{aligned} \tag{4.1}$$

along with the Poincaré generators (2.8).[4] The most important relation is the first line, that the supercharges square to space-time translations. The second line means that the Q_α's are invariant under translations, while the third implies that the Q_α transform as space-time spinors, as they should given their index. This basic algebra typically comes with a Majorana condition that $\bar{Q} = Q^T C$, and so will only be allowed in the dimensions that admit Majorana spinors.

Part of what makes supersymmetry so interesting is the variety of different algebras that can occur along with the intricate rules that determine when they can and cannot be constructed. Beyond the simple algebra above, one can construct so called N-extended algebras with more super charges where Q^I_α carries an additional index I that runs from one to N. There are also centrally extended algebras with additional "central elements" on the right hand side of the first relation. In certain curved manifolds with a high degree of symmetry, such as anti-de Sitter space, the underlying Poincaré symmetry can be replaced with a different bosonic algebra and then extended to a super Lie algebra.

We can gain much insight from the supersymmetry algebra alone. We begin with some elementary manipulations of the first line of (4.1), multiplying both sides by γ_0:

$$\{Q_\alpha, (Q^\dagger)^\beta\} = -2(\gamma_\mu\gamma_0)_\alpha{}^\beta P^\mu \tag{4.2}$$

Tracing over the spinor indices then yields

$$\mathrm{tr}(QQ^\dagger + Q^\dagger Q) = 2^{\lfloor \frac{d}{2} \rfloor + 1} P^0 \ , \tag{4.3}$$

where we have used the fact that $\mathrm{tr}(\gamma_{\mu\nu}) = 0$ and that $\mathrm{tr}(\mathrm{id}) = 2^{\lfloor \frac{d}{2} \rfloor}$. The momentum component is just the energy $P^0 = E$ and so we see that

$$E = \frac{1}{2^{\lfloor \frac{d}{2} \rfloor}} \mathrm{tr}(Q^\dagger Q) \ . \tag{4.4}$$

(In a "really real" representation, we can replace $Q^\dagger_\alpha$ with Q_α.)

The quantity $Q^\dagger Q$ is manifestly positive, and thus the energy in a SUSY theory is a positive definite quantity. States $|0\rangle$ with $E = 0$, if they exist, are the lowest energy, or vacuum states. Any such state must furthermore be annihilated by the supercharges

$$Q_\alpha|0\rangle = 0 \ , \tag{4.5}$$

and therefore preserve the supersymmetry (i.e. be invariant with respect to supersymmetry transformations).

The trace relation gives a simple diagnostic for spontaneous symmetry breaking – where the vacuum state breaks the symmetry although the action is invariant. If one finds that the vacuum state $|\Omega\rangle$ has positive energy, then the state breaks the supersymmetry $Q_\alpha|\Omega\rangle \neq 0$. Similarly if one finds that the vacuum state is not supersymmetric, $Q_\alpha|\Omega\rangle \neq 0$, then it must have positive energy.

[4] There is some arbitrariness in the normalization of the Q's. We have chosen the two on the right hand side of the first line in order to write some supersymmetry variations later on in a simpler way, with fewer factors of two.

Next, we look at the representations of the SUSY algebra. Since $[P_\mu, Q_\alpha] = 0$, it is also true that $[P^2, Q_\alpha] = 0$. As with the Poincaré group, we can use the eigenvalues of the Casimir P^2, i.e. the mass squared, to label representations of the SUSY algebra. All the members of a given irreducible representation will have the same value of m^2.[5] We pursue the same strategy that is used in classifying representations of the Poincaré algebra and treat the massive and massless cases separately. It will be simpler to perform the analysis in a "really real" representation where $C = \gamma_0$ and $Q^* = Q$.

In the massive case, we can go to a rest frame where $P^\mu = (m, 0, \ldots)$. The anti-commutation relation of the supercharges reduces to

$$\{Q_\alpha, Q_\beta\} = 2m\delta_{\alpha\beta}. \tag{4.6}$$

After a rescaling $\tilde{Q}_\alpha = m^{-1/2} Q_\alpha$, we recover the familiar Clifford algebra

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = 2\delta_{\alpha\beta} \ , \tag{4.7}$$

but now in $\mathfrak{d} = 2^{\lfloor \frac{d}{2} \rfloor}$ Euclidean dimensions. As such, it must have $2^{\mathfrak{d}/2} = 2^{2^{\lfloor \frac{d}{2} \rfloor - 1}}$ states. As the number of these Q_α matrices is even, we can construct a "gamma five" matrix as well,

$$(-1)^F = i^{\#} \tilde{Q}_1 \tilde{Q}_2 \cdots \tilde{Q}_{\mathfrak{d}} \tag{4.8}$$

which anti-commutes with the other $\tilde{Q}_\alpha$. Previously, we interpreted the ± 1 eigenvalues of "gamma five" as chirality of the state, but here they determine whether the state is fermionic or bosonic. Let $|\pm\rangle$ be an eigenstate of $(-1)^F$. As Q_α itself is fermionic, acting with it on a state will swap the state's fermionic/bosonic nature:

$$(-1)^F Q_\alpha |\pm\rangle = -Q_\alpha (-1)^F |\pm\rangle = \mp Q_\alpha |\pm\rangle \ . \tag{4.9}$$

One more remarkable thing we can learn comes from the fact that $(-1)^F$ is traceless (see problem 3.9). The trace is also the sum of the eigenvalues, and so there must be an equal number of bosonic and fermionic states in the super multiplet (irreducible representation). The pain we endured in learning about Clifford algebras and spinors is paying off!

For massless particles, the best we can do it pick a frame where $P^\mu = (E, -E, 0, 0, \ldots)$, and the SUSY algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2E(\mathrm{id} + \gamma_{01})_{\alpha\beta} \ . \tag{4.10}$$

Problem 4.1. *Show that the matrix $\frac{1}{2}(\mathrm{id} + \gamma_{01})$ acts like a projector,*

$$\left(\frac{1}{2}(\mathrm{id} + \gamma_{01})\right)^2 = \frac{1}{2}(\mathrm{id} + \gamma_{01}) \ , \tag{4.11}$$

half of whose eigenvalues are equal to zero and the other half are equal to one.

[5] There are interesting exceptions to this rule when the space-time is curved and the underlying Poincaré algebra is replaced with something else.

Using the projector, we can choose a new basis where half of the Q_α commute with one another and the other half do not:

$$\{Q_{\alpha'}, Q_{\beta'}\} = 4E\delta_{\alpha,\beta} \ , \quad \{Q_{\alpha''}, Q_{\beta''}\} = 0 \tag{4.12}$$

where the primed and double primed indices run over only $\frac{\mathfrak{d}}{2} = 2^{\lfloor \frac{d}{2} \rfloor -1}$ values. Rescaling the nontrivial Q's a bit differently this time, $\tilde{Q}_{\alpha'} = (2E)^{-1/2}Q_{\alpha'}$, we find a Clifford algebra with half as many Q's as before. We can repeat the previous argument with this smaller algebra. The massless supermultiplet has fewer states, $2^{\mathfrak{d}/4}$ instead of $2^{\mathfrak{d}/2}$. Half of these states are bosons and the other half fermions.

Finally we draw some conclusions from the commutator $[Q_\alpha, M_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta$ or equivalently the fact that Q_α has spin. Acting with Q_α thus must change the Lorentz group representation of the state. Let us focus on the 4d case, where massive states are labeled by their remaining $SO(3)$ spin quantum number while massless states are characterized by a helicity under the remaining $SO(2)$. We can think of Q_α as carrying spin (or helicity) one half. Using our quantum mechanics intuition, acting with Q_α should be like tensoring the underlying representation of the particle by a spin one half representation and should lead to new possible representations with angular momentum either larger or smaller by a quantized unit of 1/2.

In a supermultiplet, there will be a state with maximum spin $j_{\max}$ (or helicity in the massless case). The remaining states then have spins $j_{\max} - \frac{1}{2}$, $j_{\max} - 1$, etc. Acting with the appropriate lowering Q operator on the $j_{\max}$ state should lead to a new state with spin or helicity less by an amount one half, $j_{\max} - 1/2$.

An annoying complication is that if $j_{\max}$ is large enough, the multiplets tend to have more positive helicity states than negative and so are not CPT complete. The standard procedure to remedy the problem and obtain a theory that is CPT invariant is to add by hand a "mirror multiplet" with a corresponding lowest helicity state and use raising instead of lowering operators.

We can consider a few examples. In four dimensions, the smallest representation of a Clifford algebra is 4 dimensional. Focusing on massless states, we can then use two of the four Q_α's to create a multiplet, leading to one raising and one lowering operator and two states with helicities λ and $\lambda + \frac{1}{2}$. One of the states is fermionic and the other bosonic. Such a set of states is not CPT complete and needs to be supplemented with a mirror multiplet with helicities $-\lambda$ and $-\lambda - \frac{1}{2}$. One important example is the multiplet $\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ – really $\left(0, \frac{1}{2}\right)$ and its mirror – corresponding to a field theory with a complex scalar and a Majorana fermion. (Note the Dirac equation reduces the number of on-shell fermionic degrees of freedom from the size of the representation, four, down to two.) Another option is to have $\left(-1, -\frac{1}{2}, \frac{1}{2}, 1\right)$ corresponding to a gauge field and its superpartner, sometimes called a photino or gluino. A gauge field in 4d has two on-shell degrees of freedom, corresponding to two polarizations. The massive multiplets will be twice as large. One has $\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, which is the same as massless $\left(0, \frac{1}{2}\right)$ multiplet along with its mirror. We will construct actions later in the course with precisely these particle contents.

In general, going to higher dimension forces us to consider representations with larger and larger spin. While the numbers of Q's grow exponentially, the number of polarization states for a particle with a given spin tends to grow as a power law, linearly for a gauge field for example, or quadratically for a graviton. Ten dimensions turns out to be the largest dimension with a multiplet with helicities less than or equal to one. In this dimension, the smallest spinor representation is 16 dimensional Majorana-Weyl. For a massless multiplet, we can use 8 of the corresponding supercharges to construct four pairs of raising and lowering operators. Acting with at most four lowering operators will change the spin by at most an amount 2, corresponding to the helicity difference between the two polarizations of a photon. (In more detail, the 16 possible states divide up between the 8 polarization states of a photon and the 8 physical degrees of freedom of a Majorana-Weyl fermion.) In other words, supersymmetric gauge theories (without gravity) must have $d \leq 10$.

Eleven dimensions is the largest dimension with a multiplet with helicities less than or equal to two. Now the smallest representation is a 32 dimensional Majorana spinor. We get eight pairs of raising and lowering operators, corresponding to a maximum helicity difference of 4, i.e. the difference in helicity between two polarizations of a graviton. In other words, supergravity theories (without higher spin fields) must have $d \leq 11$.[6]

4.1 Two Component Notation

It is very common in four dimensions to work with a spinor basis where γ is diagonal, a so-called Weyl basis. In this case, the four component representation breaks apart into spinors with two components each. Let us discuss what the SUSY algebra looks like in this basis. We introduce projectors $\Pi_\pm = \frac{1}{2}(1 \pm \gamma)$ which satisfy the properties that $\Pi_\mp \Pi_\pm = 0$ and $\Pi_\pm \Pi_\pm = \Pi_\pm$. Note also that $\gamma_\mu \Pi_\pm = \Pi_\mp \gamma_\mu$. Using these projectors, we define spinors which are eigenvectors of γ

$$Q^\pm = \Pi_\pm Q \ . \tag{4.13}$$

There is something immediately confusing here. The claim is that Q_+ contains the same information as Q, and yet Π_+ is a projector which has a large kernel when acting on a Dirac spinor. In order for Q_+ to contain the same information, the kernel of Π_+ when acting on the space of Majorana spinors needs to be trivial. Indeed, it is, as can be seen straightforwardly in a "really real" representation where $Q^* = Q$ has real components while γ is pure imaginary. The only way for Π_+ to vanish when acting on a Majorana spinor is for all the components of Q to vanish.

As a result of this definition of $Q_\pm$, we find that $\overline{Q^\pm} = \overline{\Pi_\pm Q} = Q^\dagger \Pi_\pm^\dagger \gamma_0 = \overline{Q}\Pi_\mp$. Using further that Q is Majorana, $Q^* = BQ$, we can write $(Q^\pm)^* = \Pi_\pm^* Q^* = \Pi_\pm^* BQ = B\Pi_\mp Q = BQ^\mp$. (We have used that $-\gamma^* B = B\gamma$ in four dimensions.) In other words, complex conjugation exchanges the chirality of the spinors, which was a reason we saw earlier why in four dimensions we could not have spinors that were simultaneously Weyl and Majorana.

[6] The multiplet has 44 graviton states, 128 gravitino states, and 84 states associated with an antisymmetric three-form, for a grand total $2^8 = 256$ states. The number of degrees of freedom of a graviton map to the number of metric degrees of freedom. The metric starts out as a $d \times d$ symmetric matrix, but we can remove one row and column using diffeomorphism invariance (or the freedom to change variables). The remaining trace also drops out of the equations of motion, leading to $\frac{d(d-1)}{2} - 1 = \frac{d(d-3)}{2}$ degrees of freedom. The gravitino ψ_μ^α has a spinor and a vector index, but is "gamma traceless", $\gamma^\mu \psi_\mu^\alpha = 0$, leading to $16 \times 9 - 16 = 128$ on-shell degrees of freedom. An anti-symmetric three index tensor has $9 \times 8 \times 7/3! = 84$ on-shell degrees of freedom.

Now we act to the right and left of the first line of the SUSY algebra (4.1) with these projectors. For example, if we consider $\{(\Pi_\pm Q)_\alpha, (\bar{Q}\Pi_\mp)^\beta\}$, we obtain

$$\{Q^\pm_\alpha, (\overline{Q^\pm})^\beta\} = -2(\Pi_\pm \gamma^\mu \Pi_\mp)_\alpha{}^\beta P^\mu = -\frac{1}{2}(\Pi_\pm \gamma_\mu)_\alpha{}^\beta P^\mu \ . \tag{4.14}$$

Similarly, if we consider $\{(\Pi_\pm Q)_\alpha, (\bar{Q}\Pi_\pm)^\beta$, we obtain instead

$$\{Q^\pm_\alpha, (\bar{Q}^\mp)^\beta\} = -2(\Pi_\pm \gamma_\mu \Pi_\pm)_\alpha{}^\beta P^\mu = 0 \ . \tag{4.15}$$

We further multiply by γ_0 on both sides in order to replace the $\bar{Q}$ with Q^*, yielding

$$\begin{aligned} \{Q^\pm_\alpha, (Q^{*\pm})^\beta\} &= 2(\Pi_\pm \gamma_\mu \gamma_0)_\alpha{}^\beta P^\mu \ , \\ \{Q^\pm_\alpha, (Q^{*\mp})^\beta\} &= 0 \ . \end{aligned} \tag{4.16}$$

Because the spinor Q is Majorana, these equations are more than is necessary to reconstruct the algebra. We can reconstruct Q^- from Q^{*+} and vice versa. Let us therefore toss all of the Q^- generators and replace Q^+ with $\tilde{Q}$. We find then that

$$\begin{aligned} \{\tilde{Q}_\alpha, \tilde{Q}^{*\beta}\} &= 2(\Pi_+ \gamma_\mu \gamma_0)_\alpha{}^\beta P^\mu \\ \{\tilde{Q}^{*\alpha}, \tilde{Q}_\beta\} &= 2(\Pi_- \gamma_\mu \gamma_0)^\alpha{}_\beta P^\mu \\ \{\tilde{Q}_\alpha, \tilde{Q}_\beta\} &= 0 = \{\tilde{Q}^{*\alpha}, \tilde{Q}^{*\beta}\} \end{aligned} \tag{4.17}$$

The matrices $\Pi_\pm \gamma_\mu \gamma_0$ have a very simple form in the basis (3.17) as we now discuss.

In Weyl notation, where γ is diagonal, one typically breaks the four component Dirac spinor into two, two-component pieces

$$\psi = \begin{pmatrix} \lambda_a \\ \chi_{\dot{a}} \end{pmatrix} \tag{4.18}$$

where $a, \dot{a} = 1, 2$ and uses the Clifford algebra in the basis (3.17). To make the supersymmetry algebra look more compact, we introduce

$$\begin{aligned} \sigma^\mu_{a\dot{b}} &= (\Pi_+ \gamma^\mu \gamma_0)_{a\dot{b}} \ , \\ \bar{\sigma}^\mu_{\dot{a}b} &= (\Pi_- \gamma^\mu \gamma_0)_{\dot{a}b} \ , \end{aligned} \tag{4.19}$$

leading to

$$\begin{aligned} \{\tilde{Q}_a, \tilde{Q}_b\} &= 0 \ , & \{\tilde{Q}^*_{\dot{a}}, \tilde{Q}^*_{\dot{b}}\} = 0 \ , \\ \{\tilde{Q}_a, \tilde{Q}^*_{\dot{b}}\} &= 2\sigma^\mu_{a\dot{b}} P_\mu \ , & \{\tilde{Q}^*_{\dot{a}}, \tilde{Q}_b\} = 2\bar{\sigma}^\mu_{\dot{a}b} P_\mu \ , \end{aligned} \tag{4.20}$$

This form of the algebra appears in many text books and is known as the two component formalism.

Problem 4.2. *Show that the definition (4.19) of the σ matrices is compatible with our previous definition $\sigma_\mu = (\mathrm{id}, \sigma^i)$ and $\bar{\sigma}_\mu = (\mathrm{id}, -\sigma^i)$ used to write down the gamma matrices (3.17).*

We should reconsider our inner products in this two component notation. Note that $C = \gamma_0\gamma_2\gamma = \mathrm{diag}(\sigma^2, \sigma^2)$. Thus

$$\begin{aligned} \psi^\dagger \gamma_0 \psi &= \lambda^\dagger \chi - \chi^\dagger \lambda \\ \psi^T C \psi &= \lambda^T \sigma^2 \lambda + \chi^T \sigma^2 \chi \ . \end{aligned} \tag{4.21}$$

Finally, we write down the Majorana condition in two component notation. From the condition

$$\psi^* = B\psi \tag{4.22}$$

where $B = \gamma_2\gamma$, one finds that

$$\lambda^* = -\sigma^2 \chi \ , \quad \chi^* = \sigma^2 \lambda \ . \tag{4.23}$$

4.2 Witten Index

We argued for something above that is not always true. Specifically, we claimed that $\mathrm{tr}(-1)^F = 0$, since it was just like the "gamma five" matrix we considered in our construction of the Clifford algebra. While it is true that the trace vanishes when acting on a supermultiplet with nonzero energy, for the vacuum there is no such constraint. Witten took advantage of this fact to write down his index

$$\mathcal{W} = \mathrm{tr}_H(-1)^F \tag{4.24}$$

where the trace is over the whole Hilbert space, not just the positive energy states. Because $\mathrm{tr}(-1)^F$ is nearly always zero, this index measures the difference between the number of fermionic and bosonic vacuum states

$$\mathcal{W} = \{\# \text{ of bosonic vacua}\} - \{\# \text{ of fermionic vacua}\} \ . \tag{4.25}$$

By definition, it is an integer. It cannot be continously varied and cannot receive corrections as coupling constants are varied. It provides perhaps the simplest example of a non-renormalization theorem in supersymmetry. In a generic quantum field theory, one could well imagine that varying parameters leads to a vacuum state becoming a nonzero energy state or vice versa. In a supersymmetric theory, however, the nonzero energy states are all paired – one fermion and one boson. Thus the vacuum states have to disappear or appear in pairs, such that the Witten index remains invariant. A nonzero Witten index must also mean that supersymmetry cannot be spontaneously broken. There must always be a few vacuum states left that cannot pair off and disappear.

Problem 4.3. *It is possible to have extended supersymmetry where the Q^I_α carry an extra index $I = 1, 2, \ldots \mathcal{N}$. Assuming Majorana fermions and forgetting about central charges, the first (and most important) line of the supersymmetry algebra is modified to*

$$\{Q^I_\alpha, Q^J_\beta\} = -2\delta^{IJ}(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \ . \tag{4.26}$$

Let us restrict to the four dimensional case where we can choose the Q^I_α to be $\mathcal{N}$ copies of a four dimensional Majorana-Weyl spinor representation.

a) What is the typical size of a massive particle multiplet? of a massless one?

b) For a massless multiplet, what is the difference in helicity between the highest weight and lowest weight states? What is the largest $\mathcal{N}$ for which one can restrict to particles with spin less than or equal to one (i.e. gauge theories)? to particles with spin less than or equal to two (i.e. gravitational theories)?

c) For $\mathcal{N} = 2$, $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories, try to describe the particle content of some massless multiplets with small spin, i.e. less than or equal to two.

5 4d Wess-Zumino Model

The simplest four dimensional supersymmetric theory is often called the Wess-Zumino model. It has a Majorana Fermion $\psi(x)$ along with some scalar fields. Off-shell, $\psi(x)$ has four real components which are reduced to two real components on-shell by the Dirac equation. As we saw before, for supersymmetry, there must then be a pair of real scalar fields $A(x)$ and $B(x)$ as well.

Why does the Dirac equation reduce the number of degrees of freedom from 4 to 2? From a classical point of view, we associate a degree of freedom to the ability to choose the position and momentum of a particle. If the particle is described by a second order differential equation, those two quantities – position and momentum (or equivalently velocity) – are the integration constants of the differential equation. The Dirac equation, on the other hand, is not a single second order but a quadruplet (in 4d) of first order equations. In general, we can replace a single second order differential equation with a pair of first order equations, e.g. in place of

$$\phi''(x) = p(x)\phi'(x) + q(x)\phi(x) \ , \tag{5.1}$$

we could introduce $\pi(x) = \phi'(x)$ and write instead

$$\begin{aligned} \pi' &= p\pi + q\phi \ , && (5.2) \\ \phi' &= \pi \ . && (5.3) \end{aligned}$$

Going backward, we expect the four first order components that make up the Dirac equation should correspond to a pair of second order differential equations and thus to two degrees of freedom. Identifying which components of the spinor $\psi(x)$ correspond to "position" and which to "momentum" is unfortunately a bit ambiguous. The canonical commutation relation involves $\psi(x)$ with itself and yields no insight. Somehow, the components of $\psi(x)$ should be thought of as position and momentum at the same time, in some linear combination.

Following our nose, we test the following free theory for supersymmetry

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu A)(\partial^\mu A) - \frac{1}{2}(\partial_\mu B)(\partial^\mu B) - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \ . \tag{5.4}$$

The bosonic part leads straightforwardly the the expected equations of motion $\Box A = 0 = \Box B$. Note we can replace the pair of real scalars with a complex scalar $\phi = A + iB$ and its conjugate $\phi^* = A - iB$, in which case the Lagrangian becomes

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \ . \tag{5.5}$$

The fermionic action may seem more mysterious. The factor of i should be thought of as combining with the ∂_μ to give the Hermitian generator of translations P_μ. The resulting expression is indeed real, as we can verify explicitly. We work in a "really real" representation where the γ^μ have real coefficients and $C = \gamma_0$:

$$\begin{aligned}\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi\right)^* &= -i\left(\bar{\psi}\gamma^\mu\partial_\mu\psi\right)^* \\ &= -i\left(\psi^T C\gamma^\mu\partial_\mu\psi\right)^* \ .\end{aligned} \tag{5.6}$$

We use the result that $(\psi_\alpha\chi_\beta)^* = \chi^*_\beta\psi^*_\alpha = -\psi^*_\alpha\chi^*_\beta$, and further that $\psi^* = \psi$ in this "really real" basis for the gamma matrices. As the matrix $C\gamma^\mu$ is real, taking the complex conjugate of the expression in parentheses yields a minus sign showing that indeed

$$\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi\right)^* = i\bar{\psi}\gamma^\mu\partial_\mu\psi \ . \tag{5.7}$$

Let us also verify that we get the correct equation of motion for the fermion. Varying the action with respect to ψ, we obtain

$$\begin{aligned}\delta\mathcal{L}_0 &= -\frac{i}{2}\delta\psi^T C\gamma^\mu\partial_\mu\psi - \frac{i}{2}\psi^T C\gamma^\mu\partial_\mu\delta\psi \\ &= -\frac{i}{2}\delta\psi^T C\gamma^\mu\partial_\mu\psi + \frac{i}{2}(\partial_\mu\psi^T)C\gamma^\mu\delta\psi - \frac{i}{2}\partial_\mu(\psi^T C\gamma_\mu\delta\psi) \ .\end{aligned} \tag{5.8}$$

Note that it's important here that the fermion is real. With a complex fermion, we should vary ψ and $\bar{\psi}$ independently, similar to what we would do with a complex scalar. We now use one of the Majorana flip relations to replace $(\partial_\mu\psi)^T C\gamma^\mu\delta\psi$ with $-\delta\psi^T C\gamma^\mu(\partial_\mu\psi)$. We also throw out the total derivative term, assuming that we can implement the appropriate boundary conditions. The result is the Dirac equation:

$$\delta\mathcal{L}_0 = -i\delta\psi^T C\gamma^\mu\partial_\mu\psi \tag{5.9}$$

To investigate supersymmetry, we will vary the action by an infinitesimal Grassman valued object ϵ which transforms as a Majorana spinor. This object ϵ we can think of in rough analogy to the parameter a^μ that we used in considering infinitesimal translations. While P_μ has engineering dimension one, the infinitesimal length a_μ must have engineering dimension -1. Similarly, Q as the square root of P will have engineering dimension 1/2 while ϵ has engineering dimension $-1/2$. To translate between δ and Q, we have

$$\begin{aligned}[\delta_1, \delta_2] &= [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] \\ &= \bar{\epsilon}_1^\alpha\bar{\epsilon}_2^\beta\{Q_\alpha, Q_\beta\} \ .\end{aligned} \tag{5.10}$$

Note that $\delta = \bar{\epsilon}Q = \bar{Q}\epsilon$ is bosonic and so it is natural to take a commutator.

The supersymmetric variation should rotate a scalar into a fermionic operator and a fermion into a scalar operator. By dimension counting, we should be able to relate the variation of a scalar, e.g. $[Q, \phi]$, directly to ψ. The free scalar has engineering dimension one, while the fermion has engineering dimension 3/2. A natural guess is

$$\begin{aligned}\delta\phi &= \bar{\epsilon}(a + b\gamma)\psi, \\ \delta\phi^* &= \bar{\epsilon}(-a^* + b^*\gamma)\psi \ ,\end{aligned} \tag{5.11}$$

where a and b are constants. The $-a^*$ in the second line comes from the fact that $\bar{\epsilon}\psi$ is purely imaginary in our conventions while we find $+b^*$ because $\bar{\epsilon}\gamma\psi$ is real. If we vary ϕ twice, we would like to produce a total derivative acting on ϕ, i.e. an infinitesimal translation. A natural guess then for the variation of ψ is

$$\delta\psi = \frac{1}{2}\left((1+c\gamma)\partial\!\!\!/\phi + (1-c^*\gamma)\partial\!\!\!/\phi^*\right)\epsilon \ . \tag{5.12}$$

We have used the freedom to rescale ϵ to fix the coefficient of $\partial\!\!\!/\phi$ to be $(1+c\gamma)$ for c an undetermined constant. The coefficient of $\partial\!\!\!/\phi^*$ is then fixed by the Majorana property.

Let us begin by seeing if the constants a, b, and c can be adjusted to make this infinitesimal transformation a symmetry of the action. (We must check that the variation of the action vanishes off-shell. Of course it will vanish on-shell, because that is how the equations of motion are derived in the first place.) The variation takes the form

$$\delta\mathcal{L}_0 \;\; = \;\; -\frac{1}{2}(\partial_\mu\delta\phi^*)(\partial^\mu\phi) - \frac{1}{2}(\partial_\mu\phi^*)(\partial^\mu\delta\phi) - i\delta\psi^T C\gamma^\mu\partial_\mu\psi \ , \tag{5.13}$$

where we have used (5.9). We need look only at the terms proportional to ϕ. The result for the terms proportional to ϕ^* will follow by complex conjugation:

$$\delta\mathcal{L}_0|_\phi \;\; = \;\; -\frac{1}{2}\left(-a^*\bar{\epsilon}\partial_\mu\psi + b^*\bar{\epsilon}\gamma\partial_\mu\psi\right)\partial^\mu\phi - \frac{i}{2}\left[(\partial\!\!\!/\phi + c\gamma\partial\!\!\!/\phi)\epsilon\right]^T C\gamma^\mu\partial_\mu\psi \ . \tag{5.14}$$

Now using that $-\gamma_\mu^T C = C\gamma_\mu$ (3.43) while $\gamma^T C = C\gamma$, we see that

$$\delta\mathcal{L}_0|_\phi \;\; = \;\; \frac{1}{2}\left[a^*\bar{\epsilon}\partial_\mu\psi - b^*\bar{\epsilon}\gamma\partial_\mu\psi + i\bar{\epsilon}\gamma^\mu\gamma^\nu\partial_\nu\psi + ic\bar{\epsilon}\gamma^\mu\gamma\gamma^\nu\partial_\nu\psi\right](\partial_\mu\phi) \tag{5.15}$$

Next, we write $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] = \eta^{\mu\nu} + \gamma^{\mu\nu}$ and remark that $\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\psi = \partial_\mu(\gamma^{\mu\nu}\phi\partial_\nu\psi)$ is a total derivative, allowing us to group terms:

$$\delta\mathcal{L}_0|_\phi \;\; = \;\; \frac{1}{2}\left[(a^*+i)\,\bar{\epsilon}\partial_\mu\psi + (-b^*-ic)\,\bar{\epsilon}\gamma\partial_\mu\psi\right](\partial^\mu\phi) \ . \tag{5.16}$$

For the variation to vanish, we thus require $a = i$ and $b = ic^*$.

Returning now to the issue of whether or not we are dealing with supersymmetry, we can see if we get something sensible for $[\delta_1,\delta_2]\phi$:

$$\begin{aligned}
\delta_1\delta_2\phi \;\; &= \;\; \delta_1\bar{\epsilon}_2(a+b\gamma)\psi \\
&= \;\; \frac{1}{2}\bar{\epsilon}_2(a+b\gamma)\left[(1+c\gamma)\partial\!\!\!/\phi + (1-c^*\gamma)\partial\!\!\!/\phi^*\right]\epsilon_1 \\
&= \;\; \frac{1}{2}(a+bc)\bar{\epsilon}_2\partial\!\!\!/\phi\epsilon_1 + \frac{1}{2}(ac+b)\bar{\epsilon}_2\gamma\partial\!\!\!/\phi\epsilon_1 \\
&\quad\;\; + \frac{1}{2}(a-bc^*)\bar{\epsilon}_2\partial\!\!\!/\phi^*\epsilon_1 + \frac{1}{2}(-ac^*+b)\bar{\epsilon}_2\gamma\partial\!\!\!/\phi^*\epsilon_1 \ .
\end{aligned} \tag{5.17}$$

To simplify the commutator, we need the Majorana flip relations that $\bar{\epsilon}_2\gamma^\mu\epsilon_1 = -\bar{\epsilon}_1\gamma^\mu\epsilon_2$ along with $\bar{\epsilon}_2\gamma\gamma^\mu\epsilon_1 = \bar{\epsilon}_1\gamma\gamma^\mu\epsilon_2$:

$$[\delta_1,\delta_2]\phi = (a+bc)\bar{\epsilon}_2\partial\!\!\!/\phi\epsilon_1 + (a-bc^*)\bar{\epsilon}_2\partial\!\!\!/\phi^*\epsilon_1 \ . \tag{5.18}$$

We argued before that $\bar{\epsilon}_2\gamma^\mu\epsilon_1$ has the correct transformation properties to be a Lorentz vector. Thus, if the second term can be made to vanish, we have found that $[\delta_1, \delta_2]$ produces an infinitesimal translation when acting on ϕ, as it should if we are discussing supersymmetry. In addition to $a = i$ and $b = ic^*$ that we found in demanding $\delta\mathcal{L}_0$ vanish, we now also find $a = bc^*$, allowing us to set $c^* = \pm 1$ and $b = \pm i$. We will make the positive sign choice, leading to

$$[\delta_1, \delta_2]\phi = 2i\bar{\epsilon}_2\partial\!\!\!/\phi\epsilon_1 \ . \tag{5.19}$$

The supersymmetry transformation can be cast in a more compact form using the projectors $\Pi_\pm \equiv \frac{1}{2}(1+\gamma)$:

$$\begin{aligned} \delta\phi &= 2i\bar{\epsilon}\Pi_+\psi \ , \quad \delta\phi^* = 2i\bar{\epsilon}\Pi_-\psi \ , & (5.20) \\ \delta\psi &= \Pi_+\partial\!\!\!/\phi\epsilon + \Pi_-\partial\!\!\!/\phi^*\epsilon \ . & (5.21) \end{aligned}$$

These oeprators $\Pi_\pm$ project the fermions onto positive or negative chirality states, i.e. into the space of Weyl fermions. It perhaps makes sense that some kind of projector appears, as the fermion has more degrees of freedom than a single scalar.

Finally, we need to check that $[\delta_1, \delta_2]\psi$ is indeed an infinitesimal translation. We find that

$$\begin{aligned} \delta_1\delta_2\psi &= 2i(\partial_\mu\bar{\epsilon}_1\Pi_+\psi)\gamma^\mu\Pi_-\epsilon_2 + 2i(\partial_\mu\bar{\epsilon}_1\Pi_-\psi)\gamma^\mu\Pi_+\epsilon_2 \\ &= i(\bar{\epsilon}_1\partial_\mu\psi)\gamma^\mu\epsilon_2 - i(\bar{\epsilon}_1\gamma\partial_\mu\psi)\gamma^\mu\gamma\epsilon_2 \end{aligned} \tag{5.22}$$

This doesn't yet look much like $\bar{\epsilon}_1\gamma^\mu\epsilon_2$ multiplying $\partial_\mu\psi$, in analogy to what we obtained for the scalar, but we can use our Fierz rearrangement identities. In 4d, we have the following independent gamma matrices:

$$\mathrm{id} \ , \quad \gamma_\mu \ , \quad \gamma \ , \quad \gamma_\mu\gamma \ , \quad \gamma_{\mu\nu} \ . \tag{5.23}$$

Note that there are $1+4+1+4+6 = 16$ of these matrices, which is indeed equal to 4×4, the size of gamma matrices for these Majorana fermions in 4d. The relevant Fierz identity is then

$$\begin{aligned} (\bar{\lambda}\rho)\chi &= -\frac{1}{4}(\bar{\lambda}\chi)\rho - \frac{1}{4}(\bar{\lambda}\gamma_\mu\chi)(\gamma^\mu\rho) - \frac{1}{4}(\bar{\lambda}\gamma\chi)(\gamma\rho) \\ &\quad + \frac{1}{4}(\bar{\lambda}\gamma_\mu\gamma\chi)(\gamma^\mu\gamma\rho) + \frac{1}{8}(\bar{\lambda}\gamma_{\mu\nu}\chi)(\gamma^{\mu\nu}\rho) \ . \end{aligned} \tag{5.24}$$

The extra factor of 1/2 in the last term compensates for the overcounting from $\gamma_{\mu\nu} = -\gamma_{\nu\mu}$. We are interested in the special case $(\bar{\epsilon}_1\rho)\epsilon_2 - (\bar{\epsilon}_2\rho)\epsilon_1$ where ρ is either $\partial_\mu\psi$ or $\gamma\partial_\mu\psi$. Because of the Majorana flip relations (3.47) supplemented by a couple of extra relations that also involve γ, it turns out only the terms that involve γ_μ and $\gamma_{\mu\nu}$ survive:

$$(\bar{\epsilon}_1\rho)\epsilon_2 - (\bar{\epsilon}_2\rho)\epsilon_1 = -\frac{1}{2}(\bar{\epsilon}_1\gamma_\mu\epsilon_2)(\gamma^\mu\rho) + \frac{1}{4}(\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2)(\gamma^{\mu\nu}\rho) \ . \tag{5.25}$$

Problem 5.1. *Show that $\psi^T C\gamma\lambda = \lambda^T C\gamma\psi$ and $\psi^T C\gamma\gamma_\mu\lambda = \lambda^T C\gamma\gamma_\mu\psi$.*

We find then the following

$$
\begin{aligned}
[\delta_1,\delta_2]\psi &= -\frac{i}{2}(\bar\epsilon_1\gamma_\mu\epsilon_2)\gamma^\lambda\gamma^\mu\partial_\lambda\psi + \frac{i}{8}(\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2)\gamma^\lambda\gamma^{\mu\nu}\partial_\lambda\psi \\
&\quad +\frac{i}{2}(\bar\epsilon_1\gamma_\mu\epsilon_2)\gamma^\lambda\gamma\gamma^\mu\gamma\partial_\lambda\psi - \frac{i}{8}(\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2)\gamma^\lambda\gamma\gamma^{\mu\nu}\gamma\partial_\lambda\psi \\
&= -i(\bar\epsilon_1\gamma_\mu\epsilon_2)\gamma^\lambda\gamma^\mu\partial_\lambda\psi \\
&= -2i(\bar\epsilon_1\gamma_\mu\epsilon_2)\partial^\mu\psi + i(\bar\epsilon_1\gamma_\mu\epsilon_2)\gamma^\mu\gamma^\lambda\partial_\lambda\psi \ . \qquad (5.26)
\end{aligned}
$$

In proceeding from the first to the second line, we have used that γ anticommutes with γ^μ but commutes with $\gamma^{\mu\nu}$ and also that $\gamma^2 = 1$. In going from the second to the third equality, we used the anticommutation relations for γ^λ and γ^μ.

We haven't completely succeeded here. There is still the second term in the last line of (5.26), but notice that this second term is proportional to the equation of motion for the fermion, $\gamma^\mu\partial_\mu\psi = 0$. What is going on here is that the supersymmetry algebra has failed to close off-shell. In order to get the required translation, we need to impose the equation of motion. We say that the supersymmetry algebra here closes on-shell. More formally, we can write

$$
\begin{aligned}
\bar\epsilon_1^\alpha\epsilon_{2\beta}\{Q_\alpha,\bar Q^\beta\} &= 2\bar\epsilon_1\gamma_\mu\epsilon_2 P^\mu \\
&= 2\bar\epsilon_1^\alpha\epsilon_{2\beta}(\gamma_\mu)_\alpha{}^\beta P^\mu \ . \qquad (5.27)
\end{aligned}
$$

as expected from the first line of our original statement of the supersymmetry algebra (4.1) back in section 4.

In this particular case, there is an improved formalism where we can get the supersymmetry algebra to close off-shell as well, but it requires adding auxiliary fields, i.e. fields that do not carry dynamical degrees of freedom. In this case, we would need to add a complex scalar field traditionally called F. Then the degrees of freedom would balance off-shell – four bosonic and four fermionic.

Problem 5.2. *Consider the following improved Lagrangian*

$$
\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi^*)(\partial^\mu\phi) - \frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi + \frac{1}{2}F^*F \ , \qquad (5.28)
$$

along with the improved SUSY transformation rules

$$
\begin{aligned}
\delta\phi &= 2i\bar\epsilon\Pi_+\psi \ , \\
\delta\psi &= \gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + F\Pi_+\epsilon + F^*\Pi_-\epsilon \ , \qquad (5.29) \\
\delta F &= 2i\bar\epsilon\Pi_-\gamma^\mu\partial_\mu\psi \ . \qquad (5.30)
\end{aligned}
$$

a) *Why doesn't F show up in $\delta\phi$? Why doesn't ϕ show up in δF?*

b) *Verify that the Lagrangian is invariant under these SUSY transformations.*

c) *Verify that the SUSY algebra closes off-shell, i.e. without imposing the equations of motion. This problem is rather lengthy, requiring examining $[\delta_1,\delta_2]$ acting on ϕ, ψ, and F.*

Finding the appropriate auxiliary fields to close the SUSY algebra off-shell is in fact in general a difficult problem. In the case of $\mathcal{N} = 1$ and 2 supersymmetry, answers are usually known. For many cases with $\mathcal{N} = 4$ and 8 SUSY, the problem remains unsolved.

5.1 Interactions

We can add interactions to this model, but only in a rather limited fashion because of the constraints from supersymmetry. In the interest of simplicity, we will work with on-shell SUSY and no additional auxiliary fields. Consider the following interaction Lagrangian

$$\mathcal{L}_{\rm int} = -V(\phi,\phi^*) + U(\phi,\phi^*) i\bar{\psi}\Pi_+\psi + U(\phi,\phi^*)^* i\bar{\psi}\Pi_-\psi \ . \tag{5.31}$$

Note the second and third terms are complex conjugates of each other. Provided then that V is real, the interaction is a real quantity as well. This modified Lagrangian is not invariant under the original SUSY transformations, but it is under a minor modification of them,

$$\begin{aligned} \delta\phi &= 2i\bar{\epsilon}\Pi_+\psi \ , \quad \delta\phi^* = 2i\bar{\epsilon}\Pi_-\psi \ , \\ \delta\psi &= \gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + W(\phi,\phi^*)^*\Pi_+\epsilon + W(\phi,\phi^*)\Pi_-\epsilon \ . \end{aligned} \tag{5.32}$$

To verify SUSY of the new interacting Lagrangian, we start with the extra pieces that now do not cancel out in $\delta\mathcal{L}_0$ because of the modification of the supersymmetry transformations. From the derivation of the equation of motion for ψ, we can write the left-over piece as

$$\begin{aligned} \delta\mathcal{L}_0|_{\rm leftover} &= -i\bar{\psi}\gamma^\mu\partial_\mu\left(W^*\Pi_+\epsilon + W\Pi_-\epsilon\right) \\ &= -i\bar{\psi}\left[(\partial W^*)(\partial\!\!\!/\phi)\Pi_+\epsilon + (\bar{\partial}W^*)(\partial\!\!\!/\phi^*)\Pi_+\epsilon + (\partial W)(\partial\!\!\!/\phi)\Pi_-\epsilon + (\bar{\partial}W)(\partial\!\!\!/\phi^*)\Pi_-\epsilon\right] \ . \end{aligned} \tag{5.33}$$

Next we consider the SUSY variation of the interactions, which we break up into terms that are linear and cubic in ψ: $\delta\mathcal{L}_{\rm int} = \delta_1\mathcal{L}_{\rm int} + \delta_3\mathcal{L}_{\rm int}$. The linear terms are as follows

$$\begin{aligned} \delta_1\mathcal{L}_{\rm int} &= -(\partial V)(2i\bar{\epsilon}\Pi_+\psi) - (\bar{\partial}V)(2i\bar{\epsilon}\Pi_-\psi) \\ &\quad + iU(\gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + W^*\Pi_+\epsilon + W\Pi_-\epsilon)^T C\Pi_+\psi + c.c. \\ &\quad + iU\psi^T C\Pi_+(\gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + W^*\Pi_+\epsilon + W\Pi_-\epsilon) + c.c. \end{aligned} \tag{5.34}$$

We use that $\Pi_+\gamma_\mu = \gamma_\mu\Pi_-$, $\gamma_\mu^T C = -C\gamma_\mu$ and $\gamma^T C = C\gamma$, along with projection conditions that $\Pi_\pm\Pi_\mp = 0$ and $\Pi_\pm^2 = \Pi_\pm$:

$$\begin{aligned} \delta_1\mathcal{L}_{\rm int} &= -(\partial V)(2i\bar{\epsilon}\Pi_+\psi) - (\bar{\partial}V)(2i\bar{\epsilon}\Pi_-\psi) \\ &\quad - iU\bar{\epsilon}(\partial\!\!\!/\phi)\Pi_+\psi + iUW^*\bar{\epsilon}\Pi_+\psi + c.c. \end{aligned} \tag{5.35}$$

$$\quad + iU\bar{\psi}(\partial\!\!\!/\phi)\Pi_-\epsilon + iUW^*\bar{\psi}\Pi_+\epsilon + c.c. \tag{5.36}$$

Using the Majorana flip identities, this expression simplifies somewhat further

$$\begin{aligned} \delta_1\mathcal{L}_{\rm int} &= -(\partial V)(2i\bar{\psi}\Pi_+\epsilon) - (\bar{\partial}V)(2i\bar{\psi}\Pi_-\epsilon) \\ &\quad + 2iU\bar{\psi}(\partial\!\!\!/\phi)\Pi_-\epsilon + 2iUW^*\bar{\psi}\Pi_+\epsilon + c.c. \end{aligned} \tag{5.37}$$

The combination $\delta\mathcal{L}_0 + \delta_1\mathcal{L}_{\rm int}$ has to vanish independently of $\delta_3\mathcal{L}_\int$ because of the differing numbers of fermions in the expressions. Pairing up terms, we find that the following expressions (and their complex conjugates) must vanish

$$\begin{aligned} \partial W - 2U &= 0 \ , \\ \bar{\partial}W &= 0 \ , \\ \partial V - UW^* &= 0 \end{aligned} \tag{5.38}$$

The second equation implies the remarkable fact that W must be a holomorphic function of the fields, i.e. depend only on ϕ and not its complex conjugate ϕ^*. The first and third equations (along with their complex conjugates) can be assembled and used to solve for V as a function of W:

$$V = \frac{1}{2} W W^* \ . \tag{5.39}$$

(There are some integration constants which we have set to zero here.)

Problem 5.3. *Verify that $\delta\mathcal{L}_3 = 0$ as well, and so the action is supersymmetric. You will need some Fierz identities.*

The holomorphic function $\mathcal{W}(\phi)$ defined such that

$$\frac{\partial \mathcal{W}}{\partial \phi} = W \tag{5.40}$$

is usually given the name superpotential. The choice of $\mathcal{W}$ determines all of the interactions in the Wess-Zumino model! Since W is holomorphic, its image is the entire complex plane. Even if W never vanishes, it must come arbitrarily close to zero. Although it is possible to break SUSY spontaneously by taking $W = 1/\phi$, there will still be in some sense a SUSY vacuum at infinity that the system can roll toward, possibly by tunneling out of a non-SUSY vacuum with positive energy. A number of years ago, there was a flurry of activity concerned with this phenonenon, dubbed meta-stable SUSY breaking.

Problem 5.4. *Verify that the SUSY variations (5.32) close on-shell.*

Consider for a moment a superpotential of the form

$$\mathcal{W} = \frac{1}{2} m \phi^2 + \frac{1}{3} \lambda \phi^3 \ . \tag{5.41}$$

We see that the quadratic term proportional to ϕ^2 will produce mass terms m^2 for the scalar ϕ and m for its superpartner ψ in the original Lagranigan. The cubic term will on the other hand lead to genuine interactions, a Yukawa and its complex conjugate of the schematic form $\phi\bar{\psi}\psi$ as well as a quartic $|\phi|^4$ potential for the scalar.

The holomorphic nature of $\mathcal{W}$ along with some global symmetries can greatly constrain the way in which $\mathcal{W}$ can be renormalized as a function of energy scale. First consider a U(1) symmetry under which ϕ and $\Pi_+\psi$ have the same charge q, $\phi \to e^{i\alpha q}\phi$ and $\Pi_+\psi \to e^{i\alpha q}\Pi_+\psi$. Hence, ϕ^* and $\Pi_-\psi$ will have the opposite charge, $\phi^* \to e^{-i\alpha q}\phi^*$ and $\Pi_-\psi \to e^{-i\alpha q}\Pi_-\psi$. The correlation between ϕ and $\Pi_+\psi$ appears for consistency with the SUSY transformation rules (5.32). A more concise way of writing the transformation rule for the fermion is to use the gamma five matrix, $\psi \to e^{iq\alpha\gamma}\psi$.[7] By construction, the potential V as well as the Yukawas $U\bar{\psi}\Pi_+\psi$ and $U^*\bar{\psi}\Pi_-\psi$ will be inert under such a symmetry transformation provided $\mathcal{W}$ is inert under this U(1) as well.

[7]One might worry that such a transformation destroys the reality property of the Majorana spinor. Consider however a "really real" representation where $\psi^* = \psi$. In such a basis, γ is pure imaginary and hence $e^{iq\alpha\gamma}$ is purely real.

There is the possibility of a more subtle global symmetry as well, under which the supercharge Q transforms. This symmetry is usually called R-charge, and the conventional normalization is that $\Pi_+ Q$ should have charge -1, $Q \to e^{-i\alpha\gamma} Q$. Hence, the R-charge of $\Pi_+ \psi$ must be one less than the R-charge of ϕ for consistency with the SUSY transformation rules. If the superpotential itself $\mathcal{W}$ has R-charge 2, then the Lagrangian will again be invariant with respect to this global symmetry.

Coming back to the simple model (5.41), the first step in the renormalization argument is to assume that m and λ are not numbers but scalar fields in part of some larger supersymmetric field theory. Their only role for us, however, will be to take on expectation values $\langle m \rangle$ and $\langle \lambda \rangle$ that lead to masses and interactions of the dynamical field ϕ and its super partner ψ. Given their new interpretation as fields, we can restore a $U(1) \times U(1)_R$ symmetry to the theory. The relevant charge assignments for the individual fields such that the superpotential has charge zero and two respectively are

$$\begin{array}{c|cc} & U(1) & U(1)_R \\ \hline \phi & 1 & 1 \\ m & -2 & 0 \\ \lambda & -3 & -1 \end{array} \tag{5.42}$$

When λ is very small, the theory is nearly free, and we have good perturbative control over the behavior. However, as λ gets larger, so do the interactions, and much more complicated behavior can ensue in a generic quantum field theory. Through loop corrections one can generate additional interactions, for example ϕ^4. Non-polynomial and non-perturbative expressions like $e^{-\phi^2/\lambda}$ could appear as well. Here, however, supersymmetry and the U(1) symmetries make the rules much stricter. To respect the symmetries, the potential must be a holomorphic function with the scaling form

$$\mathcal{W} = m\phi^2 f\left(\frac{\lambda\phi}{m}\right) . \tag{5.43}$$

Without holomorphicity, we could satisfy the charge constraints much more easily by including the complex conjugate fields ϕ^*, m^*, and λ^*. In the limit where λ is very small, we can expand this function out as a power series involving only non-negative powers of λ. After all, the theory should be well-behaved with respect to λ in this nearly free limit:

$$\mathcal{W} = \sum_{n=0}^{\infty} g_n m^{1-n} \lambda^n \phi^{n+2} . \tag{5.44}$$

However, we also ought to be able to take a massless limit and expect the theory to be well-behaved. Thus we can rule out all terms with $n > 1$. The generic form of the super potential is then

$$\mathcal{W} = g_0 m\phi^2 + g_1 \lambda\phi^3 . \tag{5.45}$$

We can determine the constants g_0 and g_1 by matching to (5.41) in the weakly interacting limit $\lambda \to 0$. However, the constants g_0 and g_1 must be independent of λ, m and ϕ and so we have fixed $\mathcal{W}$ for all λ, and the super potential is not renormalized. There are some subtleties here which have to do with choice of renormalization group scheme and other subtleties associated with massless limits and Wilsonian RG, but we shall gloss over them.

This argument is easily generalizable to more complicated superpotentials. Each time we add a new coupling and new interaction term, e.g. $\lambda'\phi^4$, we also get a new U(1) symmetry to add to the mix, which constrains the renormalization of the new coupling. This argument can be further generalized to include gauge fields, which I hope we will have time to see later.

Problem 5.5. *Can you modify the argument above for a superpotential of the form* $\mathcal{W} = \frac{1}{2}m\phi^2 + \lambda\phi^r$ *for* r *some positive integer,* $r \geq 3$*?*

6 From Super Maxwell to Super Yang-Mills with Matter

Gauge symmetry is an important part of real world physics, in particular the Standard Model, and it is a critical part of these lectures to incorporate supersymmetry into gauge theories. We start with the simplest gauge theory – a Maxwell field with a U(1) symmetry group – and then proceed to make it more complicated, first generalizing to a non-abelian simple Lie group, e.g. SU(n), and then adding matter fields that are charged under the gauge group. The lessons we learn in the simpler cases will carry over to the more complicated ones and make our life easier later on.

Before embarking, let us first do some simple counting on our fingers. A gauge theory is characterized by a massless vector field A_μ^a that transforms in the adjoint representation of the gauge group. Naively, A_μ should have the same number of space-time degrees of freedom as the number of dimensions, $\mu = 0, 1, \ldots d-1$. However, this counting does not agree with our experience in four dimensions where the photon has just two polarizations. In Lorentz gauge $\partial_\mu A^\mu = 0$, the equation of motion for the photon is that of d massless scalar fields $\partial^2 A_\mu = 0$. However $\partial_\mu A^\mu = 0$ does not completely fix the gauge and we are free to perform a shift $A_\mu \to A_\mu + \partial_\mu \Lambda$ provided $\partial^2\Lambda = 0$. This freedom allows us to remove one component of A_μ, say A_1. The gauge constraint $\partial_\mu A_\mu = 0$ then removes an additional degree of freedom. If we choose a reference frame where the photon is traveling in the 1 direction, then its momentum vector will be $p^\mu = (E, E, 0, 0, \ldots)$. In momentum space, the gauge constraint implies $p^\mu A_\mu = 0$, leaving only the components A_2, A_3, …, A_{d-1} free.

For the simplest supersymmetry, we should then add the same number of fermionic degrees of freedom in the form of spin 1/2 fermions. [[Spin 3/2 particles are called gravitinos. A discussion of them would quickly lead us into supergravity which I want to avoid.]] So we should look at our table of fermions in various dimensions contained in figure 1 and see when the counting matches. Recall that the Dirac equation removes half of the degrees of freedom, and so we need to see when the numbers in the last column, divided by two, are equal to $d-2$. The match happens precisely for $d = 3$, 4, 6 and 10. We can have supersymmetric gauge theories in other dimensions as well, but they will require adding fields in other representations of the Lorentz group, for example scalars.

6.1 Maxwell Field

We will focus on the four dimensional case in what follows. Our action here is constructed from a U(1) gauge field A_μ and its corresponding field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$ and a Majorana fermion λ where $\bar{\lambda} = \lambda^T C$:

$$S_{\mathrm{SM}} = -\int \mathrm{d}^4 x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda \right) , \tag{6.1}$$

The theory is free, and perhaps not very interesting on its own. We could add charged matter fields to get a supersymmetric version of QED, which we will do later in the non-abelian case. Our interest here though is in the fact that it is supersymmetric:

$$\delta A_\mu = i\bar{\epsilon}\gamma_\mu \lambda , \tag{6.2}$$

$$\delta \lambda = -\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon . \tag{6.3}$$

Note that the transformation rules are consistent with naive engineering dimensions of the fields, where A_μ has dimension one and λ has dimension 3/2. We need to deal with the variation $\delta\bar{\lambda}$:

$$\begin{aligned} \delta\bar{\lambda}\gamma^\mu \partial_\mu \lambda &= \partial_\mu(\delta\bar{\lambda}\gamma^\mu\lambda) - (\partial_\mu \delta\bar{\lambda})\gamma^\mu \lambda \\ &= \partial_\mu(\delta\bar{\lambda}\gamma^\mu\lambda) + \bar{\lambda}\gamma^\mu \partial_\mu \delta\lambda , \end{aligned} \tag{6.4}$$

integrating by parts and using one of the Majorana flip identities (3.47). Discarding the total derivative, the total SUSY variation reduces to

$$\begin{aligned} \delta S &= -\int \mathrm{d}^4 x \left(F^{\mu\nu} \partial_\mu \delta A_\nu + i\bar{\lambda}\gamma^\rho \partial_\rho \delta\lambda \right) \\ &= -\int \mathrm{d}^4 x \left(F^{\mu\nu} i\bar{\epsilon}\gamma_\nu \partial_\mu \lambda - \frac{i}{2} \bar{\lambda}\gamma^\rho \partial_\rho F_{\mu\nu} \gamma^{\mu\nu}\epsilon \right) . \end{aligned} \tag{6.5}$$

To proceed, we use again a Majorana flip identity, this time on $\bar{\epsilon}\gamma_\nu \partial_\mu \lambda$, and rewrite $\gamma^\rho \gamma^{\mu\nu} = \gamma^{\rho\mu\nu} + \eta^{\rho\mu}\gamma^\nu - \eta^{\rho\nu}\gamma^\mu$:

$$\delta S = -\int \mathrm{d}^4 x \left(-iF^{\mu\nu}(\partial_\mu \bar{\lambda})\gamma_\nu \epsilon - \frac{i}{2} \bar{\lambda}(\partial_\rho F_{\mu\nu})(\gamma^{\rho\mu\nu} + 2\eta^{\rho\mu}\gamma^\nu)\epsilon \right) . \tag{6.6}$$

The combination $\gamma^{\rho\mu\nu}\partial_\rho F_{\mu\nu} = \gamma^{\rho\mu\nu}\partial_{[\rho} F_{\mu\nu]} = 0$ vanishes by a Bianchi identity, and the remaining bits combine to give a total derivative:

$$\delta S = \int \mathrm{d}^4 x \, \partial_\mu \left(iF^{\mu\nu} \bar{\lambda}\gamma^\nu \epsilon \right) , \tag{6.7}$$

which can be discarded assuming SUSY preserving boundary conditions.

We next verify that the SUSY algebra closes in the proper way, consistent with (4.1). The simpler task is closure on the gauge field:

$$\begin{aligned} [\delta_1, \delta_2] A_\mu &= -i\bar{\epsilon}_2 \gamma_\mu \left(\frac{1}{2} F^{\lambda\rho} \gamma_{\lambda\rho} \epsilon_1 \right) - (1 \leftrightarrow 2) \\ &= -i\bar{\epsilon}_2 \left(\frac{1}{2} \gamma_{\mu\lambda\rho} + \eta_{\mu\lambda}\gamma_\rho \right) F^{\lambda\rho} \epsilon_1 - (1 \leftrightarrow 2) . \end{aligned} \tag{6.8}$$

By the Majorana flip identities (3.47), the $\gamma_{\mu\nu\rho}$ term will cancel out of the commutator, leaving

$$\begin{aligned}
[\delta_1, \delta_2] A_\mu &= -2i\bar{\epsilon}_2 \gamma^\nu \epsilon_1 F_{\mu\nu} \\
&= (2i\bar{\epsilon}_2 \gamma^\nu \epsilon_1) \partial_\nu A_\mu - \partial_\mu (2i\bar{\epsilon}_2 \gamma^\nu \epsilon_1 A_\nu)
\end{aligned} \tag{6.9}$$

The first term is a translation and the second a gauge transformation. Thus the supersymmetry closes up to gauge transformations.

Closure on the fermions, as usual, is a more complicated story involving Fierz rearrangement identities. We find

$$\begin{aligned}
[\delta_1, \delta_2] \lambda &= -2\partial_\mu \left(\frac{i}{2} \bar{\epsilon}_1 \gamma_\nu \lambda \right) \gamma^{\mu\nu} \epsilon_2 - (1 \leftrightarrow 2) \\
&= -i\gamma^{\mu\nu} (\bar{\epsilon}_1 \gamma_\nu \partial_\mu \lambda) \epsilon_2 - (1 \leftrightarrow 2) \ .
\end{aligned} \tag{6.10}$$

We will need in particular the same 4d Fierz identity (5.24) that we used in verifying closure of the SUSY algebra for the Wess-Zumino model. We take $\lambda = \epsilon_1$, $\chi = \epsilon_2$, and $\rho = \gamma_\nu \partial_\mu \lambda$. From the Majorana flip identities (3.47), the only term on the right hand side of the Fierz identity that will contribute to the commutator are the ones that involve γ_μ and $\gamma_{\mu\nu}$. Hence we find

$$[\delta_1, \delta_2] \lambda = \frac{i}{2} (\bar{\epsilon}_1 \gamma_\rho \epsilon_2) \gamma^{\mu\nu} \gamma^\rho \gamma_\nu \partial_\mu \lambda - \frac{i}{4} (\bar{\epsilon}_1 \gamma^{\rho\sigma} \epsilon_2) \gamma^{\mu\nu} \gamma_{\rho\sigma} \gamma_\nu \partial_\mu \lambda \tag{6.11}$$

We now need to go through some rather tedious manipulations with the gamma matrices. The strategy here is to try to either get an infinitesimal translation, i.e. $\bar{\epsilon}_1 \gamma_\mu \epsilon_2 \partial^\mu \lambda$, or something that will vanish by the equations of motion, i.e. stuff times $\partial\!\!\!/ \lambda$. Here we go for the simpler one:

$$\begin{aligned}
\gamma^{\mu\nu} \gamma^\rho \gamma_\nu &= -\gamma^{\mu\nu} \gamma_\nu \gamma^\rho + 2\gamma^{\mu\rho} \\
&= -3\gamma^\mu \gamma^\rho + 2\gamma^{\mu\rho} \\
&= -3\eta^{\mu\rho} - \gamma^{\mu\rho} \\
&= -4\eta^{\mu\rho} + \gamma^\rho \gamma^\mu \ .
\end{aligned} \tag{6.12}$$

And now for the more complicated one:

$$\begin{aligned}
\gamma^{\mu\nu} \gamma^{\rho\sigma} \gamma_\nu &= [\gamma^{\mu\nu}, \gamma^{\rho\sigma}] \gamma_\nu + \gamma^{\rho\sigma} \gamma^{\mu\nu} \gamma_\nu \\
&= 2(\eta^{\nu\rho} \gamma^{\mu\sigma} - \eta^{\mu\rho} \gamma^{\nu\sigma} + \eta^{\mu\sigma} \gamma^{\nu\rho} - \eta^{\nu\sigma} \gamma^{\mu\rho}) \gamma_\nu + 3\gamma^{\rho\sigma} \gamma^\mu \\
&= 2\gamma^{\mu\sigma} \gamma^\rho + 6\eta^{\mu\rho} \gamma^\sigma - 6\eta^{\mu\sigma} \gamma^\rho - 2\gamma^{\mu\rho} \gamma^\sigma + 3\gamma^{\rho\sigma} \gamma^\mu \\
&= 2\gamma^{\mu\sigma\rho} + 2\gamma^\mu \eta^{\rho\sigma} + 4\eta^{\mu\rho} \gamma^\sigma - 2\gamma^{\mu\rho\sigma} - 2\gamma^\mu \eta^{\rho\sigma} - 4\eta^{\mu\sigma} \gamma^\rho + 3\gamma^{\rho\sigma} \gamma^\mu \\
&= 4\gamma^{\sigma\rho\mu} + 4\eta^{\mu\rho} \gamma^\sigma - 4\eta^{\mu\sigma} \gamma^\rho + 3\gamma^{\rho\sigma} \gamma^\mu \\
&= 4\gamma^{\sigma\rho} \gamma^\mu + 3\gamma^{\rho\sigma} \gamma^\mu \\
&= -\gamma^{\rho\sigma} \gamma^\mu
\end{aligned} \tag{6.13}$$

In the second line, we used the fact that $-\frac{i}{2}\gamma^{\mu\nu}$ are generators of the Lorentz algebra. Assembling the various pieces, we find for the commutator

$$[\delta_1, \delta_2] \lambda = -2i(\bar{\epsilon}_1 \gamma^\mu \epsilon_2) \partial_\mu \lambda + \frac{i}{2} (\bar{\epsilon}_1 \gamma_\rho \epsilon_2) \gamma^\rho \gamma^\mu \partial_\mu \lambda + \frac{i}{4} (\bar{\epsilon}_1 \gamma_{\rho\sigma} \epsilon_2) \gamma^{\rho\sigma} \gamma^\mu \partial_\mu \lambda \ . \tag{6.14}$$

The first term is a translation. The second two vanish by the equations of motion. There is no gauge transformation piece here because λ does not transform under gauge transformations. Thus we have demonstrated that the SUSY algebra closes on-shell.

We did not expect the algebra to close off-shell. Fixing a gauge in 4d, the gauge field has 3 off-shell degrees of freedom while the Majorana fermion has 4. Thus we need one more bosonic degree of freedom to put together an off-shell formalism. The corresponding auxiliary field is often given the name D.

Problem 6.1. *How does the calculation of δS and $[\delta_1, \delta_2]$ above get modified in three dimensions?*

6.2 Super Yang-Mills

We now generalize the supersymmetric Maxwell theory to an arbitrary (semi-simple) Lie group G. Superficially, the action appears nearly identical but where now the gauge field A_μ and Majorana fermion λ are assumed to transform in the adjoint representation of G:

$$S_{\rm SYM} = -\frac{1}{g_{\rm YM}^2} \int d^4x \left(\frac{1}{2} \operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + i \operatorname{tr}(\bar{\lambda}\gamma^\mu \mathrm{D}_\mu \lambda) \right) . \tag{6.15}$$

Each term hides some cubic and quartic interactions; the kinetic terms are constructed from covariantized derivatives, $D_\mu\lambda = \partial_\mu\lambda - i[A_\mu, \lambda]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. By judiciously rescaling the fields, the interaction strength $g_{\rm YM}$ appears only as an over-all renormalization of the action. We can move $g_{\rm YM}$ to a more conventional location by sending $\lambda \to g_{\rm YM}\lambda$ and $A_\mu \to g_{\rm YM} A_\mu$.

Some Lie Algebra and Lie Group Background

To unpack the trace tr, we need to review some facts about Lie groups, Lie algebras, and their representations. Given a Lie group G, there is a corresponding Lie algebra $\mathfrak{g}$ and a set of generators of that algebra t_a which obey the commutation relations

$$[t_a, t_b] = i f^c{}_{ab} t_c \tag{6.16}$$

where $f^c{}_{ab} = -f^c{}_{ba}$ are (real) structure constants of the algebra. The indices a, b, c run from one to $\dim(G)$ where $\dim(G)$ is the dimension of G. In fact, for the Lie groups that we are interested in, we may take the structure constants to be completely antisymmetric in all indices. We will raise and lower indices with the Kronecker delta δ_{ab}, and therefore the placement of an index, up or down, will not mean much. The generators (indeed any three elements of the algebra) obey the Jacobi relation

$$0 = [[t_a, t_b], t_c] + [[t_c, t_a], t_b] + [[t_b, t_c], t_a] , \tag{6.17}$$

which in turn puts constraints on the $f^a{}_{bc}$. We will always take the generators to be Hermitian $t_a^\dagger = t_a$.

In general, we will be interested in Lie groups which decompose as

$$G = G_1 \times G_2 \times \cdots \times G_n , \tag{6.18}$$

where G_i is either U(1); one of the classical Lie groups SU(n), $SO(n)$, or $Sp(n)$; or one of the exceptional Lie groups G_2, F_4, E_6, E_7 or E_8. In fact, most of our interest will be in just two cases, $G_i = U(1)$ which we discussed above or $G_i = SU(\text{n})$. Indeed, the standard model of particle physics involves just $G = U(1) \times SU(2) \times SU(3)$.

Given a Lie algebra, we can then in turn find representations of that Lie algebra acting on a vector space. The defining or fundamental representation is induced by the definition of the group itself. For example, SU(n) is the group of $n \times n$ unitary complex matrices with determinant 1. It naturally acts on vectors in an n dimensional vector space. This construction of SU(n) is the fundamental representation.

To go from the Lie group to the Lie algebra, we use the exponential map,

$$g = e^{i\theta^a t_a} \ .$$

Indeed, e^{iH} is a unitary matrix if H is Hermitian. Pursuing the example of SU(n), from the matrix relation that $\log \det g = \operatorname{tr} \log g$, it follows that the generators t_a in the fundamental representation must be traceless matrices. For example, for $SU(2)$, we could write down the Pauli spin matrices as generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

They are Hermitian, traceless matrices. Moreover, there are three of them, consonant with the fact that the dimension of $SU(2)$ is $2^2 - 1 = 3$. When we anti-commute them, we find

$$[\sigma_a, \sigma_b] = 2i\epsilon^c{}_{ab}\sigma_c \ , \tag{6.19}$$

yielding structure constants $2\epsilon^c{}_{ab}$ proportional to the Levi-Civita tensor. In fact, it is conventional to use a normalization of the generators where the structure constants are the components of the Levi-Civita tensor itself, without the two, which we can achieve by rescaling $\sigma_a \to \tau_a = \frac{1}{2}\sigma_a$. In general, SU($n$) is an $n^2 - 1$ dimensional group which has a fundamental representation with $n^2 - 1$ traceless generators.

Another important representation is the adjoint representation. Given that the Lie algebra in the fundamental representation can act on an n dimensional vector space on which the group also acts, there must be an induced action of the group on the Lie algebra itself, treating the Lie algebra as a vector space in its own right. For $g \in G$, $t \in \mathfrak{g}$, and $v \in \mathbb{R}^n$, being a little sloppy about the disinction between the group and algebra and their representations, we have $tv \to gtv = gtg^{-1}gv$, where gtg^{-1} must still be an element in the Lie algebra. This adjoint group action then induces an adjoint action of the Lie algebra on itself. Let us take g to be close to the identity element and expand in a Taylor series

$$gtg^{-1} = e^{is}te^{-is} \approx t + ist - its + \ldots = t + i[s,t] + \ldots \tag{6.20}$$

The adjoint action of the Lie algebra on itself is nothing but the commutator. In other words, we should be able to find an adjoint representation with generators t_a which are $\dim(G) \times \dim(G)$ matrices which act on an arbitrary element of the Lie algebra, viewed as an $\dim(G)$ dimensional vector. The defining relation of the Lie algebra (6.16) makes it clear what these matrices must be. In the adjoint representation, the generators of the Lie algebra must be the structure constants themselves,

$$(t_a)^b{}_c = i f^b{}_{ac} \ . \tag{6.21}$$

As the indices run from 1 to $\dim(G)$, we see that indeed the t_a are indeed $\dim(G) \times \dim(G)$ matrices in this representation. For the SU(2) case we considered above, the fundamental representation has generators which are 2×2 matrices, while the adjoint representation will have 3×3 matrices. More generally for SU(n), the fundamental will have $n \times n$ matrix generators while the the adjoint will have $(n^2 - 1) \times (n^2 - 1)$ matrices.

Given an object A_μ or λ which transforms in the adjoint representation of the Lie group, we secretly mean then that they have adjoint indices A^a_μ and λ^a which can be contracted with the generators of the Lie algebra. A confusing point now involves what representation to use in writing down the generators t_a and furthermore how to normalize the t_a. In computing an object like $\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ where $F_{\mu\nu} = F^a_{\mu\nu} t_a$, we need to specify the representation for the t_a and determine $\mathrm{tr}(t_a t_b)$. For a compact Lie group, this trace is proportional to a Kronecker delta function

$$\mathrm{tr}(t_a t_b) = C \delta_{ab} \ , \tag{6.22}$$

where $C > 0$ – sometimes called the quadratic invariant of the representation – depends on the representation.

We have a proposal to use the structure constants themselves for the adjoint representation:

$$\mathrm{tr}(t_a t_b) = f^c{}_{ad} f^d{}_{bc} \ . \tag{6.23}$$

Given that this trace is proportional to the delta function, we must have that

$$C(\mathrm{adj}) = \frac{f_{abc} f^{abc}}{\dim(G)} \ . \tag{6.24}$$

Unfortunately, this expression $\mathrm{tr}(t_a t_b) = C\delta_{ab}$ is still ambiguous. As we saw already in the example of SU(2), if we rescale the generators, $t_a \to \lambda t_a$ by a real number, the structure constants will also be rescaled $f^a{}_{bc} \to \lambda f^a{}_{bc}$ and hence the quadratic invariant as well, $C \to \lambda^2 C$. As the structure constants are independent of the choice of representation, what will be invariant is the ratio of the quadratic invariants for different representations. For a general representation $\mathbf{r}$, we define the index as the ratio

$$T(\mathbf{r}) = \frac{C(\mathbf{r})}{C(\mathrm{fund})} \ . \tag{6.25}$$

Returning to our example $SU(2)$ with generators in the fundamental representation $\tau_a = \frac{1}{2}\sigma_a$, we can calculate these quadratic invariants. We find that $C(\mathrm{fund}) = \frac{1}{2}$ while for the adjoint $C(\mathrm{adj}) = 2$. More generally for SU(n), we will use a normalization for the generators where $C(\mathrm{fund}) = \frac{1}{2}$ still but $C(\mathrm{adj}) = n$. Thus $T(\mathrm{adj}) = 2n$.

Problem 6.2. *Construct generators for the Lie algebra $su(3)$ normalized such that $C(\mathrm{adj}) = 3$ and $C(\mathrm{fund}) = \frac{1}{2}$.*

Returning to our super Yang-Mills theory, for $\mathrm{SU}(n)$ it is conventional to write the kinetic term for the gauge field normalized such that

$$-\frac{1}{4g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\, F_{\mu\nu}^a F^{a\mu\nu} \ ,$$

with an implicit sum over the adjoint indices a. Restoring the generators, it is further conventional to use the fundamental representation

$$-\frac{1}{2g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\, \mathrm{tr}_{\mathrm{f}}(F_{\mu\nu}F^{\mu\nu}) \ ,$$

However, sometimes people write the kinetic term using adjoint generators, so beware. The difference in normalization will be the ratio $C(\mathrm{adj})/C(\mathrm{fund}) = T(\mathrm{adj})$.

Verifying Supersymmetry

The action (6.15) is supersymmetric with respect to essentially the same transformation rules as the Maxwell action

$$\begin{aligned} \delta A_\mu &= i\bar{\epsilon}\gamma_\mu\lambda \ , && (6.26)\\ \delta\lambda &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \ . && (6.27)\end{aligned}$$

Similar to what we did for the interacting Wess-Zumino model, we can divide up the supersymmetry variation of the action into terms that are linear and cubic in λ. These linear and cubic terms must vanish independently. There is in fact only one cubic term, which comes from varying $\bar{\lambda}\gamma^\mu[A_\mu,\lambda]$ in the kinetic term for the gaugino λ. As there is only one term, we can be careless about overall normalization

$$\begin{aligned} \delta_3\mathcal{L} &\sim \mathrm{tr}(\bar{\lambda}\gamma^\mu[(\bar{\epsilon}\gamma_\mu\lambda),\lambda])\\ &= \mathrm{tr}(t_a t_b) f^b{}_{cd}\bar{\lambda}^a\gamma^\mu(\bar{\epsilon}\gamma_\mu\lambda^c)\lambda^d\\ &\sim f_{abc}\bar{\lambda}^a\gamma^\mu(\bar{\epsilon}\gamma_\mu\lambda^b)\lambda^c \ . && (6.28)\\ & && (6.29)\end{aligned}$$

For the structure constants, we know that f_{abc} is antisymmetric in the bc indices. We will use the Fierz re-arrangement identities to put λ^b and λ^c in the same spinor bilinear. Many of the terms in the identity should then cancel by anti-symmetry of the bc indices. In fact only two terms survive, the "antisymmetric" ones involving γ_μ and $\gamma_{\mu\nu}$:

$$\begin{aligned} f_{abc}\bar{\lambda}^a\gamma^\mu(\bar{\epsilon}\gamma_\mu\lambda^b)\lambda^c &= -f_{abc}\bar{\lambda}^a\gamma^\mu(\bar{\lambda}^b\gamma_\mu\epsilon)\lambda^c\\ &= \frac{1}{4}f_{abc}\bar{\lambda}^a\gamma^\mu(\bar{\lambda}^b\gamma_\nu\lambda^c)\gamma^\nu\gamma_\mu\epsilon - \frac{1}{8}f_{abc}\bar{\lambda}^a\gamma^\mu(\bar{\lambda}^b\gamma_{\nu\rho}\lambda^c)\gamma^{\nu\rho}\gamma_\mu\epsilon && (6.30)\end{aligned}$$

where we used a Majorana flip identity in the first line and the Fierz identity in 4d in the second. After some further manipulations with the gamma matrices, we will achieve the desired result:

$$
\begin{aligned}
\gamma^\mu\gamma^\nu\gamma_\mu &= -(d-2)\gamma^\nu \ , && (6.31)\\
\gamma^\mu\gamma^{\nu\rho}\gamma_\mu &= \gamma^\mu(\gamma^{\nu\rho}{}_\mu + \gamma^\nu\delta^\rho_\mu - \gamma^\rho\delta^\nu_\mu)\\
&= (d-2)\gamma^{\nu\rho} + \gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\rho\\
&= (d-4)\gamma^{\nu\rho} \ . && (6.32)
\end{aligned}
$$

The second product $\gamma^\mu\gamma^{\nu\rho}\gamma_\mu$ therefore vanishes in exactly four dimensions. (In three dimensions, the $\gamma_{\mu\nu}$ structure would not be present in the Fierz identity to begin with.) However, in higher dimensions, we are potentially in trouble here, as this cubic variation $\delta_3\mathcal{L}$ will not vanish on its own. Anyway, in 4d (and also with a slight variation in 3d) we find the following seeming contradiction:

$$
\begin{aligned}
f_{abc}\bar\lambda^a\gamma^\mu(\bar\epsilon\gamma_\mu\lambda^b)\lambda^c &= -\frac{1}{2}f_{abc}(\bar\lambda^b\gamma_\nu\lambda^c)\bar\lambda^a\gamma^\nu\epsilon\\
&= \frac{1}{2}f_{abc}\bar\lambda^b\gamma^\mu(\bar\epsilon\gamma_\mu\lambda^a)\lambda^c\\
&= -\frac{1}{2}f_{abc}\bar\lambda^a\gamma^\mu(\bar\epsilon\gamma_\mu\lambda^b)\lambda^c \ , && (6.33)
\end{aligned}
$$

where in the second line we have used a Majorana flip identity and moved some of the spinors around, and in the last line we used the antisymmetry of the structure constants. We have arrived at an equation of the form $x = -\frac{1}{2}x$. The only way for this equation to be satisfied is if the corresponding product of spinors vanishes. Hence we have found that $\delta_3\mathcal{L} = 0$. We move on now to the linear terms.

The next obstacle in our way is to generalize the integration by parts argument we used earlier in varying the kinetic term for the gaugino. Before, we had just ∂_μ, but now we have D_μ. In particular, what we would like to be true is the following

$$
\operatorname{tr}(\delta\bar\lambda\gamma^\mu D_\mu\lambda) = \partial_\mu \operatorname{tr}(\delta\bar\lambda\gamma_\mu\lambda) + \operatorname{tr}(\bar\lambda\gamma^\mu D_\mu\delta\lambda) \ . \qquad (6.34)
$$

We need to check what happens to the new connection term proportional to A_μ:

$$
\begin{aligned}
-i\operatorname{tr}(\delta\bar\lambda\gamma^\mu[A_\mu,\lambda]) &= i\operatorname{tr}([A_\mu,\bar\lambda]\gamma^\mu\delta\lambda)\\
&= -i\operatorname{tr}([\bar\lambda,A_\mu]\gamma^\mu\delta\lambda)\\
&= -i\operatorname{tr}(\bar\lambda\gamma^\mu[A_\mu,\delta\lambda]) \ . && (6.35)
\end{aligned}
$$

In the first line, we used a Majorana flip identity, and in the third cyclicity of the trace, that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

We can thus write the supersymmetric variation of the linear terms in the Lagrangian immediately a more compact way, dropping a boundary term:

$$
\delta_1\mathcal{L} = -\frac{1}{g^2_{\rm YM}}\left(\operatorname{tr}(F^{\mu\nu}\delta F_{\mu\nu}) + 2i\operatorname{tr}(\bar\lambda\gamma^\rho \mathrm{D}_\rho\delta\lambda)\right) \ . \qquad (6.36)
$$

The variation of the field strength is then

$$
\begin{aligned}
\delta F_{\mu\nu} &= \partial_\mu\delta A_\nu - i[A_\mu,\delta A_\nu] - (\mu\leftrightarrow\nu)\\
&= i\bar\epsilon\gamma_\nu \mathrm{D}_\mu\lambda - (\mu\leftrightarrow\nu) \ . && (6.37)
\end{aligned}
$$

Assembling the pieces, the variation of the Lagrangian is then

$$\delta_1\mathcal{L} = -\frac{1}{g^2_{\rm YM}}\left(2i\,\mathrm{tr}(F^{\mu\nu}\bar{\epsilon}\gamma_\nu \mathrm{D}_\mu\lambda) - i\,\mathrm{tr}(\bar{\lambda}\gamma^\rho \mathrm{D}_\rho\gamma_{\mu\nu}F^{\mu\nu}\epsilon)\right) \ . \tag{6.38}$$

As in the Maxwell case, we again employ the identity $\gamma_\rho\gamma_{\mu\nu} = \gamma_{\rho\mu\nu} + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\nu}\gamma_\mu$ and find

$$\delta_1\mathcal{L} = -\frac{1}{g^2_{\rm YM}}\left(-2i\,\mathrm{tr}(F^{\mu\nu}\mathrm{D}_\mu\bar{\lambda}\gamma_\nu\epsilon) - 2i\,\mathrm{tr}(\bar{\lambda}\mathrm{D}_\mu F^{\mu\nu}\gamma_\nu\epsilon) - i\,\mathrm{tr}(\bar{\lambda}\gamma^{\rho\mu\nu}\mathrm{D}_\rho F_{\mu\nu}\epsilon)\right) \tag{6.39}$$

The first two terms add up to a total derivative which can be dropped, and the third term vanishes by a Bianchi identity, exactly as we saw in the Maxwell case. The added wrinkle are the connection terms A_μ in the covariant derivatives D_μ. We have packaged things in a way such that these A_μ terms go harmlessly along for the ride.

Problem 6.3. *Show that* $\mathrm{tr}(F^{\mu\nu}\mathrm{D}_\mu\bar{\lambda}\gamma_\nu\epsilon) + \mathrm{tr}(\bar{\lambda}\mathrm{D}_\mu F^{\mu\nu}\gamma_\nu\epsilon) = \partial_\mu\,\mathrm{tr}(F^{\mu\nu}\bar{\lambda}\gamma_\nu\epsilon)$. *Prove that* $\mathrm{D}_{[\mu}F_{\nu\rho]} = 0$ *and hence that* $\gamma^{\mu\nu\rho}\mathrm{D}_\mu F_{\nu\rho} = 0$. *Note the definition* $\mathrm{D}_\mu F_{\nu\rho} \equiv \partial_\mu F_{\nu\rho} - i[A_\mu, F_{\nu\rho}]$.

Finally, we should verify that the supersymmetry algebra closes up to equations of motion and gauge transformations. The calculation is very similar to what we did in the abelian case. For the gauge field, we find

$$\begin{aligned}[\delta_1,\delta_2]A_\mu &= -(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1)F_{\mu\nu} \\ &= 2i(\bar{\epsilon}_2\gamma^\nu\epsilon_1)\partial_\nu A_\mu - \mathrm{D}_\mu(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 A_\nu)\end{aligned} \tag{6.40}$$

using the fact that $F_{\mu\nu} = \mathrm{D}_\mu A_\nu - \partial_\nu A_\mu$.

Problem 6.4. *Show also that* $[\delta_1,\delta_2]\lambda$ *is an infinitesimal translation, up to equations of motion and gauge transformations.*

6.3 Super Yang-Mills with Matter

We will be a little bit less thorough in our presentation of of super Yang-Mills with matter fields than we have been heretofore. Most of the requisite calculations we have seen at this point in a simpler context. What we need to do is assemble all the ingredients in one place. What we are about to write down is the super Yang-Mills action of the previous section plus the interacting Wess-Zumino model from section 5. The added wrinkle is that the scalar and fermion from the Wess-Zumino model now transform in a representation of the gauge group and hence can interact through the exchange of gauge bosons (and also, because of supersymmetry, gauginos).

In addition to the gauge field A^a_μ and gaugino λ^a of section 6.2, we add a scalar ϕ^i and Majorana fermion ψ^j transforming in an arbitrary representation $\mathbf{r}$ of the gauge group G. The indices i, j run from one to $\dim(\mathbf{r})$. For example, for the fundamental of $\mathrm{SU}(n)$, they would run from one to n. Note that ϕ^i and ψ^j must transform in the same representation $\mathbf{r}$ because of the constraints of supersymmetry.

The covariant derivatives acting on ϕ^i and ψ^j must involve Lie algebra generators in the correct representation. We write

$$\begin{aligned}\mathrm{D}_\mu\phi^i &= \partial_\mu\phi^i - iA^a_\mu(t_a)^i{}_j\phi^j \ , \\ \mathrm{D}_\mu\psi^i &= \partial_\mu\psi^i - iA^a_\mu(t_a)^i{}_j\psi^j \ .\end{aligned} \tag{6.41}$$

Scaling out an overall factor of the coupling constant, we divide up the Lagrangian into three pieces:

$$\mathcal{L} = \frac{1}{g_{\rm YM}^2} \left(\mathcal{L}_{\rm SYM} + \mathcal{L}_{\rm WZ} + \mathcal{L}_{\rm Yuk} \right) \ . \tag{6.42}$$

The first piece is the super Yang-Mills action from section 6.2:

$$\mathcal{L}_{\rm SYM} = -\left(\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{i}{2} \bar{\lambda}^a \not{D} \lambda^a - \frac{1}{2} (D^a)^2 \right) \ .$$

We have added an auxiliary field D^a, transforming in the adjoint representation of the gauge group, that will allow us to close the SUSY algebra off-shell. The SUSY transformation rules for these adjoint fields are

$$\begin{aligned} \delta A^a_\mu &= i\bar{\epsilon}\gamma_\mu \lambda^a \ , \\ \delta\lambda^a &= -\frac{1}{2} F^a_{\mu\nu}\gamma^{\mu\nu}\epsilon - i\gamma D^a \ , \\ \delta D^a &= i\bar{\epsilon}\gamma \not{D} \lambda^a \ . \end{aligned} \tag{6.43}$$

The second term in the Lagrangian is the interacting Wess-Zumino model of section 5. We have added an auxiliary field F^i transforming in the representation $\mathbf{r}$ of the gauge group, again so that we can close the SUSY algebra off-shell.

$$\begin{aligned} \mathcal{L}_{\rm WZ} &= -\frac{1}{2} ({\rm D}_\mu \phi^*_i)({\rm D}^\mu \phi^i) - \frac{i}{2} \bar{\psi}_i \gamma^\mu {\rm D}_\mu \psi^i + \frac{1}{2} F^*_i F^i \\ &\quad + F^i \partial_i \mathcal{W} + F^*_i \bar{\partial}^i \mathcal{W}^* + (\partial_i \partial_j \mathcal{W}) i\bar{\psi}^i \Pi_+ \psi^j + (\bar{\partial}^i \bar{\partial}^j \mathcal{W}^*) i\bar{\psi}_i \Pi_- \psi_j \ . \end{aligned} \tag{6.44}$$

We are using the shorthand notation that $\partial_i \mathcal{W} = \frac{\partial \mathcal{W}}{\partial \phi^i}$, and similarly $\bar{\partial}^i \mathcal{W}^* = \frac{\partial \mathcal{W}^*}{\partial \phi^*_i}$. The transformation rules for these matter fields are

$$\begin{aligned} \delta\phi^i &= 2i\bar{\epsilon}\Pi_+\psi^i \ , \quad \delta\phi^*_i = 2i\bar{\epsilon}\Pi_-\psi_i \ , \\ \delta\Pi_+\psi^i &= (\not{D}\phi^i)\Pi_-\epsilon + F^i \Pi_+\epsilon \ , \\ \delta\Pi_-\psi_i &= (\not{D}\phi^*_i)\Pi_+\epsilon + F^*_i \Pi_-\epsilon \\ \delta F^i &= 2i\bar{\epsilon}\Pi_-\not{D}\psi^i \ , \quad \delta F^*_i = 2i\bar{\epsilon}\Pi_+\not{D}\psi_i \ . \end{aligned} \tag{6.45}$$

An awkward feature of this model is that ϕ^i and ϕ^*_i will in general transform in complex conjugate representations of the gauge group G. The adjoint representation for SU(n) is real, and the issue does not appear here, but if we were to choose the fundamental representation of SU(n), the complex conjugate representation is different, often called the anti-fundamental. One might worry that we have used only one type of generator $(t_a)^i{}_j$ in writing down the Lagrangian even though we have potentially two different representations at play. In fact, everything is okay since the representation and its complex conjugate are related, $t_a^{(\bar{\mathbf{r}})} = -(t_a^{(\mathbf{r})})^T$.

As $\delta\phi^i$ transforms into one chirality of the fermion $\Pi_+\psi^i$ and $\delta\phi^*_i$ transforms into the other $\Pi_-\psi_i$, we find that the different chiral pieces of the Majorana fermion ψ will transform in different, complex conjugate representations of the gauge group! As one might expect, there is no consistent way of acting on a Majorana spinor with a complex representation and preserving the reality property of the spinor. We could restrict to real representations, but what is almost always done in the literature is to work instead with Weyl fermions. The kernel of Π_+ acting on a Majorana spinor is trivial, and hence we can think of $\psi_+ = \Pi_+\psi$ as a repackaging of the degrees of freedom of the Majorana spinor into a Weyl spinor, as we discussed in section 4.1. If ψ_+ transforms in the fundamental representation, then $\psi_- = \Pi_-\psi$, which is also ψ_+^*, will transform in the anti-fundamental. As one can see by staring at the Lagrangian above, we can consistently replace our Majorana spinor ψ everywhere by a Weyl spinor ψ_+^i and its complex conjugate ψ_{-i}, where the location of the index i implies whether it is a fundamental or anti-fundamental representation.

The one sticky point is the kinetic term, which we have written by an abuse of notation as $\frac{1}{2}\bar{\psi}_i \not{D}\psi^i$. Let us go back to the Wess-Zumino model case with no A_μ. We can perform the following manipulation:

$$\begin{aligned} \bar{\psi}_+\gamma^\mu\partial_\mu\psi_+ &= \overline{\Pi_+\psi}\gamma^\mu\partial_\mu\Pi_+\psi \\ &= \bar{\psi}\Pi_-\gamma^\mu\partial_\mu\Pi_+\psi \\ &= \bar{\psi}\gamma^\mu\partial_\mu\Pi_+\psi \\ &= \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}\bar{\psi}\gamma^\mu\gamma\partial_\mu\psi \ . \end{aligned} \tag{6.46}$$

The second term in the last line is a total derivative, by the Majorana flip identities. Thus we can replace a kinetic term for a chiral spinor $\overline{\psi_+}\not{\partial}\psi_+$ by a kinetic term for a Majorana spinor $\frac{1}{2}\bar{\psi}\not{\partial}\psi$. What we really mean by $\frac{1}{2}\bar{\psi}_i\not{D}\psi^i$ is thus $\overline{\psi_+}_i\not{D}\psi_+^i$.

One might think the story ends here, but there is a problem. Given that the matter fields interact by exchanging gauge bosons, supersymmetry implies that they must also interact by exchanging gauginos. Therefore, for the action to be invariant, we need to add by hand the following interaction terms

$$\mathcal{L}_{\text{Yuk}} = -\phi_i^*\bar{\lambda}^a(t_a)^i{}_j\Pi_+\psi^j - \bar{\psi}_i\Pi_-(t_a)^i{}_j\lambda^a\phi^j - \frac{i}{2}\phi_i^*(t_a)^i{}_j\phi^j D^a \ . \tag{6.47}$$

Having added the first two Yukawa's, we find we need to add the last term as well to get the supersymmetry variation to vanish.

There is in fact one further term one can add to this last piece of the Lagrangian when the gauge group is abelian. In this case the auxiliary field D is not charged under the U(1) and so a term ξD in the Lagrangian is gauge invariant. Furthermore, δD is a total derivative, and so ξD is supersymmetric (up to a boundary term that we neglect). Such a term is often called a Fayet-Iliopoulos term, or FI term for short.

Ideally, we should do what we have done in previous cases, i.e. verify that $\delta\mathcal{L} = 0$ up to total derivatives and also verify that $[\delta_1, \delta_2]$ acts as an infinitesimal translation. Let us briefly summarize what pieces of this calculation we have effectively already done and what pieces remain to be completed.

We verified previously that $\delta\mathcal{L}_{\text{SYM}} = 0$ and also that $[\delta_1, \delta_2]$ acting on A_μ^a was an infinitesimal translation up to gauge transformations. We left as an exercise that $[\delta_1, \delta_2]\lambda^a$ should be an infinitesimal translation as well. We did not previously include the auxiliary field D^a in these calculations, and so in principle we could do that now and generalize the calculation off-shell.

Regarding closure of the SUSY algebra for the Wess-Zumino model, we verified in the non-interacting case that $[\delta_1, \delta_2]$ is an infinitesimal translation up to the equations of motion. We left it as an exercise to show that off-shell including the auxiliary field F, $[\delta_1, \delta_2]$ functions as an infinitesimal translation as well. Even though the exercise was posed at the end of the discussion of the free case, it applies without change to the interacting case as well.

We verified previously that $\delta\mathcal{L}_{\mathrm{WZ}} = 0$ in the case when the covariant derivatives D_μ were merely partial derivatives ∂_μ. In fact, we saw in the argument surrounding (6.34) that the distinction between D_μ and ∂_μ does not really matter for the most part. The exception is when we actually need to look at terms involving δA^a_μ. In other words, some thought and reflection ought to convince the reader that $\delta\mathcal{L}_{\mathrm{WZ}}$ will continue to vanish in the present case up to terms proportional to δA_μ.

These nonvanishing terms in $\delta\mathcal{L}_{\mathrm{WZ}}$ proportional to δA_μ must then cancel against $\delta\mathcal{L}_{\mathrm{Yuk}}$. Indeed they do, but we will not check it.

7 Scale Dependence of Super Yang-Mills with Matter

In this chapter, we will try to understand some of the interplay between the renormalization group – the idea that physics can depend on energy scale in QFT – and supersymmetry.

The gauge coupling famously depends on the energy scale. The one loop computation of this dependence in QCD was performed in 1973 by David Gross and Frank Wilczek and independently by David Politzer. The three were awarded the Nobel Prize in 2004. In QCD, one finds that the coupling strength gets stronger as one goes to lower energy scales, and conversely weaker as one goes to higher energy scales. Strong coupling at low scales gives some intuition for confinement and chiral symmetry breaking that we observe in nature. Conversely, weak coupling at high scales, also called asymptotic freedom, is consistent with collider experiments.

We can express this dependence using a beta function:

$$\beta_g \equiv \mu \frac{dg}{d\mu} = -\frac{b}{16\pi^2} g^3 + O(g^5) \ , \tag{7.1}$$

where g is the coupling and μ is the energy scale. The first coefficient b is determined by a collection of one-loop diagrams. We quote the result

$$b = \frac{11}{6} T(\mathrm{adj}) - \frac{1}{3} \sum_a T(\mathbf{r}_a) - \frac{1}{6} \sum_n T(\mathbf{r}_n) \ , \tag{7.2}$$

where the sum on a is over Weyl fermions, with the ath fermion in the $\mathbf{r}_a$ representation of the gauge group. The sum on n is over complex bosons in representations $\mathbf{r}_n$. Recall the index $T(\mathbf{r})$ is the ratio of the quadratic invariants $C(\mathbf{r})/C(\mathrm{fund})$. Clearly $T(\mathrm{fund}) = 1$ while for $\mathrm{SU}(n)$, we discussed above that $T(\mathrm{adj}) = 2n$. We include in b only light species of matter, whose mass is much less than the reference scale μ_0. Heavier species are essentially Boltzmann suppressed and have a negligible effect on the running of the coupling.

Assuming we know the coupling $g(\mu_0)$ at some reference scale μ_0, we can integrate the differential equation (7.1) to give a result for the coupling at a nearby scale $\mu \approx \mu_0$:

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2} \log\left(\frac{\Lambda}{\mu}\right) \ . \tag{7.3}$$

The quantity

$$\Lambda \equiv \mu_0 \exp\left(-\frac{8\pi^2}{bg^2(\mu_0)}\right) \ . \tag{7.4}$$

is a "strong coupling scale" defined to yield $g(\mu_0)$ at $\mu = \mu_0$. Of course, as we have only computed the beta function to leading order, we should not trust this equation once $\mu = \Lambda$, but Λ still can give a rough estimate of where we expect the coupling to get large. For asymptotically free theories such as QCD where $b > 0$, this scale occurs below the reference scale $\Lambda < \mu_0$. Theories that are weakly coupled at low energy, such as QED, have $b < 0$, and the scale Λ occurs above the reference scale $\Lambda > \mu_0$. (For QED, note that $T(\text{adj}) = 0$, and the index for the matter fields is simply their electric charge, $T(q) = q$.)

From the expression (7.2) for b, clearly if there is enough light charged matter around, then $b < 0$ and the theory will be free at low energy. The usual jargon here is to refer to high energy behavior as ultraviolet or UV and low energy behavior as infrared or IR. Theories which are asymptotically free or UV free have a negative beta function ($b > 0$). Theories which are IR free have a positive beta function ($b < 0$).

For supersymmetric Yang-Mills with matter, the one loop beta function takes on a more restricted form. For each gluon, we also by supersymmetry must have a gluino in the adjoint representation. Furthermore, for each Weyl fermion in a general representation $\mathbf{r}$, we must have a complex boson in the same representation. The b coefficient simplifies to

$$b = \frac{3}{2}T(\text{adj}) - \frac{1}{2}\sum_n T(\mathbf{r}_n) \ . \tag{7.5}$$

In these notes, we are particularly interested in the case of SU(n) Yang-Mills with N_f matter multiplets, i.e. N_f Weyl fermions and N_f complex bosons, in the fundamental representation, and N_f matter multiplets in the anti-fundamental representation. In this case, we have the further simplification $b = 3n - N_f$. The choice $3n = N_f$ thus separates the IR free theories with $N_f > 3n$ from their UV free cousins with $N_f < 3n$. This analysis is one-loop and receives higher order corrections. The two loop beta function has been computed:

$$\beta_g = -\frac{g^3}{16\pi^2}(3n - N_f) + \frac{g^5}{128\pi^4}\left(2nN_f - 3n^2 - \frac{N_f}{n}\right) + O(g^7) \ . \tag{7.6}$$

If we start with $N_f > 3n$, both the $O(g^3)$ and $O(g^5)$ terms are positive, and the only nontrivial zero of β_g is at $g = 0$, corresponding to the free IR limit. However, if we approach $N_f = 3n$ from below, something interesting happens. The negative $O(g^3)$ term competes with positive $O(g^5)$ contribution (see figure 2), and there is a possibility of finding a new nontrivial zero $\beta(g_*) = 0$ with $g_*^2 > 0$, as we flow down in energy scale from the free UV limit.

There is a question whether we can trust such a zero, computed from only the first two terms in a power series in g, but we have another parameter at our disposal, n. Imagine we take both n and N_f large such that $3n - N_f = \delta$ where δ is an $O(1)$ quantity. We can find a root of β_g at

$$g_*^2 = \frac{8\pi^2\delta}{3n^2} \tag{7.7}$$

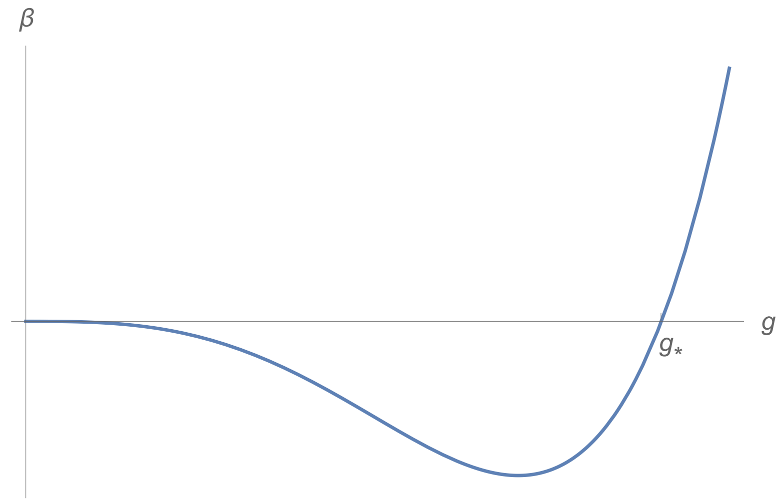

Figure 2: Schematic plot of the beta function up to two loops for super Yang-Mills theory with $N_f \lesssim 3n$.

that we can indeed trust since the next $O(g^7)$ term is down by a factor of n compared to the first two terms. In this range where $3n - N_f$ is positive and order one, we conclude that the IR limit is a weakly interacting fixed point with $g_*^2 = O(n^{-1})$.

Problem 7.1. *The beta function to two loops for QCD is*

$$\beta_g = -\frac{g^3}{48\pi^2}(11n - 2N_f) - \frac{g^5}{256\pi^4}\left(\frac{34}{3}n^2 - \frac{1}{2}N_f\left(2\frac{n^2-1}{n} + \frac{20}{3}n\right)\right) + O(g^7) \ . \qquad (7.8)$$

Try to repeat for ordinary QCD the Caswell-Banks-Zaks style analysis that we performed in the supersymmetric case.

This fixed point theory has nothing to do with supersymmetry. We could have performed the same analysis in a generic Yang-Mills theory in a large n and N_f limit, about the point where the one-loop contribution to β_g vanishes. The phenomenon was first reported by William E. Caswell in 1974, and is usually called the Banks-Zaks fixed point.

What is special about super Yang-Mills with matter is that the nature of higher loop corrections to β_g is highly constrained by holomorphicity. The arguments are similar to what we saw for the superpotential in the context of the Wess-Zumino model; we will come to these arguments somewhat later. For the moment, let us merely state the result, and try to understand some of its consequences. The claim is that in the appropriate renormalization group scheme, the beta function is exact at one loop. There are no higher loop corrections.

There seems to be an immediate contradiction, in that we wrote down a nonzero $O(g^5)$ contribution above to β_g. The scheme in which there are nonzero $O(g^5)$ terms is one in which the kinetic terms for the matter fields are canonically normalized, e.g. $|\mathrm{D}_\mu\phi|^2$. The scheme in which the $O(g^5)$ terms vanish is one in which the kinetic terms have explicit wave function renormalization factors, e.g. $|\mathrm{D}_\mu\phi|^2 Z_\phi$. The main resulting difference for β_g is the appearance of anomalous dimensions.

What do we mean by anomalous dimensions? By dimensional analysis, we can establish the classical scaling dimension of our fields by looking at their kinetic terms. Scalar fields have dimension $\Delta_\phi = 1$ in four space-time dimensions. Fermions have dimension $\Delta_\psi = \frac{3}{2}$. Interactions, however, can shift these values through renormalization effects, $\Delta(\mu) = \Delta(\mu_0) + \gamma(\mu)$ where $\gamma(\mu)$ is scale dependent. The renormalization shift, $(\partial_\mu \phi)^2 \to (\partial_\mu \phi)^2 Z_\phi(\mu)$, induces an anomalous dimension via

$$\gamma = \frac{\mu}{2} \frac{dZ}{d\mu} \ . \tag{7.9}$$

Supersymmetry then means γ must be the same for fields in the same multiplet.

In the RG scheme with canonical kinetic terms, the "exact" beta function is

$$\mu \frac{d(1/g_{\rm YM}^2)}{d\mu} = \frac{1}{2} \frac{3T(\text{adj}) - \sum_n T(\mathbf{r}_n)(1 - 2\gamma(\mathbf{r}_n))}{8\pi^2 - \frac{1}{2} T(\text{adj}) g^2} \ , \tag{7.10}$$

where $\gamma(\mathbf{r}_n)$ is the anomalous dimension associated to the (ϕ, ψ) multiplet transforming in the $\mathbf{r}_n$ representation. The numerator is very close to what it was before, with the exception of the additional dependence on the $\gamma(\mathbf{r}_n)$.

These anomalous dimensions have consequences that go beyond effects on β_g. In the superpotential $\mathcal{W}$, suppose we have an interaction term of the form $\lambda_r \phi_1 \phi_2 \ldots \phi_n$, with associated coupling constant λ_r. We argued in the context of the Wess-Zumino model that λ_r is not renormalized through quantum effects. What we meant more specifically is the following. Suppose the theory has a weakly coupled limit at some reference scale μ_0 where we can calculate all of the classical scaling dimensions of the fields. In this limit, the operator $\phi_1 \cdots \phi_n$ will have scaling dimension $d_r = \sum_i \Delta_i(\mu_0)$. Consequently, λ_r will have scaling dimension $3 - d_r$. (The superpotential results in a potential term in the Lagrangian of the form $|\partial \mathcal{W}|^2$ and thus must have scaling dimension $\Delta_{\mathcal{W}} = 3$.) Pulling out the scale μ_0, we can write the superpotential coupling in the form

$$\lambda_r(\mu_0) = \lambda_{r0} \, \mu_0^{3-d_r} \ , \tag{7.11}$$

where λ_{r0} is dimensionless. Non-renormalization of the superpotential means this form continues to be valid at an arbitrary scale μ:

$$\lambda_r(\mu) = \lambda_{r0} \, \mu^{3-d_r} \ . \tag{7.12}$$

We can further define a beta function for the superpotential couplings:

$$\beta_{\lambda_r} \equiv \mu \frac{d\lambda_r}{d\mu} = (3 - d_r)\lambda_r \ . \tag{7.13}$$

We should be careful again, however, about RG scheme dependence. This result is in a scheme where the kinetic terms are not canonically normalized. Through a field redefinition, $\phi_i \to Z_i^{-1/2} \phi_i$, we can get canonically normalized kinetic terms at the price of introducing anomalous dimensions into the beta functions β_{λ_r} for the super potential couplings. In the canonically normalized scheme, we have instead

$$\beta_{\lambda_r} = (3 - d_r - \sum_i \gamma_i)\lambda_r \ . \tag{7.14}$$

This result has the possibility of giving you the wrong idea about how composite operators behave in general interacting theories. In the supersymmetric case, one consequence of this argument is the demonstration that if we know the scaling dimensions $\Delta_i(\mu)$ of the ϕ_i at an arbitrary scale μ, then the dimension of the composite operator $\phi_1 \cdots \phi_n$ will be $\sum_i \Delta_i(\mu)$. While true in the free limit, this additivity is almost never true in a typical QFT scenario.

Some nomenclature and further insight from physics. Physics must depend on dimensionless quantities. At a scale μ, we expect λ_r to appear in the combination $\lambda_r \mu^{\sum_i \Delta_i - 3}$. If $\sum_i \Delta_i - 3 > 0$, then perturbative effects, which come with positive powers of the coupling λ_r should get weak at small values of μ, i.e. in the IR. We call such a coupling irrelevant. Similarly, for $\sum_i \Delta_i - 3 < 0$, perturbative effects should get strong in the IR. We call such a coupling relevant. The critical case, $\sum_i \Delta_i = 3$, where λ does not change with scale, is called marginal.

Leigh and Strassler proposed a clever strategy for thinking about supersymmetric gauge theories where the couplings g and λ_r can be made independent of scale. Suppose the conditions $\beta_r = 0$ and $\beta_g = 0$ are not all linearly independent as functions of g and λ_r. Then there will be a submanifold in the the (g, λ_r) coupling space where the beta functions vanish. Changing the couplings in such a way as to stay on this manifold corresponds to turning on an exactly marginal operator. The presence of these marginal operators in turn means these scale invariant theories will have nontrivial interactions.

As an example, consider an SU(n) gauge theory with $2n$ (Q_i, q_i) multiplets transforming in the fundamental of SU(n) and $2n$ $(\tilde{Q}^j, \tilde{q}^j)$ multiplets transforming in the anti-fundamental. We introduce a third multiplet (Φ, Ψ) transforming in the adjoint. This matter content allows for a superpotential term $\mathcal{W} = \lambda \operatorname{tr}(\tilde{Q}^j \Phi Q_j)$ which is classically marginal. The beta function for the gauge coupling vanishes as well at one loop since $b = 0$. Now let us consider what happens once we allow for anomalous dimensions γ_Φ and γ_Q. By symmetry, it must be that $\gamma_Q = \gamma_{\tilde{Q}}$. Both β_g and β_λ will vanish when $\gamma_\Phi + 2\gamma_Q = 0$, putting one condition on two variables, g and λ_r, and allowing for a line of fixed points which goes through the origin, where the coupling does not depend on scale. The exact curve turns out to be $\lambda = g$, and along this line, the theories have $\mathcal{N} = 2$ extended supersymmetry.

Another example is Yang-Mills with three adjoint multiplets (Φ_i, Ψ_i), $i = 1, 2, 3$. Consider a superpotential of the form

$$\mathcal{W} = a \operatorname{tr} \Phi_1 \Phi_2 \Phi_3 + b \operatorname{tr} \Phi_3 \Phi_2 \Phi_1 + c \operatorname{tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^2) \ . \tag{7.15}$$

There is a weak coupling limit where $\mathcal{W}$ is classically marginal and also $b = 0$. The anomalous dimensions $\gamma_i = \gamma$ are all equal by symmetry. The beta functions β_g, β_a, β_b and β_c will all vanish provided $\gamma = 0$, putting one condition on four variables and leaving a three dimensional space of fixed points. A one dimensional sub-space of this manifold, where $g = a = -b$ and $c = 0$, has $\mathcal{N} = 4$ extended supersymmetry.

Problem 7.2. *This pair of questions involves an SU$(n) \times$ SU(n) supersymmetric gauge theory with three types of field multiplets. We have scalars A_i, $i = 1, 2$, and their fermionic super partners a_i transforming in the $(n, \bar{n})$ representation of SU$(n) \times$ SU(n), where the notation $(n, \bar{n})$ is short for (fundamental, anti-fundamental). We have the multiplet (B_j, b_j), $j = 1, 2$, transforming in the $(\bar{n}, n)$. And we have two types of adjoint multiplets (Φ_k, ψ_k), $k = 1, 2$, Φ_1 transforming in the adjoint of the first SU(n) and Φ_2 transforming in the adjoint of the second.*

a) We start with the superpotential

$$\mathcal{W} = \lambda_1 \operatorname{tr}(\Phi_1(A_1 B_1 + A_2 B_2)) + \lambda_2 \operatorname{tr}((B_1 A_1 + B_2 A_2)\Phi_2)$$

Do the beta functions vanish in the perturbative limit? What is the dimension of the conformal manifold on which the beta functions vanish?

b) Consider the special case $\lambda_1 = \lambda_2 = \lambda$. Let us add a mass term to the super potential $\delta\mathcal{W} = M \operatorname{tr}(\Phi_1^2 - \Phi_2^2)$. At energies low compared to the mass, the adjoint fields cannot be produced and we can "integrate them out". Assuming that at low energies there is a classical and supersymmetric solution to the equations of motion that involves extremizing $\mathcal{W}$ with respect to the Φ_i, use the "equations of motion" to remove the Φ_i from the super potential and calculate a new effective superpotential at low energy. For this new superpotential, is there a choice of anomalous dimensions for which the beta functions vanish? What is the new dimension of the conformal manifold on which the beta functions vanish?

7.1 The Theta Parameter and Non-Renormalization

To explain why the beta function in SYM is, for a particular choice of RG scheme, one-loop exact, we need to back up and discuss the theta angle in gauge theories. There is a possibility of an additional term in a gauge theory of the form

$$S_\theta = \int \mathrm{d}^4x \frac{\theta}{16\pi^2} \operatorname{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \ , \tag{7.16}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$ is often called the dual field strength. Note that $F_{\mu\nu}F^{\mu\nu} = -\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$, and so there is only one new linearly independent combination to consider here. In the presence of a boundary, such a theta term reduces to a Chern-Simons term on the boundary:

$$S_\theta = \int \mathrm{d}^4x \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu \operatorname{tr}\left(A_\nu\partial_\rho A_\sigma - \frac{2i}{3}A_\nu A_\rho A_\sigma\right) \ . \tag{7.17}$$

The issue is that even thought this second expression for S_θ looks like a total derivative, A_μ need not be globally defined throughout the space-time. (In the context of supersymmetry, we would also need a boundary term for the gaugino, λ, of the rough form $\theta\bar{\lambda}\Pi_+\lambda$.)

Consider instead of our usual Minkowski space, the Euclidean setting. Instead of working in an infinite space, we imagine the system in a box with spherical boundary S^3. We furthermore demand our field strength $F_{\mu\nu}$ die off at infinity and so the corresponding gauge potential should be pure gauge

$$A_\mu = -i(\partial_\mu h)h^{-1} \ , \tag{7.18}$$

where $h(x^\mu)$ is an element of the gauge group. Plugging this expression into the total derivative form (7.17) and using the identity $\partial_\mu h^{-1} = -h^{-1}(\partial_\mu h)h^{-1}$, we find

$$\begin{aligned} S_\theta &= -\frac{\theta}{24\pi^2}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\partial_\mu \operatorname{tr}\left[h^{-1}(\partial_\nu h)h^{-1}(\partial_\rho h)h^{-1}(\partial_\sigma h)\right] \\ &= -\frac{\theta}{24\pi^2}\int_{S^3} \mathrm{d}^3\xi\, \epsilon^{abc} \operatorname{tr}\left[h^{-1}(\partial_a h)h^{-1}(\partial_b h)h^{-1}(\partial_c h)\right] \ , \end{aligned} \tag{7.19}$$

where in the second line, we express the bulk integral as a boundary integral over the S^3.

We claim this integral (7.19) is always an integer. All simple non-Abelian gauge groups contain $SU(2)$ subgroups. So consider the following form for h, living in an $SU(2)$ subgroup of a more general group,

$$h = (h_1)^n \ , \quad h_1(x_\mu) = \frac{t + i x^i \sigma_i}{\sqrt{t^2 + x_i x_j \delta^{ij}}} \ . \tag{7.20}$$

One can compute that

$$S_\theta[h] = n\theta \ . \tag{7.21}$$

The integer n is usually called the instanton number of the gauge field. It is a topological invariant which does not change under arbitrary continuous deformations of the fields. In other words, we could deform our choice of h in a continuous way, but we would still get n.

Problem 7.3. *Verify that $S_\theta[h] = n\theta$.*

(As $SU(2)$ is not a subgroup of U(1), the physics of theta angles in Abelian theories is conceptually rather different. One needs instead of an S^3, a nontrivial S^1 along which one can integrate the gauge potential.)

Back in the Minkowski setting, the path integral involves e^{iS_θ}. As a result, the path integral cannot be sensitive to the shift $\theta \to \theta + 2\pi$. In other words, the coupling θ is defined only modulo 2π. Anticipating the holomorphic arguments to come, it is convenient to define a complexified coupling that depends on both the gauge coupling g and θ:

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \ . \tag{7.22}$$

With respect to the complexified coupling, the equivalence is $\tau \sim \tau + 1$.

To understand the reason for the word instanton, we can replace our S^3 with a cylinder. The caps of the cylinder we orient along constant time slices. If we work in a temporal gauge where $A_t = 0$, then the only contribution to the integral (7.19) will come from the caps of the cylinder. The fact that the difference in the value of the integral over the caps is nonzero means that something nontrivial happened in the middle, an instanton. Nontrivial here means that in the middle of the cylinder, the field strengths cannot vanish. There is some nonzero energy barrier that must be overcome to take the theory from one global configuration for A_μ to another.

The goal here is to be able to use the same non-renormalization arguments for $g_{\rm YM}$ that we used for the superpotential $\mathcal{W}(\phi)$ in the context of the interacting Wess-Zumino model. The claim is that through the supersymmetry constraints, the kinetic term for the field strength and gaugino can be packaged into the same superpotential that we unearthed in discussing the Wess-Zumino model. To the original superpotential $\mathcal{W}(\phi)$, we add a gaugino bilinear

$$\mathcal{W}(\phi) \to \mathcal{W}(\tau, \bar\lambda\lambda, \phi) = \mathcal{W}(\phi) + \frac{\tau}{8\pi i}\,\mathrm{tr}(\bar\lambda\lambda) \ . \tag{7.23}$$

Roughly, in the construction of the action, the role of ϕ for this extra term is now played by the Majorana fermion λ. The role of ψ is played by the field strength $F_{\mu\nu}$, and the F term is swapped out for a derivative of the gaugino $\bar{\partial\!\!\!/}\lambda$. In the context of the action, the $(\partial\mathcal{W})F$ term from before is replaced by the gaugino kinetic term $\bar{\lambda}\bar{\partial\!\!\!/}\lambda$. The Yukawa's $(\partial^2 W)\bar{\psi}\Pi_+\psi$ and $\bar{\partial}^2\mathcal{W}^*\bar{\psi}\Pi_-\psi$ become the field strength kinetic terms $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$.

We should be a bit more careful in this last step as we want to emphasize the holomorphic nature of this construction. In addition to the holomorphic coupling τ, we should introduce self-dual and anti-self-dual field strengths,

$$F^{\pm}_{\mu\nu} = F_{\mu\nu} \mp i\tilde{F}_{\mu\nu} \ . \tag{7.24}$$

With these objects, we can write the kinetic term for the gauge field in a "more holomorphic" manner. Note that

$$\frac{1}{2}F^+_{\mu\nu}F^{+\mu\nu} = F_{\mu\nu}F^{\mu\nu} - iF_{\mu\nu}\tilde{F}^{\mu\nu} \ , \tag{7.25}$$

and so

$$\frac{\tau}{16\pi i}F^+_{\mu\nu}F^{+\mu\nu} - \frac{\bar{\tau}}{16\pi i}F^-_{\mu\nu}F^{-\mu\nu} = \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \ . \tag{7.26}$$

In slightly more detail then, it is the self-dual field strength $F^+_{\mu\nu}$ that appears in $\mathcal{W}$ and the anti-self-dual field strength $F^-_{\mu\nu}$ that appears in $\mathcal{W}^*$. One uses both $\mathcal{W}$ and $\mathcal{W}^*$ in the construction of the action, and the sum leads to the standard kinetic term for the $F_{\mu\nu}$ along with the θ term.

We are bending over a little bit backwards not to introduce superfields and superspace notation here. The rules for converting a superpotential into an action are much more natural in superspace, where one performs an integral of $\mathcal{W}$ over a pair of Grassman variables, and the various terms emerge after some Taylor expansion. Having come so far without it, however, it would be a needless distraction at this point to introduce it here.

Having packaged the the τ, λ, and $F_{\mu\nu}$ dependence of the overall super Yang-Mills action into a superpotential, we can use the same holomorphicity arguments that we employed before in the interacting Wess-Zumino model. We promote τ to the expectation value of some external field, in which case $\mathcal{W}$ must depend holomorphically on τ. In fact, we have more than holomorphicity here. We also have the equivalence $\tau \sim \tau + 1$, which must be true at any scale. The claim is that the only way the superpotential can be renormalized is through non-perturbative effects:

$$\mathcal{W}(\tau, \bar{\lambda}\lambda, \phi) = \left(\frac{\tau}{8\pi i} + O(e^{2\pi i n\tau})\right)\mathrm{tr}(\bar{\lambda}\lambda) + \mathcal{W}(\phi) + O(e^{2\pi i n\tau}) \ , \tag{7.27}$$

where n is a positive integer. The exponential factors are designed to be invariant under $\tau \to \tau+1$. At the same time, the exponents scale as $1/g^2$, which are invisible in perturbation theory, where g is small. Hence the name non-perturbative. We cannot allow negative powers of n (or $n = 0$ for that matter) because we expect the theory to have a well-behaved weakly coupled limit.

The claim then is that τ is renormalized only at one-loop perturbatively and by non-perturbative corrections. No higher loop perturbative corrections are allowed. This is a weaker non-renormalization theorem than we had for the original superpotential, but it is still quite useful and allows us to find nontrivial fixed points of the renormalization group for gauge theories, as we have seen already.

There is one lacuna here, that of converting the one-loop beta function to the canonical result (7.10) that involved anomalous dimensions of the fields. We gave some hint of how the argument should go, that we need to rescale the kinetic terms for the (ϕ, ψ) matter multiplets by the appropriate wave function renormalization factors Z_ϕ and also, it turns out, the kinetic terms for the gauge field itself, to clear out $1/g^2$ from the kinetic term and redistribute $g_{\rm YM}$ factors among the various cubic and quartic interactions.

To see how the argument goes in detail requires some quantum field theory knowledge. Let us try to give a flavor of it. The beta function for our super Yang-Mills theory with matter is determined by the self-energy diagram for the gluon. (The contributions from the vertices and wave-function renormalization of the charged fields must cancel by gauge invariance.) In computing the one-loop contribution to this self-energy, we have to consider the various charged fields that run in this loop – the (ϕ_n, ψ_n) multiplets and, since this is a non-abelian theory, the gluons and gluinos themselves. When we rescale these fields by $g_{\rm YM}$ or the wave-function renormalization factors Z_n, these rescalings will affect the loop computation. Let us distinguish the "physical" $g_{\rm YM}$ we get after rescaling from the g associated with the exact one-loop beta function. These extra rescalings of the fields in the one-loop gluon propagator introduce the following relation between g and $g_{\rm YM}$:

$$\frac{1}{g_{\rm YM}^2} = \frac{1}{g^2} + \frac{1}{16\pi^2}\left(T({\rm adj}) \log\left(\frac{1}{g^2}\right) + \sum_n T(\mathbf{r}_n) \log(Z_n) \right) . \tag{7.28}$$

From this shift, taking a derivative with respect to μ, one can deduce that the canonical beta function (7.10) follows from the one loop result (7.1).

7.2 Anomalies

Classically, Noether's theorem guarantees that for every continuous global symmetry, there is a corresponding conserved current. Quantum mechanically, however, not only the action must be invariant under the symmetry transformation; the measure of the path integral must be invariant as well. Occasionally there are problems on this latter front, and these problems are called anomalies.

Let us suppose we have a global U(1) symmetry and corresponding current j^μ which classically should be conserved, $\partial_\mu j^\mu = 0$. Furthermore, let us suppose we have fermions ψ_I which have charge q_I under this U(1) as well as transform in some $\mathbf{r}_I$ representation of a gauge group G. One such anomaly, which plays an important role for our study of gauge theories, is

$$\partial_\mu j^\mu = \frac{\sum_I q_I T(\mathbf{r}_I)}{16\pi^2} \, {\rm tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) . \tag{7.29}$$

If this sum $\sum_I q_I T(\mathbf{r}_I) \neq 0$ fails to vanish, we say the U(1) symmetry is anomalous. There are several ways of getting at this result. The most time honored and tested is the computation of a triangle diagram in perturbation theory with external legs corresponding to the gluons of the gauge group G and the photon of the U(1) were we to weakly gauge it. The anomaly actually makes it impossible to couple the j^μ current to an external photon and "gauge the symmetry," so perhaps a better way of thinking about this third leg is simply as the insertion of the corresponding current.

Anomalies are an important, intricate, and subtle story in QFT. We gather here just a few details that will be important in what follows.

- In the case of several U(1) global symmetries, the structure of the right hand side of (7.29) means that we can always take linear combinations of the U(1)'s so that at most one is anomalous.

- Violations in charge conservation will be associated with changes in the instanton number. As a result, such violations are non-perturbative and highly suppressed at weak coupling, proportional to $e^{-8|n|\pi^2/g^2}$.

- We saw the $F_{\mu\nu}\tilde{F}^{\mu\nu}$ structure previously, in the θ term in the action for the gauge field. The appearance of this structure means that the effects of the anomaly are equivalent to an explicit breaking of the U(1) symmetry that we can realize by assigning a transformation rule to the θ angle. While $\psi_i \to e^{iq_i\alpha}\psi_i$ under the U(1) symmetry, we can obtain the same violation of current conservation by sending

$$\theta \to \theta + \alpha \left[\sum_I q_I T(\mathbf{r}_I) \right] . \tag{7.30}$$

Let us consider our SQCD theory with N_f massless squarks Q_I^i where i is a fundamental index of SU(n) and I is a flavor index that runs from one to N_f. We also have their fermionic quark superpartners, Weyl fermions q_I^i. Similarly we have have N_f massless quarks in the antifundamental of SU(n), $\tilde{Q}_i^I$ and their superpartners. The symmetry groups are the SU(n) gauge group and two U(N_f) global flavor groups. We can separate out a pair of abelian factors from the flavor groups U(N_f) = SU(N_f) × U(1). These two U(1)'s are then frequently re-diagonalized into a U$(1)_A$ and a U$(1)_B$, A for axial and B for baryon.

Let us discuss this rediagonalization in more detail in a way that will help explain the nomenclature, axial and baryon. When we have massless fermions, there are two types of U(1) symmetries that we can associate with them, one that uses a gamma five matrix $\psi \to e^{i\alpha\gamma}\psi$ often called axial, and one that doesn't $\psi \to e^{i\alpha}\psi$ often called vector. It this second one that we couple to the photon and that gives rise to electric charge. The first is often anomalous.

In our case, we start out with a U(1) that acts on the Weyl fermion q_I^i and oppositely on its complex conjugate $q_I^{i\dagger}$ but not on the $\tilde{q}_i^I$ or $\tilde{q}_i^{I\dagger}$. We have a second U(1) that acts on $\tilde{q}_i^I$ and its complex conjugate but not on q_I^i or $q_I^{i\dagger}$. Rediagonalizing, we will have a U$(1)_A$ under which q_I^i and $\tilde{q}_i^I$ have the same charge (but $q_I^{i\dagger}$ and $\tilde{q}_i^{I\dagger}$ have opposite charge because of the complex conjugation). We will have a $U(1)_B$ under which q_I^i and $\tilde{q}_i^I$ have opposite charge. In analogy with QED, we call U$(1)_A$ axial because q_I^i and $\tilde{q}_i^I$ have the same U$(1)_A$ charges but are in conjugate representations of SU(n). We call the second $U(1)_B$ baryonic because, as we will discuss in more detail shortly, there is the possibility of constructing gauge invariant baryonic like objects from anti-symmetrized products of the Q_I^i only which hence have positive U$(1)_B$ charge (or similarly from the $\tilde{Q}_i^I$ only which have negative U$(1)_B$

	SU(n)	SU(N_f)	SU(N_f)	U(1)$_B$	U(1)$_A$	U(1)$_{R'}$	U(1)$_R$
Q^i_I	$\mathbf{n}$	$\mathbf{N}_f$	$\mathbf{1}$	1	1	1	$1 - \frac{n}{N_f}$
$\tilde{Q}^I_i$	$\overline{\mathbf{n}}$	$\mathbf{1}$	$\overline{\mathbf{N}}_f$	-1	1	1	$1 - \frac{n}{N_f}$

Figure 3: Charge assignments for the quarks fields in SQCD.

charge). From our expression for the anomaly (7.29), we see that U(1)$_A$ will be anomalous while U(1)$_B$ is not.

In these supersymmetric theories, as we saw in the discussion of the interacting Wess-Zumino model, there is the possibility of another global U(1) symmetry, often called R-symmetry, under which the super charges transform. The superpotential must have R-charge two, and our superpotential is proportional to $\bar{\lambda}\lambda$, from which we conclude that λ should have R-charge one. The R-charge assignments for the quark fields are somewhat arbitrary at this point. If we want to allow for the possibility of adding mass terms to the superpotential $m^J_I \tilde{Q}^I_i Q^i_J$ (without giving the mass matrix m^J_I and R-charge assignment), then we should assign $\tilde{Q}^I_i$ and Q^i_J R-charge one. Let us call this R-symmetry R' because it is anomalous, according to our result (7.29).

We can always define a new R-symmetry by taking a linear combination of U(1)$_{R'}$ with U(1)$_A$. In fact, there is a unique combination which is not anomalous. The charge assignments for this U(1)$_R$ are given in the last column in the table in figure 3.

The limit $N_f \to 0$ is special regarding the anomaly. In this limit, we have pure super Yang-Mills and no flavor symmetries. We cannot get rid of the anomaly in U(1)$_{R'}$ by taking a linear combination with U(1)$_A$. The effect of the anomaly in U(1)$_{R'}$ is equivalent to shifting the theta angle by

$$\theta \to \theta + \alpha T(\mathrm{adj}) \tag{7.31}$$

where $T(\mathrm{adj}) = 2n$ for SU(n). Because θ is an angle, shifts by $\alpha = \frac{2\pi}{T(\mathrm{adj})}$ are symmetries of the theory. We say that the U(1)$_{R'}$ is broken by the anomaly to a discrete $\mathbb{Z}_{T(\mathrm{adj})}$ subgroup. In the case of SU(n), this subgroup is $\mathbb{Z}_{2n}$.

7.3 Higgs vs. Coulomb vs. Confinement

A goal of these notes is to explore the low energy behavior of supersymmetric gauge theories. We have already done a little in this direction, determining conditions that allow for the beta functions to vanish in the IR. This type of low energy behavior is among the most interesting – a low energy, interacting, scale invariant fixed point. In these theories, the Poincaré group is enlarged to a conformal symmetry group – one of the loop holes we mentioned in the Coleman-Mandula Theorem – or in the supersymmetric case, a superconformal symmetry group.

Terms often used in a low energy description are confined phase, Higgsed phase, and Coulomb phase. Confinement is sometimes interchanged with "mass gap" and "trivial" – trivial in the sense that at low enough energy, there are no excitations because of the mass gap. In QCD, there is observed to be a mass gap associated with the lightest meson in the theory, a pion. In a Higgsed vacuum, on the other hand, a charged scalar forms a condensate, the corresponding Goldstone mode gets "eaten" by the gauge boson, and there is partial or complete breaking of the gauge symmetry. A third possibility is a Coulomb phase, like QED, where unbroken Abelian factors have photons that mediate long range interaction. We need a more precise characterization of these different behaviors. It would also be useful to know whether there are sharp boundaries between the various behaviors or whether one can be continuously deformed into another.

One way to characterize these phases is through the potential between two charged objects interacting via the gauge field. In QED, the potential between two massive electrons is the usual Coulomb potential, at least for distances large compared to the inverse mass scale set by the electrons,

$$V(R) = \frac{e^2}{R} \ . \tag{7.32}$$

Because of the running of the coupling, at very small distances, or for massless electrons, $e(R)$ will depend on the separation, getting weaker in the IR (at long distances). For a Higgsed phase, the gauge bosons become massive, and the Coulomb potential gets screened

$$V(R) \sim \frac{e^{-mR}}{R} \ , \tag{7.33}$$

although there may be some residual constant potential left over at long distances. For confinement, the expectation is that thin narrow flux tubes connect the charged particles, leading to a potential that grows linearly with separation

$$V(R) \sim \Lambda^2 R \ . \tag{7.34}$$

An issue with these descriptions in a gauge theory is that they assume a single charged particle is a good observable, when in fact it's not. While the action of a global symmetry changes the state of a theory, gauge symmetry is better thought of as a redundancy in the description. States which are related by gauge transformations are actually equivalent states. A better set of observables are gauge invariant quantities.

One important such quantity, which will help to reformulate the description of various types of potentials between charged objects (which are not gauge invariant on their own), is the Wilson loop. A Wilson loop is a non-local operator constructed by integrating the gauge potential A_μ^a along a closed contour,

$$\mathcal{A} = \operatorname{tr} \mathcal{P} e^{i \oint A} \tag{7.35}$$

where $\mathcal{P}$ stands for path ordering the insertions of A_μ^a along the contour. Various representations of the gauge group can be used, giving rise to a small zoo of representation dependent Wilson loops. Morally, however, one should think about a Wilson loop as a quark traveling along a contour through space-time. As the loop is closed, perhaps a more accurate interpretation is as a quark-antiquark pair that is created at some time t_i, propagates for a while, and then annihilates, closing the contour at some later time t_f. For a rectangular loop of width R and length T, in the confining phase, the action should evaluate to $TV(R)$ and give rise to an "area law" for the Wilson loop $\langle \mathcal{A} \rangle \sim e^{-cLR}$ where c is some constant. In the Higgs phase, on the other hand, the potential vanishes at large separation, leading to a perimeter law $\mathcal{A} \sim e^{-c'(L+R)}$.

8 The Low Energy Behavior of SQCD

8.1 The Classical Moduli Space

Let us focus on the terms in the super Yang-Mills action with matter that depend on the auxiliary field D^a:

$$\frac{1}{2}(D^a)^2 + \frac{i}{2}\phi_i^*(t_a)^i{}_j\phi^j D^a \ . \tag{8.1}$$

Integrating out D^a will lead to a quartic potential for the scalar fields ϕ^j. We are interested in a solution to the equations of motion that preserves supersymmetry, namely a static solution for which the potential will vanish. Remember nonzero energy solutions break supersymmetry. The analysis will be classical; these types of terms in a supersymmetric action tend to get altered through renormalization effects. Nevertheless, it is an important first step toward an understanding of the low energy behavior of SQCD.

These static supersymmetry preserving solutions are given by the condition that $\phi_i^*(t_a)^i{}_j\phi^j$ appropriately summed over the matter fields vanish. In the context of SQCD, we can expand this condition out

$$0 = \sum_a (t_a)^i{}_j \left(Q_i^{I\dagger} Q_I^j - \tilde{Q}_i^I \tilde{Q}_I^{j\dagger} \right) \tag{8.2}$$

where we have used that for complex conjugate representations, the generators obey $(t_a^{(\mathbf{r})})^T = -(t_a^{(\bar{\mathbf{r}})})$.

In the case that $N_f < n$, we can use color and flavor rotations to put the Q and $\tilde{Q}$ matrices in the diagonal form

$$Q = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{N_f} \\ & & \\ & & \end{pmatrix} \ ; \quad \tilde{Q}^T = \begin{pmatrix} \tilde{a}_1 & & \\ & \ddots & \\ & & \tilde{a}_{N_f} \\ & & \\ & & \end{pmatrix} \ . \tag{8.3}$$

Note these are not square matrices. They have $n - N_f$ more rows than columns. Plugging these diagonal matrices into the the "D-flatness" condition (8.2), we find a supersymmetric solution provided the difference $|a_I|^2 - |\tilde{a}_I|^2$ is independent of I. The difference need not vanish completely as the $(t_a)^i{}_j$ matrices are traceless.

We should use gauge invariant operators to describe this moduli space of vacua, and one obvious choice here are mesons

$$M_J^I = \tilde{Q}_i^J Q_I^i \ , \tag{8.4}$$

providing N_f^2 quantities to parametrize the space of vacua.

The underlying physics here is the Higgs mechanism. We are giving expectation values to the fields M_J^I which classically break the SU(n) gauge symmetry down to SU($n - N_f$). This breaking implies that $(n^2 - 1) - ((n - N_f)^2 - 1) = 2N_f n - N_f^2$ bosons get a mass.

By the Higgs mechanism, to get a mass, each of these bosons ate a squark Q^i_J or $\tilde{Q}^I_i$. Of the original $2N_f n$ superfields, then only $2N_f n - (2N_f n - N_f^2) = N_f^2$ survive, matching the number of meson fields M^I_J that we found.

For $N_f > n$, some additional gauge invariant objects can be created. In this case, we can diagonalize the squark fields into the form

$$Q = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{N_f} \end{pmatrix} \quad ; \quad \tilde{Q}^T = \begin{pmatrix} \tilde{a}_1 & & \\ & \ddots & \\ & & \tilde{a}_{N_f} \end{pmatrix} . \tag{8.5}$$

Again, we require that $|a_I|^2 - |\tilde{a}_I|^2$ is independent of I to satisfy the D-flatness condition (8.2). We can form the mesons M^I_J as before, but now we also have the possibility of making baryonic objects

$$B^{I_1 \cdots I_n} = Q^{I_1}_{i_1} \cdots Q^{I_n}_{i_n} \epsilon^{i_1 \cdots i_n} , \tag{8.6}$$

$$\tilde{B}_{I_1 \cdots I_n} = \tilde{Q}^{i_1}_{I_1} \cdots \tilde{Q}^{i_n}_{I_n} \epsilon_{i_1 \cdots i_n} . \tag{8.7}$$

These objects justify the earlier $\mathrm{U}(1)_B$ nomenclature, as they are charged under this global U(1) symmetry.

Giving expectation values to these objects B, $\tilde{B}$, and M^I_J will in general completely Higgs the gauge group SU(n). Unlike in the case $N_f < n$, however, they form an overcomplete basis with which to describe the moduli space. One such relation between the fields is

$$B^{I_1 \cdots I_n} \tilde{B}_{J_1 \cdots J_n} = M^{[I_1}_{J_1} \cdots M^{I_n]}_{j_n} , \tag{8.8}$$

but in general there are many more. From the perspective of the Higgs mechanism, only $n^2 - 1$ squarks can get eaten, leaving $2N_f n - (n^2 - 1)$ degrees of freedom, but there are in general $2\binom{N_f}{n} + N_f^2$ meson and baryon fields. In the special case $n = N_f$, the relation (8.8) is the only one between the mesons and baryons, leaving a $N_f^2 + 1$ dimensional moduli space.

We can also envision adding mass terms to the superpotential for the squarks, $m^J_I \tilde{Q}^i_J Q^I_i$. We can select the mass matrix m^J_I at will, giving mass to some or all or none of the Q and $\tilde{Q}$. We will take advantage of this freedom later. We think of these mass terms as setting a scale in the RG flow. For energy scales large compared to the mass, the mass is a negligible effect, and the quarks are effectively massless. For energy scales small compared to the mass, the quarks are not produced and get effectively removed from the theory, leading to an SQCD with a smaller value of N_f.

$N_f < n$

Classically, the moduli space for the mesonic expectation values M^J_I is flat. However, quantum mechanically there is famously a non-perturbative superpotential:

$$\begin{aligned} \mathcal{W}_{np} &= (n - N_f) \left[\frac{\Lambda^{3n-N_f}}{\det M} \right]^{\frac{1}{n-N_f}} \\ &= (n - N_f) \left[\frac{|\Lambda|^{3n-N_f} e^{in\theta}}{\det M} \right]^{\frac{1}{n-N_f}} . \end{aligned} \tag{8.9}$$

We have used the complexified strong coupling scale

$$\Lambda \;=\; \mu \exp\left[\frac{2\pi i}{b}\left(\frac{\theta}{2\pi}+i\frac{4\pi}{g^2(\mu)}\right)\right] = \mu\exp\left(\frac{2\pi i}{b}\tau\right) \tag{8.10}$$

where $b = 3n - N_f$ for SQCD. This expression is in addition to the classical $\tau\bar\lambda\lambda$ term that generates the kinetic term in the action for the gauge field. The form $\mathcal{W}_{np}$ is highly constrained by symmetry and holomorphy. We are forced to use the meson fields M^I_J by gauge invariance and holomorphy. Moreover, we require a super potential with R-charge two, which fixes the power of $\det M$. The anomalous $\mathrm{U}(1)_A$ must act on $\det M$ and θ in an equivalent way, which then forces the power of Λ in the numerator.

We can check that this super potential is consistent with an RG flow generated by adding a mass term for a meson. Imagine we add a mass term such that the superpotential becomes

$$\mathcal{W} = \frac{\tau}{8\pi i}\operatorname{tr}(\bar\lambda\lambda) + (n-N_f)\left[\frac{\Lambda_{N_f}^{3n-N_f}}{\det M}\right]^{\frac{1}{n-N_f}} + m_I^J M_J^I \ . \tag{8.11}$$

We choose the mass matrix $m_I^J = m\delta^{I1}\delta_{J1}$ such that it has a single very large entry m in the upper left hand corner, giving a mass to precisely one of the mesons. We have also labeled the strong coupling scale with a subscript N_f, allowing for the fact that it depends on the theory. The off-diagonal components M^1_j and M^j_1, $j \neq 1$ get set to zero by the mass. The equation of motion for the meson M^1_1 is then

$$m - \frac{1}{M_1^1}\left[\frac{\Lambda_{N_f}^{3n-N_f}}{M_1^1 \det \hat M}\right]^{\frac{1}{n-N_f}} = 0 \ . \tag{8.12}$$

where $\hat M$ is the meson matrix with the first column and row removed. Replacing M^1_1 with its low-energy expectation value,

$$m\langle M_1^1\rangle = \left[\frac{m\Lambda_{N_f}^{3n-N_f}}{\det \hat M}\right]^{\frac{1}{n-N_f+1}} \ , \tag{8.13}$$

we obtain a new superpotential

$$\mathcal{W} \;=\; \frac{\tau}{8\pi i}\operatorname{tr}(\bar\lambda\lambda) + (n-N_f+1)\left[\frac{m\Lambda_{N_f}^{3n-N_f}}{\det \hat M}\right]^{\frac{1}{n-N_f+1}} \tag{8.14}$$

where it remains to make the identification $\Lambda_{N_f-1}^{3n-N_f+1} \equiv m\Lambda_{N_f}^{3n-N_f}$.

In other words, if we have this non-perturbative super potential for some value of $N_f < n$, it follows we will continue to have this super potential for all smaller values of N_f as well. Establishing that we have this non-perturbative super potential in the first place is a bit trickier. In the case $N_f = N_c - 1$, the superpotential depends on τ as $e^{2\pi i\tau}$, a one instanton effect. The superpotential was calculated using an instanton approach by Finnell and Pouliot in 1995, for the $N_c = 2$ and $N_f = 1$ case, finding precisely a result of the form (8.9). Because we can always break the SU(n) gauge symmetry down to SU(2) by appropriately giving expectation values to the squark fields and moving out on the Higgs branch, this SU(2) result in turn implies the result for larger n.

A couple of comments are in order:

- The presence of the superpotential implies that SQCD has no vacuum for $0 < N_f < n$. The ground state of the theory is off at $|M| \to \infty$, and the mesonic fields will all roll off there.

- This type of superpotential cannot exist for $N_f > n$. We expect the theory to have a well behaved weakly coupled limit, when $\Lambda \to 0$. If $N_f > n$, this non-perturbative contribution to the super potential would diverge as $\Lambda \to 0$.

The fractional power $\frac{1}{n-N_f}$ in the non-perturbative superpotential implies it is multi-valued as a function of $\langle \det M \rangle$. In the $N_f = 0$, case, there would be n distinct roots. Indeed, we can consider a situation in which the N_f^2 mesons have large and generic expectation values, breaking the gauge group down to SU($n - N_f$). If the expectation values are large enough, we can take the scale to be high enough that everything is weakly coupled and a classical analysis remains valid. Quite generically then, a pure SU($n - N_f$) super Yang-Mills theory is associated with having an $(n - N_f)^{th}$ root in the superpotential.

Let Λ be the scale of the original theory with N_f flavors and SU(n) gauge group. Let $\hat{\Lambda}$ be the scale of the Higgsed theory with SU($n - N_F$) gauge group and no flavors. We can match the RG flows at the scale set by M by relating $\Lambda^{3n-N_f} = \hat{\Lambda}^{3(n-N_f)} \det M$. This matching has the interesting consequence of making the superpotential for the Higgsed SU($n - N_F$) theory, which has the standard form $\sim \tau \operatorname{tr}(\bar{\lambda}\lambda)$, depend on M through the scale $\hat{\Lambda}$. (Recall that τ implicitly depends logarithmically on the scale $\hat{\Lambda}$.)

In the context of the interacting Wess-Zumino model, we saw that a superpotential leads to a term in the full action of the form $F\partial_\phi \mathcal{W}$ where F is the auxiliary scalar associated with ϕ, needed to make the algebra close on-shell. Here, we have two equivalent ways of writing the superpotential, either for the Higgsed theory using just the $\tau \operatorname{tr}(\bar{\lambda}\lambda)$ piece or the original theory which involves the non-perturbative contribution (8.9) but where Λ does not have an implicit dependence on $\det M$. Matching the $F_M \partial_M \mathcal{W}$ contributions to the action, we conclude that the gaugino bilinear gets an expectation value

$$\operatorname{tr}(\bar{\lambda}\lambda) = 64\pi^2 \left(\frac{\Lambda^{3n-N_f}}{\det M} \right)^{\frac{1}{n-N_f}} = 64\pi^2 \hat{\Lambda}^3 \ . \tag{8.15}$$

This result shows there are actually $n - N_f$ equivalent solutions, corresponding to $n - N_f$ vacua of SU($n - N_f$) super Yang-Mills theory.

Of course, we are free to set $N_f = 0$ at this point and make a slightly more general claim. Super Yang-Mills with SU(n) gauge group has n equivalent vacua, corresponding to the n different choices of the root of (8.15). The $\mathbb{Z}_{2n}$ residual discrete U$(1)_R$ symmetry that we discussed above is broken further to $\mathbb{Z}_2$ by this condensation.

$N_f = n$, confinement with chiral symmetry breaking

The superpotential can be written with the help of a Lagrange multiplier A:

$$\mathcal{W}_{np} = \left[\det M - B\tilde{B} - \Lambda^{2n}\right] A \; . \tag{8.16}$$

Note that the Λ^{2n} term deforms the classical moduli space, that obeyed the constraint $\det M = B\tilde{B}$. Symmetries and the constraint of having a classical limit forbid any more elaborate B and M dependent deformation of the moduli space.

Problem 8.1. *We leave it as an exercise to show that adding a mass matrix for the mesons, we can recover the non-perturbative superpotential (8.9) in the $N_f < n$ cases.*

$N_f = n + 1$, confinement without chiral symmetry breaking

The classical moduli space is not modified in this case, but we shall not go through the analysis. We leave it as an exercise to show that the constraints of the M_I^J, B_I, and $\tilde{B}^j$ fields can be recovered from the superpotential

$$\mathcal{W}_{np} = \frac{1}{\Lambda^{2n-1}}(M_J^I B_I \tilde{B}^J - \det M) \; . \tag{8.17}$$

Problem 8.2. *Verify that adding a mass mM_1^1 and integrating out the $(n+1)^{th}$ flavor will lead to the constraint (8.8) appropriate for the $N_f = n$ theory.*

$N_f > n + 1$

There are three distinct regimes. One we have discussed before, $3n < N_f$, where the theory is IR free. Another regime where the theory is also IR free is $n + 2 \leq N_f < \frac{3n}{2}$. Perhaps the most interesting regime is $\frac{3n}{2} < N_f < 3n$. We discussed already that in the large n limit, there is a fixed point theory with a controllably small coupling g_* and $N_f \lesssim 3n$, where the beta functions vanish. It turns out that this fixed point theory extends all the way down to $N_f = \frac{3n}{2}$ although the couplings do not remain small as N_f decreases.

This $3n > N_f > n+1$ area of our "phase diagram" was first mapped out by Seiberg with the help of a duality named after him. The idea is that the low energy physics of our SQCD theory is equivalent to the low energy physics of a different SQCD theory with gauge group

$$\tilde{n} = N_f - n \tag{8.18}$$

and again $\tilde{N}_f = N_f$ flavors but also the mesons M_J^I of the original theory. The new N_f flavors are taken to transform under conjugate representations of the flavor symmetries. We write down the analog of figure 3 for the dual theory in figure 4. There is also a term in the superpotential in the dual theory $\mathcal{W} = \tilde{\mathcal{Q}}^I M_I^J \mathcal{Q}_J$. Consistent with supersymmetry, there must be fermionic mesinos, partners to the M_J^I.

Consistent with our proposed phase diagram, the conformal window of one theory $\frac{3n}{2} < N_f < 3n$ is mapped to the conformal window of the dual theory $3\tilde{n} > \tilde{N}_f > \frac{3\tilde{n}}{2}$. On the other hand, the upper IR free region of the first theory $N_f > 3n$ gets mapped to the lower IR free region of the second theory $\tilde{n} + 2 \leq N_f < 3\tilde{n}$.

	$\mathrm{SU}(N_f-n)$	$\mathrm{SU}(N_f)$	$\mathrm{SU}(N_f)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
$\mathcal{Q}^i_I$	$\mathbf{N}_f-\mathbf{n}$	$\mathbf{1}$	$\mathbf{N}_f$	-1	$\frac{n}{N_f}$
$\tilde{\mathcal{Q}}^I_i$	$\overline{\mathbf{N}}_f-\overline{\mathbf{n}}$	$\overline{\mathbf{N}}_f$	$\mathbf{1}$	1	$\frac{n}{N_f}$
M	$\mathbf{1}$	$\mathbf{N}_f$	$\overline{\mathbf{N}}_f$	0	$2\left(1-\frac{n}{N_f}\right)$

Figure 4: Charge assignments for the quarks fields in the dual SQCD.

Problem 8.3. *One typical check people make in this business is to match the anomalies. If we sum over the R-charges of the fermions in the theory, the following two quantities must remain invariant through the duality, the so-called R- and R^3-anomalies.*

$$\sum_i R(\psi_i) \ , \quad \sum_i R(\psi_i)^3 \ . \tag{8.19}$$

Verify that these quantities do not change as a result of the Seiberg duality.

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A Sources

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  I have set the level and course material largely using these notes.

- J. Polchinski, *String Theory*, vol. 2, Appendix B.

  All you need to know about supersymmetry in 35 pages. The style is very dense, and you can spend hours working out the equations on each page.

- D. Z. Freedman and A. Van Proeyen, *Supergravity*.

  Supergravity does not start until page 185, and many of the early chapters duplicate material that we will cover in a nicer and more thorough fashion than we have time for.

- S. Weinberg, *Quantum Field Theory*, vol. 3.

  Technical, thorough, and index heavy. Weinberg uses four component fermions. An early chapter includes a proof of Coleman-Mandula. Another reasonable looking source for a proof is a Scholarpedia page,

  http://www.scholarpedia.org/article/Coleman-Mandula_theorem

  apparently written by Mandula himself.

- J. Wess and J. Bagger, *Supersymmetry and Supergravity.*

  The canonical reference. They use two component fermions. The book is easy to read, but one often wishes for more text and fewer equations. As equation heavy as it is, much of the technical detail is left to exercises.

- P. Argyres, *An Introduction to Global Supersymmetry*, 2001.

  A very nice set of lecture notes (essentially a text book) from a course that Phil Argyres taught at Cornell nearly 20 years ago. It was the canonical reference when I was a graduate student. It is available free from his website

  http://homepages.uc.edu/∼argyrepc/cu661-gr-SUSY/index.html

- K. Intriligator and N. Seiberg, "Lectures on Supersymmetric Gauge Theories and Electric-Magnetic Duality," arXiv:hep-th/9509066.

  A canonical reference for the low energy behavior of SQCD.