Supersymmetry and Conformal Field Theory (CMMS40)

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Contents

1 Introduction

Symmetry plays a critical role in quantum field theory, and we often distinguish several different types. There are gauge symmetries – the $SU(3) \times SU(2) \times U(1)$ of the standard model for instance. There are global symmetries; consider the approximate SU(2) flavor symmetry of the up and down quarks. There are discrete symmetries, for example charge conjugation C, parity P, and time T reversal. Most important of all, perhaps, are the space-time symmetries of special relativity, also known as the Poincaré group. After all, relativistic quantum field theories were developed out of an intent to wed quantum mechanics and special relativity.

Given the prominence of the Poincaré group in relativistic quantum field theory, one is led to ask whether this group might in certain contexts be a subgroup of some larger group. The contexts in which the Poincaré group can be enlarged turn out to be surprisingly limited. There is in fact a theorem, proven in 1967 by Coleman and Mandula, that the Poincaré group can be combined with internal, continuous symmetries, such as the $SU(3)$ of the standard model, in only a trivial way, as a direct product. In other words, if one takes an element g from the Poincaré group and an element h from a continuous internal symmetry group, then $gh = hg$.

This module is about two important loop holes to the Coleman-Mandula Theorem: supersymmetry and conformal symmetry. The proof of the theorem involves the scattering or S matrix, and if the theory contains only massless particles, for which the S matrix is a somewhat problematic concept, the Poincaré group can be enlarged to the conformal symmetry group. The proof further assumes the symmetry is generated by a Lie algebra, while supersymmetry involves a generalization of a Lie algebra, called a Lie super-algebra. There are other loop holes to the Coleman-Mandula Theorem which we will not discuss here. Discrete symmetries and spontaneously broken symmetries can both be used to extend the Poincaré group.

That the Poincaré group can be extended in these special ways suggests special roles for both conformal symmetry and supersymmetry. Conformal symmetry is important for critical phenomena in condensed matter and statistical physics. It also plays a central role in the renormalization group for quantum field theory. Last but not least, it is an essential technical tool in the development of string theory. For supersymmetry, two sets of arguments outline its importance. The first is experimental: For many years people have held out hope that the next generation particle accelerators (or indeed other high energy particle experiments such as cosmic ray detectors, dark matter detectors, neutrino detectors, precision electroweak experiments, etc.) would see evidence for supersymmetry in the world around us. The second is theoretical: the presence of a symmetry in physics often helps in solving a problem, and supersymmetry is no exception. We now provide some more detail for each of these motivations, starting with conformal symmetry.

- *Critical Phenomena:* There are many statistical and condensed matter systems which undergo second order phase transitions. At the critical point, these systems often admit effective field theory descriptions which have conformal symmetry. One oft cited example is the Ising model in two dimensions, with spins $\sigma_i = \pm 1$ on sites of a square lattice. The nearest neighbor spins are allowed to interact, leading to a Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \ . \tag{1.1}$$

At high temperature, the spins are disordered. Their average value vanishes: $\langle \sigma \rangle = 0$. On the other hand, at low temperature the spins will pick an orientation $\langle \sigma \rangle \neq 0$. In fact, the two phases are related by Kramers-Wannier duality, and there is a second order phase transition between the high and low temperature phases at the self-dual point. At this critical point, there are fluctuations at all scales, and the theory is invariant under changes of scale. These scale transformations are an important generator of the full conformal symmetry group as we will see later.

- *Renormalization Group:* Perhaps the most difficult aspect of quantum field theory (QFT) is that the rules depend on the energy scale. A famous example of this phenomena is the energy dependence of quantum chromodynamics (QCD). A theory or quarks and gluons, at high energies these quarks and gluons are nearly free particles. However, at low energy, the interaction strength grows and they condense to form mesons and baryons, for example the pions observed in cosmic rays or the protons and neutrons in the nucleus of the atom. This dependence on energy scale is called the renormalization group.

  In the limit of very high energy (UV) or low energy (IR), the QFT has a fixed point behavior, where it no longer depends on scale.[1] In the case of QCD, these fixed points have a simple nature. There is a free fixed point in the UV, where the particles cease to interact with each other, and a trivial fixed point in the IR. We say the IR fixed point is trivial because all of the bound states formed have mass. If we go to an energy scale below the mass of the lightest particle (a pion), there is not enough energy to produce any excitations, and the theory is empty, or trivial. Free and trivial are not the only options, however. It is possible to have a scale invariant, interacting theory of massless particles. These interacting conformal field theories are a major subject of these lectures and provide the generic fixed point behavior of a Lorentz invariant QFT. They are thus important starting points from which to begin the analysis of a general QFT.

- *String Theory:* The renormalization group is a way of curing the divergences that appear in generic QFT calculations. Intuitively, the problem is that point-like particles of relativistic QFTs are singular objects. The self energy of a charged point particle is infinite, and many other processes, for example scattering, generate similar infinities. Renormalization emerges from adding counter-terms to cure the divergences but that introduce a scale dependence to various physical quantities such as masses and coupling strengths. One might take the reasonable point of view that QFT, with its singular behavior and consequent scale dependence, is the wrong starting point for a fundamental description of the physical world. A theory of extended objects, for

[1] People have speculated about more general behavior, for example limit cycles, but such QFTs usually have additional pathologies. For example, they may be non-unitary.

example strings, is somewhat less singular. Indeed string theory has emerged as one of the leading frameworks in which to unify the Standard Model of Particle Physics (open strings) with gravity (closed strings) at a quantum level. The string, as it propagates through time, traces out a 1+1 dimensional world sheet which hosts its own QFT. This QFT is a conformal field theory.

Moving on now to supersymmetry, the experimental set of arguments concerns what has come to be known as the Standard Model of Particle Physics. This relativistic quantum field theory describes essentially everything that we have observed in nature that is not gravitational. It postulates that the world around us is made of particles. In particular, the building blocks are fermionic spin 1/2 particles – electrons, muons, taus, neutrinos, and quarks – which interact by exchanging bosonic vector particles – gluons, W and Z bosons, and photons. The Standard Model is a gauge theory, which means it has a local continuous symmetry described by a Lie Group, in this case $SU(3) \times SU(2) \times U(1)$. Only an unbroken U(1) is observed at low energies, the U(1) associated with the photon of electricity and magnetism. A last critical ingredient is thus to explain the symmetry breaking pattern. The fact that we don't observe an SU(3) at low energies is associated with the imperfectly understood physics of confinement in quantum chromodynamics (QCD). The breaking of $SU(2) \times U(1)$ to a diagonal U(1) on the other hand is associated with a last critical ingredient of the Standard Model: the Higgs particle, a spin zero bosonic particle.

Despite its successes, there are a few key unsatisfactory aspects of the Standard Model:

- *Hierarchy Problem:* From a modern standpoint, the Standard Model is an effective field theory – something that accurately describes the physics at the relatively low energies available in today's particle accelerators. The mass of the heaviest observed fundamental particle, the top quark at 172 GeV, gives an order of magnitude estimate of the energy scales at and below which the Standard Model can be trusted to give accurate results. In contrast, we have no reason to expect the Standard Model to be accurate if extrapolated to very high energy scales, for example the Planck scale $E_P = \hbar^{1/2} G_N^{-1/2} c^{5/2} \sim 10^{19}$ GeV at which quantum gravitational affects are expected to become important. A symptom of the Standard Model's limitations are divergences that appear in loop corrections to the mass of the Higgs. A naive but standard way of regulating these divergences is to cut-off the integration at an energy scale where we expect new physics. If that scale is really E_P, then mass corrections will be huge, and the Higgs mass should be of the same order of magnitude as E_P. Through the Higgs mechanism, the other Standard Model particles will get huge masses as well. Of course, we do not observe such huge masses, and so, without some fine tuning that will arrange for cancellation between the various diagrams, there must be new physics at some lower scale. Supersymmetry provides for precisely such new physics, introducing a new class of particles that can run in loops and partially cancel these large corrections to the Higgs mass.

  The current experimental situation is not promising for supersymmetry. The LHC has observed the Higgs to have a mass of 125 GeV, but has not observed any supersymmetric partners. If such partners exist, they are then likely to have masses larger than can be observed by the LHC. 125 GeV is relatively low, but the new physics is coming in at a relatively high scale, naively at least several hundred GeV, where the loop cancellations will not be particularly effective.

- *Unification:* The existence of three gauge groups $SU(3) \times SU(2) \times U(1)$ has long seemed inelegant to theorists. How much nicer would it be if the Standard Model could be embedded in a theory with a single gauge group, for example $SU(5)$, $SO(10)$, or even E_8. Giving some support to this idea is that the gauge couplings for the three gauge groups run with energy scale and become all roughly equal at 10^{15} GeV. Supersymmetry has the remarkable property of making the three couplings much closer together at this unification scale. In addition, it provides an accurate prediction for the Weinberg angle, i.e. the way in which the U(1) of the photon sits inside the original $SU(2) \times U(1)$ gauge group.

- *Dark Matter:* Roughly 70% of the matter in the universe is not particles in the Standard Model. Astrophysicists have come to this conclusion from a variety of observations, for example from looking at rotation of individual galaxies, rotation of clusters of galaxies, and the cosmic microwave background radiation. The new class of particles introduced by supersymmetry provide a host of dark matter candidates, the most serious of which is often called the LSP, the lightest super partner.

The second set of arguments for supersymmetry is that it helps solve various problems. Maybe the real question we are interested in does not involve supersymmetry, but if we add supersymmetry, we can often find solutions and then hopefully learn something about what to expect in answer to the original question.

- *String Theory:* An issue with non-supersymmetric versions of string theory is that we have not been able to find stable vacuum states. Supersymmetry cures this problem.

- *Confinement:* You can make yourself a million dollars in the Clay Mathematics Prize Competition if you successfully explain why Yang-Mills theory (i.e. QCD without the quarks) develops a mass gap at low energies. Add supersymmetry, and the problem has already been solved. The vacuum structure of a very large variety of supersymmetric gauge theories has at this point been successfully analyzed, giving us some insight into the original problem of confinement in QCD.

- *Partition Functions and Localization:* The basic problem of quantum field theory is to compute the path integral (or partition function). In supersymmetric theories, this path integral can sometimes be computed exactly on special manifolds, for example spheres. Indeed, one can go further and compute correlation functions of certain supersymmetric operators as well.

1.1 Dimensional Analysis

Dimensional analysis is a powerful tool in physics. It often allows you to deduce the answers to questions about which you have at best a foggy grasp of the details. A case in point is deducing the velocity of surface waves on a liquid – so-called capillary waves. These are the waves that you see moving away from a small stone that you toss in a lake, that travel maybe at a few dozen centimeters per second. Let's begin with the assumption that this speed should have something to do with the density of the liquid ρ, the surface tension σ, and the acceleration due to gravity g. If you further know that ρ is measured in mass per unit volume kg m^{-3}, σ in force per unit length kg s^{-2}, and acceleration in distance per unit of time squared m s^{-2}, then the unique quantity with units of velocity that can be constructed from these numbers is $(g\sigma/\rho)^{1/4}$. Plugging in the numbers for water, for which $\sigma = 72.8$ mN m^{-1}, one gets 16 cm s^{-1}, not bad for a back of the envelope estimate. Or one could turn this calculation around and estimate the surface tension for water from a stone throwing experiment at your local pond.

Similar dimensional analysis estimates will be crucial in our discussion of conformal and super symmetry in this class. We include a couple of problems to tone your skills.

Problem 1.1. *Using only the quantities $\hbar$, G_N, and c, construct quantities that have the units of length, mass, and time. Compute the corresponding Planck length, Planck mass, and Planck time, using SI units.*

Problem 1.2. *Another proposed source of extra physics is extra dimensions. Assume that we live not in a four dimensional world but a $(4+p)$-dimensional one where the extra dimensions are all extremely small circles of length ℓ.*

a) Noting that the dimensionality of G_N is different in $(4+p)$ dimensions, what is the new expression for the Planck energy E_P in terms of $\hbar$, c, and G_N?

b) Find a relationship between G_N and the observed 4d value G_N^{4d}. Given the observed 4d value for G_N^{4d}, how small must ℓ be in order to have $E_P = 1$ TeV? Are there some values of p that you can rule out?

For a relativistic quantum field theory, we almost always work in units where $\hbar$ and c are dimensionless quantities set equal to one. This choice gives time and distance the same units. It also gives momentum, energy, and mass the same units, and relates mass to one over distance, leaving us precisely one unit to work with, which we could either call length or mass.

To put these notions to work consider the action for a free scalar field:

$$S = -\int \mathrm{d}^d x \left((\partial_\mu \phi)(\partial^\mu \phi) + m^2 \phi^2 \right) \ . \tag{1.2}$$

From the knowledge that that action is dimensionless – after all e^{iS} must be a sensible expression in computing the path integral now that $\hbar = 1$ – we can conclude that ϕ has mass dimension

$$(\text{mass})^{\frac{d-2}{2}} \ .$$

We will often write this fact as $\Delta_\phi = \frac{d-2}{2}$, where Δ_ϕ is the scaling dimension of the field ϕ.

We can introduce an interaction to the theory by adding a $g\phi^n$ term to the Lagrangian. (Usually n is restricted to positive integer values to preserve analyticity.) Note that the coupling g will in general be dimensionful. To keep the interaction under control, we can try to keep it small and compute processes in a Taylor expansion in g. However, one should ask small compared to what? To address this question, we can make a dimensionless ratio

$g/E^{d-n\Delta_\phi}$ where E is a characteristic energy of the process under consideration. The sign of $d-n\Delta_\phi$ then becomes of crucial importance. For $d-n\Delta_\phi>0$, this dimensionless ratio becomes arbitrarily small at high energies but very large at low energies. Such an interaction is said to be *relevant* (i.e. relevant at low energies). In contrast, if $d-n\Delta_\phi<0$, then the ratio becomes arbitrarily small at low energies but large at high energies. We say such an interaction is *irrelevant* (i.e. irrelevant at low energies). Note the mass m is *relevant* in this language, like in the case of QCD where the fact that all the particles have masses drives the theory to a gapped or trivial fixed point in the IR. (There is older nomenclature you may run into: *relevant* = *normalizable* and *irrelevant* = *non-normalizable*.)

The final case $d-n\Delta_\phi=0$, called a classically marginal coupling, is relevant for our study of conformal symmetry and conformal field theory. In this case, g itself is dimensionless. Unfortunately, just because we can write such a term in a Lagrangian doesn't mean that g stays dimensionless at a quantum level. Typically loop corrections give anomalous dimensions to the quantum fields in a theory. And then $d-n\Delta_\phi$ is no longer zero.

Problem 1.3. *Consider an interacting scalar field*

$$S=-\int \mathrm{d}^dx\,((\partial_\mu\phi)(\partial^\mu\phi)+g\phi^n)\ . \tag{1.3}$$

where n is a positive integer. For what pairs (n,d) can the coupling g be dimensionless?

Problem 1.4. *Consider the Lagrangian for a Dirac spinor in d dimensions*

$$S=-\int \mathrm{d}^dx\left(\frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi+g(\bar\psi\psi)^n\right) \tag{1.4}$$

What is the scaling dimension of ψ? (You may assume the conjugate spinor $\bar\psi$ has the same scaling dimension as ψ. Moreover, the gamma matrices are dimensionless.) For what (n,d) can g be made dimensionless, assuming n is a positive integer? Considering now also the scalar field of the previous problem. In what dimensions do $\phi\bar\psi\psi$ and $\phi^2\bar\psi\psi$ lead to classically marginal couplings?

Problem 1.5. *Start with the assumption that the supersymmetry transformation Q squares to the momentum operator $Q^2\sim P$ and moreover converts fermions into bosons and bosons into fermions. Try to guess how Q acts on ϕ and ψ, purely based on dimensional analysis.*

Problem 1.6. *Consider QED in d dimensions*

$$S=-\int \mathrm{d}^dx\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{i}{2}\bar\psi\gamma^\mu(\partial_\mu+igA_\mu)\psi\right) \tag{1.5}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. What is the scaling dimension of g? What is special about $d=4$?

These kinds of finger counting exercises will be very valuable for us, in figuring out the constraints on supersymmetry transformations, in deciding whether or not a quantum field theory is conformally invariant, and in other situations as well. There is in fact an argument to be made that this subsection of the notes is the most important in the entire module, with implications far beyound theoretical physics.

2 Coleman-Mandula Theorem

In this chapter, we will review the Poincaré group, the conformal group, and continuous internal symmetry groups, and then discuss how supersymmetry and conformal symmetry evade the Coleman-Mandula theorem. (Space and time prevent us from including a proof of the theorem.) A presentation of the supersymmetry algebra will have to wait until Chapter 7. In the intervening chapters, we will instead develop conformal field theory. The second half of the module will be devoted to supersymmetry.

The Poincaré group is a Lie group that is generated by space-time translations along with Lorentz transformations (which in turn consist of rotations and boosts). The infinitesimal version (or Lie algebra version) of this group action, under which the theory is invariant, can be written

$$x^\mu \to x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu \ , \tag{2.1}$$

where the quantity $\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$ is taken to be small.

In special relativity, the space-time proper distance $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ between two points must be invariant under these transformations, which in turn places a constraint on $\omega^\mu{}_\nu$:

$$\begin{aligned} \Delta s^2 &\to \eta_{\mu\nu}(\Delta x^\mu + \omega^\mu{}_\lambda \Delta x^\lambda)(\Delta x^\nu + \omega^\nu{}_\rho \Delta x^\rho) \\ &= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu + \eta_{\mu\nu}\omega^\mu{}_\lambda \Delta x^\lambda \Delta x^\nu + \eta_{\mu\nu}\omega^\nu{}_\rho \Delta x^\mu \Delta x^\rho + \ldots \\ &= \Delta s^2 + (\omega_{\mu\nu} + \omega_{\nu\mu})\Delta x^\mu \Delta x^\nu + \ldots \ . \end{aligned} \tag{2.2}$$

In other words, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is antisymmetric under exchange of its indices.[2]

While elements of the Poincaré group compose to give new elements in the group, the infinitesimal version of this statement is that the commutator of two infinitesimal elements (i.e. elements of the corresponding Lie algebra) yields a new infinitesimal element. We consider infinitesimal elements δ_1 and δ_2 and compute

$$[\delta_1, \delta_2] x^\mu \equiv \delta_1 \delta_2 x^\mu - \delta_2 \delta_1 x^\mu \ . \tag{2.3}$$

To compute $\delta_2 \delta_1 x^\mu$, it is perhaps clearer to start with the arrow notation

$$\begin{aligned} x^\mu &\to x^\mu + a_1^\mu + \omega_{1\nu}^\mu x^\nu \\ &\to x^\mu + a_1^\mu + \omega_{1\nu}^\mu x^\nu + a_2^\mu + \omega_{2\nu}^\mu (x^\nu + a_1^\nu + \omega_{1\lambda}^\nu x^\lambda) \ , \end{aligned}$$

from which it follows that

$$\delta_2 \delta_1 x^\mu = \omega_2{}^\mu{}_\nu a_1^\nu + \omega_2{}^\mu{}_\lambda \omega_1{}^\lambda{}_\nu x^\nu + a_1^\mu + a_2^\mu + \omega_{1\nu}^\mu x^\nu + \omega_{2\nu}^\mu x^\nu \ . \tag{2.4}$$

[2]We will use a Minkowski metric with mostly plus signature:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} .$$

Note the terms in red will drop out of the commutator. The commutator then must be

$$[\delta_1, \delta_2]x^\mu = (\omega^\mu_{1\,\lambda} a_2^\lambda - \omega^\mu_{2\,\lambda} a_1^\lambda) + (\omega^\mu_{1\,\lambda}\omega^\lambda_{2\,\nu} - \omega^\mu_{2\,\lambda}\omega^\lambda_{1\,\nu})x^\nu \ . \tag{2.5}$$

The new infinitesimal Poincaré transformation is

$$a^\mu = \omega^\mu_{1\,\lambda} a_2^\lambda - \omega^\mu_{2\,\lambda} a_1^\lambda \ , \quad \omega^\mu{}_\nu = \omega^\mu_{1\,\lambda}\omega^\lambda_{2\,\nu} - \omega^\mu_{2\,\lambda}\omega^\lambda_{1\,\nu} \ . \tag{2.6}$$

Note that $\omega_{(\mu\nu)} = \frac{1}{2}(\omega_{\mu\nu} + \omega_{\nu\mu}) = 0$, consistent with the requirement that Δs^2 is invariant.

We would like to be able to act not just on space-time points x^μ with the Poincaré group but on quantum fields as well. To that end, we introduce the linear operators P_μ and $M_{\mu\nu}$ which act on the coordinates such that

$$\delta x^\mu = i a^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda} M_{\nu\lambda}(x^\mu) \ . \tag{2.7}$$

The factor of 1/2 is introduced because of the anti-symmetry so that, for example, $\omega_{12} = -\omega_{21}$ is only counted once. The factors of i allow the generators to be Hermitian rather than anti-Hermitian operators. The commutator (2.5) can be written more abstractly as

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \ , \\ [P_\mu, M_{\nu\lambda}] &= i\eta_{\mu\nu}P_\lambda - i\eta_{\mu\lambda}P_\nu \ , \\ [M_{\mu\nu}, M_{\lambda\rho}] &= i\eta_{\mu\lambda}M_{\nu\rho} - i\eta_{\nu\lambda}M_{\mu\rho} - i\eta_{\mu\rho}M_{\nu\lambda} + i\eta_{\nu\rho}M_{\mu\lambda} \ . \end{aligned} \tag{2.8}$$

Problem 2.1. *Reproduce the result (2.5) using P_μ and $M_{\nu\lambda}$ and in particular (2.7) and the commutator algebra (2.8).*

In general, we would like to be able to represent the action of P_μ and $M_{\mu\nu}$ not just on x^μ but on a quantum field $\Phi_I(x^\mu)$ which transforms under a representation of Poincaré and is additionally a function of a space-time point. Here I is some generalized index allowing for an arbitrary representation of the group. An infinitesimal group element of Poincaré g consisting of the data $(a_\mu, \omega_{\mu\nu})$ and acting on $\Phi_I(x^\mu)$ thus has two pieces, one g^{IJ} acting by matrix multiplication on the generalized index of the field I and the second acting on x^μ,

$$\delta\Phi_I(x^\mu) = g_I{}^J \Phi_J(x^\mu) + \Phi_I(x^\mu + \delta x^\mu) - \Phi_I(x^\mu) \ . \tag{2.9}$$

By a Taylor series, we can write the second two terms, to leading order, as a derivative

$$\Phi_I(x^\mu + \delta x^\mu) - \Phi_I(x^\mu) = (a^\mu + \omega^\mu{}_\nu x^\nu)\partial_\mu \Phi_I(x^\rho) \ . \tag{2.10}$$

Now it turns out that g^{IJ} simplifies as well and depends only on the Lorentz part of the Poincaré group. Because of the nontrivial commutator $[P_\mu, M_{\nu\lambda}]$, the Poincaré group is not a direct but a semi-direct product of translations and Lorentz transformations. Translations by themselves are straightforward to understand. They form an abelian and non-compact subgroup of the full group. Their irreducible representations are always one dimensional, and the corresponding matrices just constants. In fact, as far as I'm aware, for fields of physical interest, these constants always vanish. For example, for tensor fields, shifting the location of the origin of spacetime clearly should not affect the structure of the tangent and cotangent bundles, leaving the space-time indices on some general tensor field $T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}$ invariant.

The nontrivial data in g^{IJ} is then a representation of the Lorentz algebra only, and $P_\mu = -i\partial_\mu$ reduces to a derivative acting on the fields, controlling how the shift in x^μ in turn affects the field Φ_I. Smooth functions can be expanded in terms of a Taylor series:

$$\begin{aligned} f(x+a) &= f(x) + a^\mu \partial_\mu f(x) + \ldots \\ &= f(x) + i a^\mu P_\mu f(x) + \ldots \end{aligned} \tag{2.11}$$

Finite translations can be obtained as an exponential of P_μ:

$$\begin{aligned} f(x+a) &= e^{i a^\mu P_\mu} f(x) \\ &= f(x) + a^\mu \partial_\mu f(x) + \frac{1}{2} a^\mu a^\nu \partial_\mu \partial_\nu f(x) + \ldots \end{aligned} \tag{2.12}$$

The action of the Lorentz group on the coordinate dependence of Φ_I can be written in a similar derivative fashion, as $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Indeed, using this representation of $M_{\mu\nu}$ along with $P_\mu = -i\partial_\mu$, one can recover the commutation relations (2.8). However, this representation of the action of the Lorentz group on functions is not the whole story. The Lorentz group is non-abelian and admits more interesting representations. The Standard Model that we discussed briefly in the first section contains a Higgs field $H(x)$ in the trivial representation, vector fields such as the photon $A_\mu(x)$, and many spinor fields, such as the electron $\psi_\alpha(x)$. In general, a nontrivial representation of the Lorentz group implies that the field carries some kind of index, for example μ and α for the vector and spinor fields respectively. Different representations imply that there are different choices of matrices which satisfy the commutation relations (2.8) of the Poincaré group.

Problem 2.2. *For a vector representation, one takes*

$$(M_{\mu\nu})^\lambda{}_\rho = i\eta_{\mu\rho}\delta^\lambda_\nu - i\delta^\lambda_\mu \eta_{\nu\rho} \; . \tag{2.13}$$

(Notice that the indices μ and ν take a dual role, labeling both the Lorentz generator and its matrix components.) For the spinor representation, one takes instead

$$(M_{\mu\nu})_\alpha{}^\beta = -\frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)_\alpha{}^\beta \; , \tag{2.14}$$

where $(\gamma_\mu)_\alpha{}^\beta$ are the Dirac γ-matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Verify that these two representations of the Lorentz group obey the commutation relations (2.8).

Quantum field theories often possess additional symmetries, most notably gauge symmetries. Associated with the gauged Lie group, there is a Lie algebra with commutation relations of the form

$$[T_a, T_b] = i f_{ab}{}^c T_c \; , \tag{2.15}$$

where the T_a are Hermitian generators, and $f_{ab}{}^c$ are the structure constants. The fields transform in representations of this algebra and carry associated indices. For example, the quarks ψ^a_α in the standard model in addition to a spinor index α carry an index a indicating that they transform in a fundamental representation of $SU(3)$.

The component P_t is both an energy and also a generator of infinitesimal translations in time. Because P_t exists as a well defined, time independent quantity, we expect that the total energy is conserved. Often a good first step in approaching a physics problem is to work out a complete set of conserved charges. In the context of our commutator algebra of P_μ, $M_{\mu\nu}$ and T_a, the set of conserved charges is the set which commutes with P_t. In the context of the Poincaré group, we expect the full four momentum P_μ to be conserved, along with angular momenta corresponding to M_{xy}, M_{yz}, and M_{zx}. The boosts M_{ti} on the other hand do not commute with P_t. Having written down the full set, as is typical in quantum mechanics one has to worry about whether the generators mutually commute as well. Otherwise, the operators will not all be simultaneously diagonalizable. In the context of spatial rotations, for example, one typically chooses $J_z = M_{xy}$ and the Casimir operator $J^2 = M_{xy}^2 + M_{yz}^2 + M_{zx}^2$.

From Noether's theorem, we expect that continuous symmetries are associated with conserved charges and more generally conserved currents. It should follow from Noether's theorem that $[P_t, T_a] = 0$. The content of the Coleman-Mandula theorem is much stronger, that the generators T_a commute with all of the generators of the Poincaré group:

$$[T_a, P_\mu] = 0 = [T_a, M_{\mu\nu}] \ . \tag{2.16}$$

Thus the T_a are not only conserved but transform under the trivial representation of the Poincaré group.

Theorem. (Coleman-Mandula) *In any spacetime dimension greater than two, the only interacting quantum field theories have Lie algebra symmetries which are a direct product of the Poincaré algebra with an internal symmetry.*

2.1 Supersymmetry

Supersymmetry requires the violation of a key assumption of the Coleman-Mandula theorem – that the symmetry algebra should be a Lie algebra. Recall that a Lie algebra is the tangent space at the identity element of a Lie group, with an infinitesimal group transformation of the form

$$g = 1 + i\epsilon A \tag{2.17}$$

where A is an element of the Lie algebra (e.g. P_μ, $M_{\mu\nu}$ or T_a from before) and ϵ is an infinitesimal parameter. The algebra is closed under an antisymmetric bilinear operation called the Lie bracket

$$[A, B] = -[B, A] \tag{2.18}$$

which is subject to the Jacobi identity

$$[A, [B, C]] + [B, [C, A] + [C, [A, B]] = 0 \ . \tag{2.19}$$

Supersymmetry replaces the Lie algebra with a superalgebra. A superalgebra should already be a familiar notion to you if you have worked with a quantum field theory that contains both fermions and bosons. By the spin statistics theorem, bosons carry representations of the Lorentz group with integer spin and their field operators must commute outside of the light cone. On the other hand, fermions carry half-integer spin representations and anti-commute outside the light cone. A standard Lie algebra can be constructed from bosonic generators and commutators $[,]$, but once we involve fermions, it is very natural to throw anti-commutators $\{,\}$ into the mix along with the rule that the product of two fermions is a boson, the product of a fermion and a boson is a fermion, and the product of two bosons is again a boson.

We can formalize this notion of a superalgebra as a $\mathbb{Z}_2$ graded Lie algebra where fermions have odd grading and bosons have even grading. It is also convenient to write a generalized commutator $[,\}$ where the decision to anti-commute or commute is based on what is inside the brackets:

$$[B, B\} = [B, B'] \sim B'' , \tag{2.20}$$

$$[B, F\} = [B, F] \sim F' , \tag{2.21}$$

$$[F, F'\} = \{F, F'\} \sim B . \tag{2.22}$$

B here is for boson and F for fermion, and the notation is schematic. There is furthermore a generalized Jacobi identity

$$(-1)^{ac}[A, [B, C\}\} + (-1)^{ba}[B, [C, A\}\} + (-1)^{cb}[C, [A, B\}\} = 0 . \tag{2.23}$$

where $a, b, c \in \mathbb{Z}_2$ are the gradings of A, B, and C respectively.

In this course, the even generators will be the P_μ and $M_{\mu\nu}$ generators of the Poincaré group. The odd generators, or supersymmetries Q_α, are then in a sense the square roots of the Poincaré generators, schematically

$$\{Q, Q\} = P + M . \tag{2.24}$$

There is thus a symmetry that is "deeper" than Poincaré and is surely therefore worthy of study.

One more comment needs to be made. While we have found a way to nontrivially enlarge the Poincaré algebra, the supercharges Q_α still generally commute with other internal continuous symmetry generators $[Q_\alpha, T_a] = 0$. This refined version of the Coleman-Mandula theorem is due to Haag, Sohnius, and Lopuszanski and was proven in 1975.

2.2 Conformal Symmetry

The proof of the Coleman-Mandula theorem relies on the existence of an S-matrix (or scattering matrix), which contains the data of all of the scattering amplitudes in the theory. A definition of the S-matrix requires the notion of asymptotic initial and final states, where the ingoing and outgoing particles are far from each other and essentially non-interacting. However, if the underlying theory is scale invariant, then there is no notion of "far", and there are difficulties in defining the S-matrix. One issue for the S-matrix is the presence of long range forces that occur when the particles that mediate those forces are massless. (Indeed, for a scale invariant theory, all the particles must be massless because a mass would define a scale.) You may have seen similar issues in a quantum mechanics class, in looking at the scattering cross section of a charged particle in a Coulomb potential. These problems stem from the masslessness of the photon. Scale invariant theories provide another important loop hole to the Coleman-Mandula theorem.

The Poincaré group was the set of transformations which left the Minkowski tensor $\eta_{\mu\nu}$ invariant. The conformal group is the set of coordinate transformations which leave the Minkowski tensor invariant up to a position dependent rescaling

$$\eta'_{\mu\nu} \equiv \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta} = \Omega(x)\eta_{\mu\nu} \ . \tag{2.25}$$

Note the Poincaré group, for which $\Omega(x) = 1$ forms a subgroup of the conformal group. A further generator of the conformal group is the scale transformation $x^\mu \to x'^\mu = \lambda x^\mu$ for which $\Omega = \lambda^{-2}$. The rule (2.25) relates the Jacobian of the transformation to the scaling factor Ω, to wit $\Omega^d = J^2$. The word conformal is used to imply that the action of the group does not change the angle between intersecting curves. In the Euclidean context, when $\eta_{\mu\nu} = \delta_{\mu\nu}$, the cosine of the angle between two vectors is given by $v \cdot w/|v||w|$, and indeed, whether one is in the Euclidean or Minkowski signature, this quantity is invariant under conformal transformation.

Let us try to construct the infinitesimal elements of the conformal group. Consider a general coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \ , \tag{2.26}$$

assuming $\epsilon^\mu(x)$ is small. Using the rule

$$\eta'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta} \ ,$$

we find to linear order that

$$\eta_{\mu\nu} \to \eta_{\mu\nu} - (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) + O(\epsilon^2) \ . \tag{2.27}$$

From the definition of a conformal transformation (2.25) with $\Omega(x) \approx 1 - f(x)$, we can make the identification

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = f(x)\eta_{\mu\nu} \ . \tag{2.28}$$

Taking a trace fixes

$$f(x) = \frac{2}{d}\partial_\rho\epsilon^\rho \ . \tag{2.29}$$

We would now like to establish what kinds of ϵ satisfy the constraint (2.28). To this end, we take a partial derivative ∂_ρ of (2.28) and permutations and construct the linear combination

$$\partial_\mu(\partial_\nu\epsilon_\rho + \partial_\rho\epsilon_\nu - f\eta_{\nu\rho}) + \partial_\nu(\partial_\mu\epsilon_\rho + \partial_\rho\epsilon_\mu - f\eta_{\mu\rho}) - \partial_\rho(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu - f\eta_{\mu\nu}) = 0 \tag{2.30}$$

from which we conclude

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\nu\rho}\partial_\mu f + \eta_{\mu\rho}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f \ . \tag{2.31}$$

We further take a trace, which produces

$$2\partial^2\epsilon_\rho = (2-d)\partial_\mu f \ , \tag{2.32}$$

indicating something rather special about conformal symmetry in two dimensions. We will specialize to the case $d > 2$ in the remainder of this argument.

We combine a symmetrized version of ∂_ν of (2.32)

$$\partial^2(\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu) = (2-d)\partial_\mu\partial_\nu f$$

along with ∂^2 of (2.28) to find

$$(2-d)\partial_\mu\partial_\nu f = \partial^2 f\eta_{\mu\nu} \ . \tag{2.33}$$

Finally taking a trace tells us that $(d-1)\partial^2 f = 0$ and hence that $\partial_\mu\partial_\nu f = 0$ vanishes, provided $d > 2$. In other words, f can be at most linear in the coordinates,

$$f = A + B_\mu x^\mu \ , \tag{2.34}$$

and ϵ_μ at most quadratic,

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \ , \tag{2.35}$$

with the restriction $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Plugging this ansatz into the constraint (2.28) yields the following conditions:

- a_μ is unconstrained and generates infinitesimal translations.
- $b_{\mu\nu} = \lambda\eta_{\mu\nu} + \omega_{\mu\nu}$ where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ generate the Lorentz group and the trace part is an infinitesimal scale transformation.
- $c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu$ for a constant vector b_μ. These transformations are called special conformal transformations and act on coordinates as

$$x^\mu \to x'^\mu = x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2 \ . \tag{2.36}$$

We give the finite versions of these infinitesimal transformations as table 2.2. Translations and Lorentz transformations generate the Poincaré group, as we have discussed at length. In total, we have d translations, $\frac{d(d-1)}{2}$ Lorentz transformations, one dilatation, and d generators of special conformal transformations for $\frac{(d+1)(d+2)}{2}$ generators in total. It is no accident that this number is the same as the dimension of the special orthogonal group $SO(d+2)$. There is an exercise a little later on to demonstrate that the conformal symmetry group is equivalent to $SO(d,2)$ (or $SO(d+1,1)$ in the Euclidean setting).

Problem 2.3. *Verify that $b_{\mu\nu} = \lambda\eta_{\mu\nu} + \omega_{\mu\nu}$ and $c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu$ are the only solutions for $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ consistent with (2.28).*

Problem 2.4. *Verify that the infinitesimal versions of the transformations in table 2.2 recover a_μ, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$.*

- translations: $x'^\mu = x^\mu + a^\mu$
- Lorentz: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$
- dilatations (scale transformations): $x'^\mu = \lambda x^\mu$
- special conformal transformations: $x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot x+b^2x^2}$

Figure 1: The finite versions of the generators of the conformal symmetry group.

As the special coordinate transformations are somewhat ugly, it is often useful to introduce one further discrete element of the conformal group, the inversion

$$I : x^\mu \to x'^\mu = \frac{x^\mu}{x^2} \ , \tag{2.37}$$

where clearly I^2 is the identity element.

Problem 2.5. *Demonstrate that an inversion followed by a translation followed by a further inversion is equivalent to a special coordinate transformation.*

Parallel to the earlier discussion of the Poincaré group, it is useful to have a more abstract presentation of the conformal group and its corresponding Lie algebra in terms of a set of generators and their commutation relations. Extending the Poincaré group to include dilatations D and special conformal transformations K_μ, we can write the transformation rule on a coordinate as

$$\delta x^\mu = ia^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda}M_{\nu\lambda}(x^\mu) + ib^\nu K_\nu(x^\mu) + i\lambda D(x^\mu) \ . \tag{2.38}$$

From this expression, we infer how these transformations act on functions. We have $P_\mu = -i\partial_\mu$ and $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ as we had before in the case of the Poincaré group, to which we add two more:

$$D = -ix^\mu\partial_\mu \ , \quad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \ . \tag{2.39}$$

From this representation, it is then a straightforward although tedious exercise to work out how to extend the Poincaré group commutation relations to include the conformal group:

$$\begin{gathered}[D, P_\mu] = iP_\mu \ , \quad [D, K_\mu] = -iK_\mu \ , \quad [D, M_{\mu\nu}] = 0 \ , \quad [K_\mu, K_\nu] = 0 \ , \\ [M_{\mu\nu}, K_\rho] = i(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu) \ , \quad [P_\mu, K_\nu] = -2i(\eta_{\mu\nu}D + M_{\mu\nu}) \ .\end{gathered}$$

Importantly, P_μ does not commute with dilatation or special conformal transformations, in apparent contradiction of the Coleman-Mandula theorem and also implying that massive states are not good eigenstates of the full conformal group.

One further remark is that $M_{\mu\nu}$, P_μ and D form a subgroup, and it is a subtle point whether there may exist theories which have scale invariance and Poincaré symmetry without also having the special conformal transformations. People have looked at this question in detail, and the lore seems to be that examples with scale but not conformal invariance are not physically interesting – they are non-unitary or have an unbounded spectrum or are non-interacting.

Problem 2.6. *Compute the commutator of P^2 with K_μ and D. What happens to a massive particle state $|p\rangle$ (where $P^2|p\rangle = m^2|p\rangle$, $m^2 \neq 0$) under the infinitesimal special conformal transformation K_μ?*

Problem 2.7. *If $\mu, \nu = 0, \ldots, d-1$, then define $J_{\mu\nu} = M_{\mu\nu}$ along with $J_{\mu,d} = \frac{1}{2}(P_\mu - K_\mu)$, $J_{\mu,d+1} = \frac{1}{2}(P_\mu + K_\mu)$, and $J_{d,d+1} = D$, along with the constraint that $J_{ab} = -J_{ba}$ is antisymmetric. Show that the commutators of these generators are the same as for a $(d+2)$-dimensional orthogonal group, with metric signature $(2,d)$, i.e. $SO(2,d)$.*

Problem 2.8. *Write out the consistency relations (2.28) in $d = 2$ in the coordinate system $x_\pm = x \pm t$. What can you conclude about the allowed form of ϵ_μ?*

Problem 2.9. *Compute $\Omega(x)$ for the (finite) special conformal transformations.*

3 Constraints of Conformal Symmetry

We would like to understand how the conformal symmetry group acts on quantum states and fields. In the case of the Poincaré group, it is often convenient to choose fields that are eigenvectors of the momentum operator P_μ. In the context of the conformal symmetry group, P_μ no longer plays as privileged a role. P_μ does not commute with K_μ nor with D.

In the case of conformal symmetry, dilatation D largely replaces the privileged role of P^t. The commutation relations $[D, P_\mu] = iP_\mu$ and $[D, K_\mu] = -iK_\mu$ are suggestively close to the commutation relations for the raising and lower operators of the harmonic oscillator with the identifications $H \sim D$, $P_\mu \sim a^\dagger$ and $K_\mu \sim a$. Recall that for the harmonic oscillator, the raising and lower operators commute to give $[a, a^\dagger] = 1$ and the Hamiltonian can be written as a combination of these raising and lower operators: $H = a^\dagger a + E_0$, where E_0 is a constant (the ground state energy). A short computation leads to the conclusion $[H, a] = -a$ and $[H, a^\dagger] = a^\dagger$. If there is a lowest weight state $|0\rangle$, such that $a|0\rangle = 0$, then $H|0\rangle = E_0|0\rangle$. Moreover, the relation $H(a^\dagger)^n|0\rangle = (E_0 + n)(a^\dagger)^n|0\rangle$ follows from the commutation relations of H with $a^\dagger$.

We can play a very similar game with the conformal group. We declare a lowest weight state – or *primary* state – to be an eigenvector of the dilatation operator and also annihilated by special conformal transformations

$$D|\phi_I\rangle = i\Delta|\phi_I\rangle \ , \tag{3.1}$$

$$K_\mu|\phi_I\rangle = 0 \ . \tag{3.2}$$

The factor of i is rather funny and is a consequence of the fact that the conformal group has indefinite signature, either $SO(d+1,1)$ or $SO(d,2)$, depending on whether we include a time-like direction. The dilatation operator does not have real eigenvalues!

If we like, we can also associate with the state an irreducible representation of the Lorentz group, indicated by the generalized index I. In this case, we have the rule that

$$\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}|\phi_I\rangle = \frac{i}{2}\omega^{\mu\nu}(M_{\mu\nu})_J{}^I|\phi_I\rangle = g_I{}^J\phi_J\rangle \ , \tag{3.3}$$

where by placing further I and J indices on $M_{\mu\nu}$, we have converted it from a generalized operator to a specific matrix representation of the Lorentz group. The rules for dealing with these more general representations of the Lorentz group quickly get involved, and so for the most part we will content ourselves with representations with no, one, or two vector indices.

Just as the harmonic oscillator has excited states that are formed by acting with $a^\dagger$ on the ground state, conformal primary states have *descendant* states which are constructed by acting with derivatives $P_\mu = -i\partial_\mu$ on the conformal primary state. Acting with P_μ n times increases the conformal weight $\Delta \to \Delta + n$. Acting with K_μ decreases the weight.

Most of the conformal field theory literature is phrased in terms of operators and correlation functions rather than states. We thus replace these conformal primary states with operators at the origin acting on the vacuum that create these states. A conformal primary operator $\phi_I(x)$ is one such that

$$\phi_I(0)|0\rangle = |\phi_I\rangle \ . \tag{3.4}$$

Part of the definition of the vacuum is that it is conformally invariant; it is annihilated by all of the generators of the conformal group. We could have chosen any point in space-time to insert the operator as all points are related via the conformal group. However, our choice of generators, for example $D = -ix^\mu\partial_\mu$, make the origin a simpler choice.

The action of the group on the operator is then given in terms of commutation relations:

$$[D, \phi_I(0)] = i\Delta\phi_I(0) \ , \tag{3.5}$$

$$[M_{\mu\nu}, \phi_I(0)] = (M_{\mu\nu})_I{}^J\phi_J(0) \ , \tag{3.6}$$

$$[K_\mu, \phi_I(0)] = 0 \ . \tag{3.7}$$

To recover the action of D, $M_{\mu\nu}$ and K_μ on $\phi_I(x)$ away from the origin, we use the fact that $\phi_I(x) = e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}$ and the commutator algebra of the conformal group. For instance

$$\begin{aligned}
[D, \phi_I(x)] &= De^{iP\cdot x}\phi_I(0)e^{-iP\cdot x} - e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}D \\
&= e^{iPx}(e^{-iP\cdot x}De^{iP\cdot x}\phi_I(0) - \phi_I(0)e^{-iP\cdot x}De^{iP\cdot x})e^{-iP\cdot x} \\
&= e^{iP\cdot x}[\hat{D}, \phi_I(0)]e^{-iP\cdot x} \ ,
\end{aligned} \tag{3.8}$$

where we have defined $\hat{D} = e^{-iP\cdot x}De^{iP\cdot x}$. We then compute $\hat{D}$ explicitly,

$$\begin{aligned}
\hat{D} &= \left(1 - ix\cdot P - \frac{(x\cdot P)^2}{2} + \ldots\right) D \left(1 + ix\cdot P - \frac{(x\cdot P)^2}{2} + \ldots\right) \\
&= D - ix^\mu[P_\mu, D] - \frac{1}{2}x^\mu x^\nu[P_\mu, [P_\nu, D]] + \ldots
\end{aligned} \tag{3.9}$$

and from the commutator algebra conclude that $[P_\mu, [P_\mu, D]]$ and all higher order terms vanish. In short $\hat{D} = D - x^\mu P_\mu$ and

$$[D, \phi_I(x)] = i(\Delta + x^\mu\partial_\mu)\phi_I(x) \ . \tag{3.10}$$

A similar simplification occurs for the other elements of the conformal group.

Problem 3.1. *Verify that*

$$[K_\mu, \phi_I(x)] = -2ix_\mu \Delta \phi_I(x) - i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)\phi_I(x) - 2x^\rho {(M_{\rho\mu})_I}^J \phi_J(x) \ . \tag{3.11}$$

From the infinitesimal action of the conformal group, one can in principle reconstruct the finite action on the field $\phi_I(x)$. For scalar fields (trivial representation of the Lorentz group), the rule is that

$$\phi'(x') = \Omega^{\Delta/2}\phi(x) \ . \tag{3.12}$$

Instead of constructing the finite version of the transformation from the infinitesimal one, it is more straightforward to check that the infinitesimal action of D and K_μ can be recovered from the finite transformations. Let us check D and leave K_μ for the reader. We want to look at the variation of the field at a particular point,

$$\delta\phi \equiv \phi'(x) - \phi(x) \ . \tag{3.13}$$

Note carefully which objects are primed and which are not in comparing this expression with (3.12). Now consider the dilatation $x' = (1+\lambda)x$ for small $\lambda \ll 1$. The infinitesimal change in the field is given by (3.10):

$$\delta\phi = i\lambda[D, \phi(x)] = -\lambda(\Delta + x^\mu \partial_\mu)\phi(x) \ . \tag{3.14}$$

For the dilatation $x' = (1+\lambda)x$, we know $\Omega = (1+\lambda)^{-2}$ and therefore from (3.12)

$$\phi'(x') = (1+\lambda)^{-\Delta}\phi(x) \approx (1 - \Delta\lambda)\phi(x) \ . \tag{3.15}$$

We could equally well consider the variation of the field at x' as at x:

$$\delta\phi(x') = \phi'(x') - \phi(x') \ . \tag{3.16}$$

We then expand out $\phi(x')$ in a Taylor series, $\phi(x') \approx \phi(x) + \lambda x^\mu \partial_\mu \phi(x)$, yielding

$$\delta\phi(x') \approx -\Delta\lambda\phi(x) - \lambda x^\mu \partial_\mu \phi(x) \ . \tag{3.17}$$

Then, because we are already working at linear order in λ, we are free to replace x on the right hand side with x', yielding the desired transformation rule.

Problem 3.2. *Verify that the rule (3.12) for the finite conformal symmetry transformations is also consistent with the infinitesimal transformation rule (3.11) for the special conformal transformations K_μ.*

For tensor fields, the power of Ω in the transformation rule is adjusted by the spin of the operator:

$$T'^{\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_m}(x') = \Omega^{\frac{\Delta+n-m}{2}} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x'^{\nu_n}}{\partial x^{\beta_n}} \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\alpha_m}}{\partial x'^{\nu_m}} T^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_m}(x) \ . \tag{3.18}$$

3.1 Conformal Symmetry from Curved Space

The conformal symmetry group is complicated, and it is often valuable to try to find conceptually more efficient ways of representing it. One method which we shall not touch in these lectures is the embedding space or null-cone formalism, where one uplifts the d-dimensional CFT to $d+2$ dimensions where $SO(d,2)$ acts linearly, as matrix multiplication. Another is to think about d-dimensional flat space as a limit of the QFT in curved space. It is often a surprisingly simple exercise to write a flat space space action in a diffeomorphism invariant form in curved space. For example, the massless scalar field in flat space

$$S = -\frac{1}{2}\int \mathrm{d}^d x (\partial_\mu \phi)(\partial^\mu \phi) \ , \tag{3.19}$$

where indices are raised and lowered with the Minkowski tensor $\eta_{\mu\nu}$, becomes

$$S = -\frac{1}{2}\int \mathrm{d}^d x \sqrt{-g} (\partial_\mu \phi)(\partial^\mu \phi) \tag{3.20}$$

in curved space where now indices are raised and lowered with the full metric tensor $g_{\mu\nu}$ and $\sqrt{-g}$ is shorthand for $\sqrt{-\det(g_{\mu\nu})}$. Such a naive approach will miss terms that vanish in flat space, for example $R\phi^2$ where R is the Ricci scalar, which can turn out to be very important.

Conformal symmetry from the curved space perspective are the set of diffeomorphisms which leave the metric invariant up to rescaling by a local function:

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \Omega(x) g_{\mu\nu} \ . \tag{3.21}$$

As diffeomorphism is trivially a symmetry of the theory in curved space, what we require from this perspective for conformal symmetry is an additional symmetry under rescaling of the metric. The additional symmetry has a name – Weyl symmetry.

Problem 3.3. *In the case of the free scalar field, the simple $(\partial\phi)^2$ action is not Weyl symmetric. However, if one adds the $R\phi^2$ term*

$$S = -\frac{1}{2}\int \mathrm{d}^d x \sqrt{-g} \left[(\partial_\mu \phi)(\partial^\mu \phi) + \xi R \phi^2\right] \tag{3.22}$$

where $\xi = \frac{(d-2)}{4(d-1)}$, the action is Weyl symmetric. Verify this fact, assuming $\phi \to \Omega^{\frac{d-2}{4}}\phi$ and $g_{\mu\nu} \to \Omega^{-1} g_{\mu\nu}$ under Weyl rescaling.

A convenient side effect of moving to curved space is a simple method for computing the stress-energy tensor. This tensor, which describes the flow of energy and momentum, is usually introduced in the context of Noether's theorem and translation invariance. The stress-energy tensor is the conserved current associated with translation symmetry. However, an alternate definition is the response of the action to variation of the metric:

$$\delta S = \int \mathrm{d}^d x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \ . \tag{3.23}$$

Diffeomorphisms are symmetries for which infinitesimally $x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x)$. The metric changes infinitesimally as $\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$. That this transformation is a symmetry means that δS should vanish in this case. Integrating by parts, we conclude the stress-tensor is conserved $\nabla_\mu T^{\mu\nu} = 0$. If we also insist Weyl scaling $\delta g_{\mu\nu} = \lambda(x) g_{\mu\nu}$ is a symmetry, then we conclude that the stress-tensor is traceless, $T^\mu_\mu = 0$.[3,4]

Tracelessness of the stress tensor is an oft cited property of conformal field theories. In fact, Weyl symmetry is almost always anomalous. In other words it is a symmetry classically but spoiled by quantum effects, when we consider the full path integral for the field theory. One finds in general curved space-time that the trace of the stress tensor is proportional to a sum over curvature invariants with special properties. These "trace anomalies" feature prominently in the stud of conformal field theory, but we do not have space to discuss them in detail. In 2d CFT, for example, $\langle T^\mu_\mu \rangle = \frac{c}{24\pi} R$ where R is the Ricci scalar curvature and c is a constant, the central charge of the CFT. The Zamolodchikov c-theorem states in part that if two CFTs are connected by a renormalization group flow, c at the UV fixed point must be larger than c at the IR fixed point.

Problem 3.4. *Compute the stress tensor in the flat space limit $g_{\mu\nu} = \eta_{\mu\nu}$ for the scalar field of problem 3.3 with the conformal coupling $\xi = \frac{(d-2)}{4(d-1)}$. Check that $T^{\mu\nu}$ is conserved and traceless on-shell in the flat space limit.*

3.2 Correlation Functions

In the study of quantum field theory, a central role is played by the notion of a correlation function. These correlation functions are defined through the path integral

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle \equiv \frac{1}{Z}\int [d\phi]\phi(x_1)\cdots\phi(x_n) e^{iS[\phi]} , \qquad (3.24)$$

for a generic action $S[\phi]$ that is a functional of a field $\phi(x)$. Here $Z = \int [d\phi] e^{iS[\phi]}$. We are interested in QFTs that are invariant with respect to a symmetry. That means, at a quantum level, both S and the measure $[d\phi]$ should be invariant with respect to the action of the symmetry group. (Theories where S is invariant but the measure fails to be invariant are said to have the symmetry classically but possess an anomaly.) These symmetries have consequences for the correlation functions, consequences which are called Ward identities.

Let us suppose that the symmetry acts on ϕ via $\phi \to R(\phi)$. We would like to understand how the symmetry affects the correlation function:

$$\langle R(\phi(x_1))R(\phi(x_2))\cdots R(\phi(x_n))\rangle = \frac{1}{Z}\int [d\phi] R(\phi(x_1))\cdots R(\phi(x_n)) e^{iS[\phi]} . \qquad (3.25)$$

[3]Note that global scale invariance, where λ is a constant, is not enough to guarantee tracelessness. It only guarantees that T^μ_μ is a total derivative. The special conformal transformations, where λ depends on x, are needed to guarantee tracelessness.

[4]We are playing a little fast and loose here. In analyzing the transformation of the action with respect to the symmetry, either Weyl rescaling or diffeomorphism, the fields will transform as well. However, these conditions on the stress tensor are expected to hold only on-shell, after applying the equations of motion. The equations of motion are derived by varying the action with respect to the fields. Thus the equations of motion can be used to zero out the contribution from varying the fields in computing δS for the symmetry transformation, leaving only the contribution from $\delta g_{\mu\nu}$.

With respect to earlier notation $R(\phi(x)) = \phi'(x')$. Because the measure and the action are invariant under the symmetry, we can make the replacements $[d\phi] = [dR(\phi)]$ and $S[\phi] = S[R(\phi)]$ without changing the value of the correlation function:

$$\langle R(\phi(x_1))R(\phi(x_2))\cdots R(\phi(x_n))\rangle = \frac{1}{Z}\int [dR(\phi)]R(\phi(x_1))\cdots R(\phi(x_n))e^{iS[R(\phi)]} .$$

Further, it is important to realize that $R[\phi]$ is just a dummy integration variable. We are free to replace it with ϕ itself. However, for space-time symmetries, this replacement will not affect the action of the symmetry group on the locations x_i of the $\phi(x_i)$ insertions:

$$\langle R(\phi(x_1))R(\phi(x_2))\cdots R(\phi(x_n))\rangle = \frac{1}{Z}\int [d\phi]\phi(R(x_1))\cdots \phi(R(x_n))e^{iS[\phi]} .$$

We are left with the result, slightly generalizing to the case where the fields are distinct,

$$\langle \phi_1(R(x_1))\phi_2(R(x_2))\cdots \phi_n(R(x_n))\rangle = \langle R(\phi_1(x_1))R(\phi_2(x_2))\cdots R(\phi_n(x_n))\rangle . \tag{3.26}$$

For conformal symmetry and scalar primary operators, we can put (3.26) and (3.12) together to learn that

$$\langle \phi_1(x_1')\cdots\phi_n(x_n')\rangle = \left(\prod_{i=1}^{n}\Omega^{\Delta_i/2}(x_i)\right)\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle . \tag{3.27}$$

In the case of translations and Lorentz transformations, we have that $\Omega = 1$. For translations more particularly, we find that the correlation function depends only on the relative positions of the insertions

$$\langle \phi(x_1+a)\cdots\phi(x_n+a)\rangle = \langle\phi(x_1)\cdots\phi(x_n)\rangle . \tag{3.28}$$

For Lorentz transformations, indices must be contracted in a Lorentz invariant way.

But we have two more transformations at our disposal – dilatations and special conformal transformations – which turn out to be strong enough to fix the form of two and three point functions of scalar primaries up to constants. Let us see how these constraints arise in more detail.

Two Point Functions

For two point functions of scalars, Poincaré invariance implies the correlation function can only depend on the Lorentz invariant distance between the insertions

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1-x_2|) . \tag{3.29}$$

Scale transformations $x \to x' = \lambda x$ further imply

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle\phi_1(\lambda x)\phi_2(\lambda x)\rangle , \tag{3.30}$$

from which we conclude $f(|x_1-x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x_1-x_2|)$. The only way to satisfy this constraint is to choose

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} . \tag{3.31}$$

Finally, we consider special conformal transformations. From Problem 2.9, you should have learned that

$$\Omega = (1 - 2b \cdot x + b^2 x^2)^2 \tag{3.32}$$

Let us define $\gamma_i \equiv 1 - 2b \cdot x_i + b^2 x_i^2$. A remarkable property about special conformal transformations is that

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{\gamma_1^{1/2} \gamma_2^{1/2}} \ , \tag{3.33}$$

from which we can see that

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} = \frac{1}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2}} \frac{C_{12}}{|x_1' - x_2'|^{\Delta_1 + \Delta_2}} = \frac{(\gamma_1 \gamma_2)^{\frac{\Delta_1 + \Delta_2}{2}}}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2}} \frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \tag{3.34}$$

where in the first equality, we used the Ward identity. This expression can only make sense if $\Delta_1 = \Delta_2$ or if $C_{12} = 0$, since γ_1 and γ_2 are independent quantities. The final result for the correlation function of two scalar primary operators is thus

$$\langle \phi_1(x) \phi_2(x) \rangle = \begin{cases} 0 & \Delta_1 \neq \Delta_2 \\ \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \ . \end{cases} \tag{3.35}$$

Often it is possible to normalize the fields such that $C_{12} = 1$. For example, for the free scalar field $\phi(x)$, a kinetic term in the action normalized with a 1/2 in front will lead to a particular value of $C_{\phi\phi}$. However, by sending $\phi \to \phi' = c\phi$, one will shift the normalization $C_{\phi\phi} \to C_{\phi\phi}/c^2$.

Problem 3.5. *Verify the remarkable property (3.33).*

Three Point Functions

Three point functions $\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle$ of scalar primary operators are fixed in a similar manner. Poincaré plus scale invariance fix the correlation function to be a sum over terms of the form

$$\frac{1}{|x_{12}|^a |x_{23}|^b |x_{13}|^c} \ , \tag{3.36}$$

where $a+b+c = \Delta_1 + \Delta_2 + \Delta_3$. We have also introduced the compact notation $x_{ij} = |x_i - x_j|$. Special conformal invariance then fixes one particular choice of the constants a, b, and c. In particular, one finds the constraint

$$\frac{C_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c} = \frac{(\gamma_1 \gamma_2)^{a/2} (\gamma_2 \gamma_3)^{b/2} (\gamma_3 \gamma_1)^{c/2}}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2} \gamma_3^{\Delta_3}} \frac{C_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c} \tag{3.37}$$

For this ratio of gamma factors to be unity,

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 \ , \\ b &= \Delta_2 + \Delta_3 - \Delta_1 \ , \qquad (3.38) \\ c &= \Delta_3 + \Delta_1 - \Delta_2 \ . \qquad (3.39) \end{aligned}$$

The final result is that

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \ . \tag{3.40}$$

While the coefficients of two-point functions can often be absorbed through changing the normalization of the fields, the ratios of three point function coefficients C_{ijk} to two-point coefficients C_{ij} contain physical information.

Four Point Functions

Once we have four positions at our disposal, something new occurs. We can form the invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ , \tag{3.41}$$

which are invariant under the full conformal group. Unlike the two and three point functions, the four point function is not completely fixed by conformal invariance

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{F(u,v)}{\prod_{i<j}|x_{ij}^2|^{\delta_{ij}}} \ , \tag{3.42}$$

where $\sum_{j\neq i}\delta_{ij} = \Delta_i$. The function $F(u,v)$ is not constrained in any obvious way from this point of view.

Vector and Tensor Operators

One can play the same game with operators in nontrivial representations of the Lorentz group. Two important example worth mentioning are a conserved current J^μ and the stress tensor $T^{\mu\nu}$. Conservation here implies that $\partial_\mu J^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$, which places further constraints on the correlation functions.

Let us begin with $\langle J_\mu(x)J_\nu(0)\rangle$, where by translation invariance, we are free to put the second current at the origin without loss of generality. The game is played by trying to construct the most general symmetric two index tensor out of the elementary building blocks available to us, in this case $\eta_{\mu\nu}$ and x^μ. Poincaré and scaling symmetry tell us that the two-point function must have the form

$$\langle J^\mu(x)J^\nu(0)\rangle = \tau\frac{\eta^{\mu\nu} + \alpha\frac{x^\mu x^\nu}{x^2}}{|x|^{2\Delta}} \tag{3.43}$$

where τ and α are constants. The general transformation rule for a vector field is

$$J'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\Omega^{\frac{\Delta+1}{2}} J^\nu(x) \ . \tag{3.44}$$

Combined with the Ward identity $\langle J^\mu(x')J^\nu(y')\rangle = \langle J'^\mu(x')J'^\nu(y')\rangle$ for special conformal transformations, we find $\alpha = 2$. The tensor

$$I^{\mu\nu}(x) = \eta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2} \ , \tag{3.45}$$

called the inversion tensor, plays an important role in conformal field theory.

Finally, we enforce the conservation condition $\partial_\mu \langle J^\mu(x) J^\nu(0)\rangle = 0$, which tells us either $\tau = 0$ or $\Delta = d-1$. In other words, conserved currents must have scaling dimension $d-1$, which makes sense from a dimensional analysis point of view. The time component J^0 is a charge density, which carries some units of dimensionless charge per unit volume.

The stress tensor two-point function can also be expressed in terms of the inversion tensor. One finds, after a similar analysis,

$$\langle T^{\mu\nu}(x) T^{\rho\sigma}(0)\rangle = \frac{c}{|x|^{2d}} \left(\frac{1}{2}(I^{\mu\sigma}(x) I^{\nu\rho}(x) + I^{\mu\rho}(x) I^{\nu\sigma}(x)) - \frac{1}{d}\eta^{\mu\nu}\eta^{\sigma\rho} \right) . \tag{3.46}$$

The conservation condition $\partial_\mu \langle T^{\mu\nu}(x) T^{\rho\sigma}(0)\rangle = 0$ fixes the dimension $\Delta = d$, which again makes sense from a dimensional analysis point of view. The component T^{00} is the energy density, which has units of mass per unit volume, or in our relativistic field theory framework where $\hbar = c = 1$, dimensions of mass to the d power. Unlike the case of conformal primary operators, whose normalization can often be adjusted, the normalization of the two-point function of the stress tensor is a physical quantity. The stress-tensor is a composite operator, made up of a product of conformal primaries. It is thus secretly a higher point correlation function in a limit where some of the points are taken to be coincident and divergences subtracted. The number c is called the central charge. The numbers τ and c play an important role in characterizing CFTs.

4 Radial Quantization and the Operator Product Expansion

In introducing the notion of a conformal primary state $|\phi_I\rangle$ and conformal primary operator $\phi_I(x)$ in the previous chapter, the origin played a special role: $|\phi_I\rangle = \phi_I(0)|0\rangle$. The origin plays such a role because in defining the dilatation operator on function space, $D = x^\mu \partial_\mu$, we chose to think about it as scale transformations with respect to the origin. (Of course, we could equally well have chosen to dilate space about some other point $\hat{D}(x) = e^{-iPx} D e^{iPx}$.)

There is a different and useful way of thinking about the origin. Let's instead return to the standard QFT framework, where we can create in and out states by acting on the vacuum in the far past and far future, $|\psi_{\rm in}\rangle = \lim_{t\to-\infty} \hat\psi(t)|0\rangle$ and $|\psi_{\rm out}\rangle = \lim_{t\to\infty} \hat\psi(t)|0\rangle$, with some local operator $\hat\psi(t)$.

In a conformal field theory, in a Euclidean context where all the coordinates are spatial, people often choose to think about the radial coordinate as a time-like coordinate. Suppose we write the line element of flat space as a foliation of spheres

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \tag{4.1}$$

where $\mathrm{d}\Omega^2$ is the line element on a $(d-1)$-dimensional sphere of unit radius and $r > 0$. Then we could equally well decide to define a new radial coordinate $r = e^\tau$ in which case the line element becomes

$$\mathrm{d}s^2 = e^{2\tau} \left(\mathrm{d}\tau^2 + \mathrm{d}\Omega^2 \right) . \tag{4.2}$$

In the new coordinate system τ ranges from $-\infty < \tau < \infty$. We can think about the point $r = 0$ as the far past. Similarly $r \to \infty$ is the far future.

We mentioned before that in a CFT context, the dilatation operator D largely replaces the Hamiltonian P^0. While in QFT, we can use the evolution operator $U(t) = e^{itP^0}$ to move from time slice to time slice, in a CFT framework, we can use instead the operator $U(r) = e^{i\tau D}$ to move from radial slice to radial slice. In QFT, we have $P^0 = -i\partial_t$. In CFT, on the other hand, we have $D = -ix^\mu\partial_\mu = -ir\partial_r = -i\partial_\tau$.

From this point of view of "radial quantization", the conformal primary state $|\phi_I\rangle$, created at the origin by $\phi_I(0)$, can be thought of as a standard in-state in a usual QFT context. Similarly, there are out states which are created by inserting operators at large radial distance from the origin.

There are some technical perils in this program which we will not dwell on overly. The first is that $e^{i\tau D}$ is not unitary. Another is how exactly to define a useful inner product with the out states.

4.1 Operator Product Expansion

The next exercise is to consider the state

$$|\psi\rangle = \phi_1(x)\phi_2(0)|0\rangle \ . \tag{4.3}$$

For simplicity, we can consider the case where both operators are scalars. Because ψ is a state and because the space of states is spanned by eigenstates of the dilatation operator, we can expand ψ in a basis of such states:

$$|\psi\rangle = \sum_n c_n|\Delta_n\rangle \ . \tag{4.4}$$

Moreover, we know that these eigenstates come in multiplets. Each multiplet contains a conformal primary state $|\phi_I\rangle$ and its descendants $P^{\mu_1}\cdots P^{\mu_n}|\phi_I\rangle$. We can therefore write the state $|\psi\rangle$ in the form

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{\phi_I} C_{\Delta,I}(x,\partial)\phi_I(0)|0\rangle \ , \tag{4.5}$$

where we will discuss the precise form of $C_{\Delta,I}(x,\partial)$ momentarily. Here Δ is the scaling dimension of ϕ_I.

In fact, we can promote this operator product expansion from a discussion of states to a discussion of the operator algebra itself:

$$\phi_1(x)\phi_2(0) = \sum_{\phi_I} C_{\Delta,I}(x,\partial)\phi_I(0) \ , \tag{4.6}$$

where implicitly the equality holds only inside correlation functions, and also only where the additional operators inside the correlation function are inserted outside the sphere, centered at the origin, of radius $|x|$. Said another way, the insertion of a third operator $\phi(x')$ in the correlation function $\langle\phi(x')\phi_1(x)\phi_2(0)\rangle$ sets a radius of convergence for the small x expansion, namely $|x| < |x'|$.

Now let us try to pin down the form of $C_{\Delta,I}$. By dimensional analysis, for a scalar operator $\phi(x)$ of dimension Δ, we can see that

$$C_\Delta(x,\partial)\phi(0) = \frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots) \ . \tag{4.7}$$

The ellipsis refers to all of the descendants of $\phi(x)$. To check this guess, we can act with the dilatation operator. Acting on the left hand side of (4.6) yields

$$\begin{aligned} D\phi_1(x)\phi_2(0)|0\rangle &= i(\Delta_1 + x^\mu\partial_\mu)\phi_1(x)\phi_2(0)|0\rangle + i\Delta_2\phi_1(x)\phi_2(0)|0\rangle \\ &= i(\Delta_1+\Delta_2)\phi_1(x)\phi_2(0)|0\rangle + x^\mu\partial_\mu\phi_1(x)\phi_2(0)|0\rangle \ . \end{aligned} \tag{4.8}$$

We now substitute the guess (4.7) for $\phi_1(x)\phi_2(0)$ in the second term, focusing on the contribution of dimension Δ to this operator product expansion:

$$D\phi_1(x)\phi_2(0)|0\rangle \sim i(\Delta_1+\Delta_2-(\Delta_1+\Delta_2-\Delta))\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots)|0\rangle \ . \tag{4.9}$$

Acting directly on the right hand side of (4.7) with D yields the same result to leading order in a small x expansion:

$$D\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots)|0\rangle = i\Delta\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots)|0\rangle \ . \tag{4.10}$$

Problem 4.1. *Continuing the small $|x|$ expansion of $C_\Delta(x,\partial)$, we find at next order*

$$C_\Delta(x,\partial)\phi(0) = \frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(1+\alpha x^\mu\partial_\mu+\ldots\right)\phi(0) \ .$$

By acting with K^μ on boths sides, show that $\alpha = \frac{\Delta_1-\Delta_2+\Delta}{2\Delta}$.

In fact conformal invariance completely fixes the form of $C_{\Delta,I}(x,\partial)$ up to an overall constant, which we called c in the discussion above. A more efficient way to compute $C_\Delta(x,\partial)$ is as follows. Consider expanding the following three point function of three scalar operators using the operator product expansion

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \sum_{\Delta'} C_{12\Delta'}C_{\Delta'}(x,\partial_y)\langle\phi_{\Delta'}(y)\phi_\Delta(z)\rangle_{y=0} \ . \tag{4.11}$$

The constant c has now been renamed $C_{12\Delta'}$ and pulled out of the definition of $C_\Delta(x,\partial)$. All the higher spin primaries in the operator product expansion will have vanishing expectation value with ϕ_Δ and so we can restrict the sum to scalar primaries. In fact only scalar primaries with dimension $\Delta' = \Delta$ will contribute to the sum:

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = C_{12\Delta}C_\Delta(x,\partial)\langle\phi_\Delta(y)\phi_\Delta(z)\rangle_{y=0} \ . \tag{4.12}$$

Conformal invariance forces the two and three point functions to have the form

$$\langle\phi_\Delta(y)\phi_\Delta(z)\rangle = \frac{1}{|y-z|^{2\Delta}} \ , \tag{4.13}$$

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \frac{C_{12\Delta}}{|x|^{\Delta_1+\Delta_2-\Delta}|z|^{\Delta_2+\Delta-\Delta_1}|x-z|^{\Delta_1+\Delta-\Delta_2}} \ . \tag{4.14}$$

where we have taken the liberty of fixing the normalization of the two-point function in a conventional CFT way. By expanding out the left hand side of (4.12) for small $|x|$ and matching to the right hand side, we can fix the form of $C_\Delta(x, \partial)$. Note that having normalized the two-point function to unity, the constant $C_{12\Delta}$ in the operator product expansion and in the three point function are naturally identified, fixing a normalization for $C_\Delta(x, \partial)$.

Problem 4.2. *Use this procedure to compute the first three terms in* $C_\Delta(x, \partial)$.

4.2 Conformal Blocks

We now apply this notion of the operator product expansion to higher point functions. For simplicity, let us consider the correlation function of four identical scalar primaries $\Phi(x)$ with dimension η. From the discussion at the end of section 3, we saw that conformal symmetry constrains the four point correlation function to have the form

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \frac{G(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} \tag{4.15}$$

where u and v were the invariant cross ratios (3.41) formed from combinations of the differences x_{ij} between the insertion locations.

Given the technology of the operator product expansion, we can take x_1 close to x_2 and x_3 close to x_4 and write pairs of the operators in the four point function as sums over conformal primaries:

$$\phi(x_1)\phi(x_2) = \sum_{\Delta,I} c_{\Delta,I} C_{\Delta,I}(x_{12}, \partial_y)\phi_{\Delta,I}(y)\big|_{y=x_2} \ , \tag{4.16}$$

$$\phi(x_3)\phi(x_4) = \sum_{\Delta,I} c_{\Delta,I} C_{\Delta,I}(x_{34}, \partial_z)\phi_{\Delta,I}(z)\big|_{z=x_4} \ . \tag{4.17}$$

The $c_{\Delta,I}$ are the OPE coefficients, or equivalently the coefficients in the three point functions if we normalize the two-point functions in the conventional way. Inserting these decompositions into the four point function, we obtain the sum (see fig. 2a)

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \sum_{\Delta,I} c^2_{\Delta,I} \left[C_{\Delta,I}(x_{12}, \partial_y) C_{\Delta,I}(x_{34}, \partial_z)\langle \phi_{\Delta,I}(y)\phi_{\Delta,I}(z)\rangle\right]\Big|_{y=x_2, z=x_4} \tag{4.18}$$

Note the double sum collapses to a single sum because the two point function between two conformal primaries vanishes unless the operators have the same conformal dimension and spin.

The important point here is that the term in brackets is completely fixed by conformal invariance. By convention, we define a *conformal block* $G_{\Delta,I}(u,v)$ such that

$$\left[C_{\Delta,I}(x_{12}, \partial_y) C_{\Delta,I}(x_{34}, \partial_z)\langle \phi_{\Delta,I}(y)\phi_{\Delta,I}(z)\rangle\right]\big|_{y=x_2, z=x_4} = \frac{G_{\Delta,I}(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} \ . \tag{4.19}$$

The conformal block is defined in a theory independent fashion by a choice of Lorentz representation I and conformal dimensions η and Δ. The theory dependent data in the four

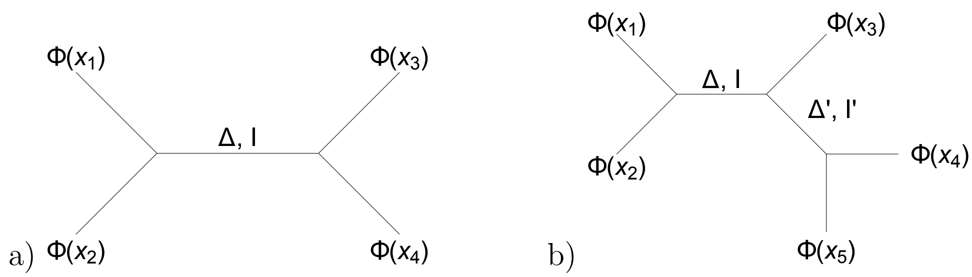

Figure 2: The decomposition of a) a four-point function and b) a five-point function into a sum over conformal blocks.

point function reduces to the operator product coefficients $c_{\Delta,I}$ and the conformal dimensions Δ.

A similar story holds true for higher point functions as well (see fig. 2). By bringing the insertions close together pairwise, one can decompose an arbitrary correlation function into a sum over conformal blocks. One can make thus a stronger statement that a conformal field theory is defined by the data – the spin and scaling dimension – of its conformal primaries along with the coefficients in their three point functions. With those in hand, one can reconstruct any correlation function in a conformal partial wave decomposition. In the case of the four point function, we can write

$$G(u,v) = \sum_{\Delta,I} c_{\Delta,I}^2 G_{\Delta,I}(u,v) \ . \tag{4.20}$$

We will see in the next section how to further constrain the operator spectrum and OPE coefficients that define a CFT by examining a particular constraint on this sum.

To be more concrete, we can give $G_{\Delta,I}$ for four identical scalars in four dimensions:

$$G_{\Delta,\ell}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \ , \tag{4.21}$$

where we have defined

$$k_\beta(z) = z^{\frac{\beta}{2}}{}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta,z\right) \ .$$

We have also introduced $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$. To understand these new coordinates geometrically, one can place $x_1 = (0,0,\ldots)$, $x_3 = (1,0,0,\ldots)$ and x_4 at infinity. Then rotate the coordinate system to put x_2 in the xy-plane. The z coordinate is x_2, thinking of the xy-plane as a complex coordinate system (see fig. 3).

Problem 4.3. *By explicitly computing the first few terms in a small z expansion, verify the form of the conformal block for $\ell = 0$ and $d = 4$ by comparing it against your previous small x expansion of $C_\Delta(x,\partial)$.*

4.3 Deriving the Conformal Blocks

One method for deriving the expression (4.21) for the conformal blocks is to find a differential equation satisfied by $G_{\Delta,I}(u,v)$ and solve it. The claim is that $G_{\Delta,I}(u,v)$ is an eigenvector of the Casimir operator for the conformal group.

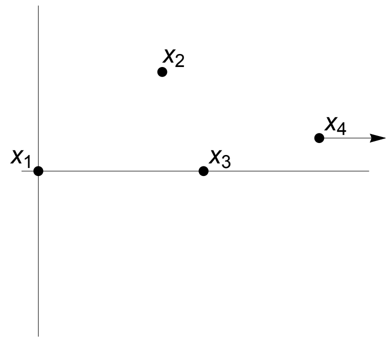

Figure 3: A useful configuration for understanding the z and $\bar{z}$ cross ratios.

What is the Casimir operator? You have seen this object for the SO(3) rotation group in quantum mechanics. In that case the Casimir operator was also called J^2 and it was equal to the sum $J^2 = J_x^2 + J_y^2 + J_z^2$. This operator had eigenvalues $\ell(\ell+1)$ for a $2\ell+1$ dimensional representation of SO(3). More generally for a rotation (or Lorentz) group, we can write

$$\text{cas} = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} \ . \tag{4.22}$$

In the case of SO(3), we have the relations $J_x = M_{yz}$, $J_y = M_{zx}$, and $J_z = M_{xy}$. The claim is that $[\text{cas}, M_{\mu\nu}] = 0$. Therefore everything in the same irreducible representation of the group will have the same eigenvalue with respect to the action of the Casimir operator.

In problem 2.7, we saw that the conformal group was also a rotation group, in particular the group $SO(d+1,1)$ (in the Euclidean case), with the identifications

$$M_{-10} = D \ , \quad M_{0i} = \frac{P_i + K_i}{2} \ , \quad M_{-1i} = \frac{P_i - K_i}{2} \tag{4.23}$$

with the metric $\eta_{-1,-1} = -1$ and $\eta_{00} = \eta_{ii} = 1$. The remaining elements M_{ij} are the generators of the usual Lorentz (or rotation) group inside the conformal group.

If we expand the Casimir operator out in terms of our more familiar P_i and K_j, we find that

$$\begin{aligned} \text{cas} &= \frac{1}{2} M^{\mu\nu} M_{\mu\nu} \\ &= \frac{1}{2} M^{ij} M_{ij} - D^2 + \frac{1}{2} P_i K^i + \frac{1}{2} K_i P^i \\ &= \frac{1}{2} M^{ij} M_{ij} - D(D - i\,d) - P_i K_i \ , \end{aligned} \tag{4.24}$$

where in the second line, I used the commutator $[K_i, P_j]$. We now apply this object to a primary state $|\phi_{\Delta,I}\rangle$ in order to learn its eigenvalue. (Note $i\,d$ is i times the the dimension d, not the identity matrix.) For simplicity, let us assume that $\phi_{\Delta,I}$ transforms in a symmetric, traceless representation of the M_{ij} with spin ℓ. The claim is that

$$\text{cas}|\phi_{\Delta,I}\rangle = [(\ell(\ell + d - 2) + \Delta(\Delta - d)]\, |\phi_{\Delta,I}\rangle \ . \tag{4.25}$$

The first part of the eigenvalue $\ell(\ell+d-2)$ is the generalization of the $\ell(\ell+1)$ result for the $SO(3)$ group. The second term $\Delta(\Delta-d)$ can be read off by acting with D on $|\phi_{\Delta,I}\rangle$.

We are now ready to return to the question of conformal blocks for the four point function $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ of four identical scalar operators. Let us insert a resolution of the identity into the four point function:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_\psi \langle\phi(x_1)\phi(x_2)|\psi\rangle\langle\psi|\phi(x_3)\phi(x_4)\rangle \ . \tag{4.26}$$

We then restrict the sum to $|\phi_{\Delta,I}\rangle$ and its descendants, i.e. a representation of the conformal group, every member of which will have the same eiegenvalue with respect to the action of the Casimir operator. This restriction is by definition the contribution of one conformal block to the four point function:

$$\sum_\psi{}' \langle\phi(x_1)\phi(x_2)|\psi\rangle\langle\psi|\phi(x_3)\phi(x_4)\rangle = \frac{G_{\Delta,I}(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} \ . \tag{4.27}$$

The claim is that $\text{cas}|\psi\rangle = [\ell(\ell+d-2)+\Delta(\Delta-d)]|\psi\rangle$ where $|\phi_{\Delta,I}\rangle$ is in a symmetric, traceless, spin ℓ representation of the Lorentz group and $|\psi\rangle$ in the multiplet with $|\phi_{\Delta,I}\rangle$. Inserting a Casimir operator and defining $\lambda_{\Delta,\ell} \equiv \ell(\ell+d-2)+\Delta(\Delta-d)$, we see that

$$\langle\phi(x_1)\phi(x_2)|\text{cas}|\psi\rangle = \lambda_{\Delta,\ell}\langle\phi(x_1)\phi(x_2)|\psi\rangle \ . \tag{4.28}$$

But we can also act with the Casimir operator to the left, using the representation of the conformal group on $\phi(x_1)\phi(x_2)$. (Of course, we could run the same argument with $\phi(x_3)\phi(x_4)$ as well, and will get the same answer.) When the dust settles, we find a second order, linear partial differential equation in the cross ratios u and v of the form

$$\text{cas}\, G_{\Delta,\ell}(u,v) = \lambda_{\Delta,\ell} G_{\Delta,\ell}(u,v) \ . \tag{4.29}$$

The solution to this differential equation are the conformal blocks of (4.21).

5 The Conformal Bootstrap

Say we just took out of a hat a random set of conformal primaries and OPE coefficients. Would such a selection provide the data to define a CFT? We would quickly find that such a random selection would lead to an inconsistent procedure for generating four and higher point correlation functions.

Consider the correlation function of four identical scalars of dimension η:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \ . \tag{5.1}$$

At the end of the previous section, we decomposed this object into a sum over conformal blocks by taking x_1 close to x_2 and x_3 close to x_4. However, we could equally well have proceeded by taking instead x_1 close to x_4 and x_2 close to x_3. This alternate procedure is

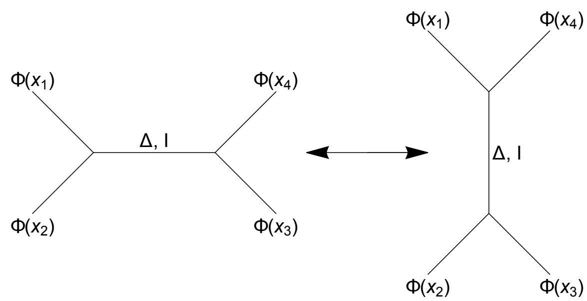

Figure 4: The basic crossing symmetry constraint.

equivalent to swapping x_2 and x_4 in the original decomposition. From the form of the cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ ,$$

this swap also exchanges u and v. We learn that

$$\frac{G(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} = \frac{G(v,u)}{|x_{14}|^{2\eta}|x_{23}|^{2\eta}} \tag{5.2}$$

or equivalently $v^\eta G(u,v) = u^\eta G(v,u)$. Inserting the partial wave decomposition, this relation becomes

$$v^\eta \sum_{\Delta,I} c_{\Delta,I}^2 G_{\Delta,I}(u,v) = u^\eta \sum_{\Delta,I} c_{\Delta,I}^2 G_{\Delta,I}(v,u) \ . \tag{5.3}$$

The exchange is illustrated pictorially in figure 4.

Now there is one operator in the spectrum of every CFT on whose presence we can rely, the identity operator. This operator has no descendants because the momentum operator annihilates constant valued functions. The OPE coefficient of $\phi \times \phi$ with the identity can be taken to be one, assuming we have normalized our two point functions conventionally, to have the form $|x-y|^{-2\eta}$. Removing the identity operator from the partial wave decomposition, we find

$$v^\eta \left(1 + {\sum_{\Delta,I}}' c_{\Delta,I}^2 G_{\Delta,I}(u,v) \right) = u^\eta \left(1 + {\sum_{\Delta,I}}' c_{\Delta,I}^2 G_{\Delta,I}(v,u) \right) \ . \tag{5.4}$$

The conformal bootstrap equation is then the following slight massage of the previous expression:

$${\sum_{\Delta,I}}' c_{\Delta,I}^2 \left(\frac{v^\eta G_{\Delta,I}(u,v) - u^\eta G_{\Delta,I}(v,u)}{u^\eta - v^\eta} \right) = 1 \ . \tag{5.5}$$

Generically, a random selection of conformal primaries and their OPE coefficients will be inconsistent with this relation. One could take a step back and insist only on a random selection of conformal primaries. Perhaps then the $c_{\Delta,I}$ can be adjusted to make the equation

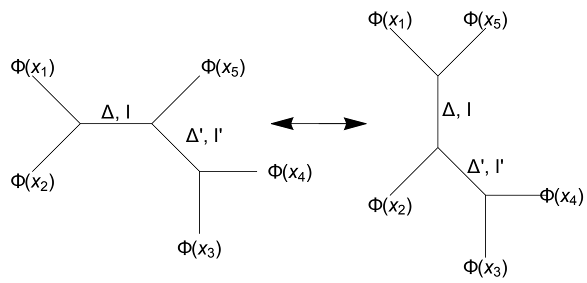

Figure 5: A crossing symmetry constraint for a five-point function.

true. In fact, however, one can use this expression to place bounds on the operator spectrum as well!

Before we proceed further with trying to constrain the operator spectrum, a natural question to ask is whether considering higher point functions will lead to further constraints on the set of possible conformal field theories. The answer is no. By imposing four point crossing symmetry on intermediate channels of higher point functions, one can access all possible ways of decomposing the higher point functions into conformal blocks. The case of a five point function is illustrated in fig. 5. From a more formal standpoint, we are making a statement about the associativity of the operator algebra.

5.1 Interlude on Unitarity Bounds

In order to determine these bounds on the operator spectrum, one imposes additionally unitarity. Unitarity implies that the dimension of a field of spin ℓ in a symmetric traceless representation is bounded below by

$$\begin{aligned}
\Delta &\geq \ell + d - 2 \;\text{ if }\; \ell = 1, 2, 3, \ldots \\
\Delta &\geq \frac{d-1}{2} \;\text{ if }\; \ell = \frac{1}{2}\,, \\
\Delta &\geq \frac{d-2}{2} \;\text{ if }\; \ell = 0\,.
\end{aligned}$$

It also imposes that the OPE coefficients are real, so that $c^2_{\Delta,I} \geq 0$. Note that the minimum dimension of a scalar $\frac{d-2}{2}$ is the engineering dimension of a free scalar in d dimensions. The minimum dimension for $\ell = \frac{1}{2}$ is the engineering dimension of a free spin one half fermion. The minimum dimension for a vector field $\ell = 1$ is in fact the dimension of a conserved current. Similarly, the minimum dimension of a symmetric, traceless spin two field is the same as the dimension of the stress tensor. In other words, the dimensions of a generic field in CFT must be, according to its spin, greater or equal to that of a free scalar, free fermion, conserved current or stress tensor. There is a pattern here, that the multiplets generated from a primary of the smallest conformal dimension tend to be smaller. There is a shortening condition, where some of the descendants vanish. In the case of the free scalar, the condition is that $\Box\phi = 0$. For the fermion, it's the Dirac equation. For the conserved current and stress tensor, that $\partial_\mu J^\mu = 0 = \partial_\mu T^{\mu\nu}$.

Let us try to understand where these bounds come from in more detail. When we talk about unitarity for a Euclidean CFT (where time is like all the other spatial coordinates), what we really mean is reflection positivity:

$$\langle \mathcal{R}(\mathcal{O})\mathcal{O}\rangle \geq 0 \ , \tag{5.6}$$

where $\mathcal{O}$ is some arbitrary operator, possibly composite, and $\mathcal{R}$ is a reflection operator that reflects all of the insertions in $\mathcal{O}$ about some plane, for example $\{(x_1, x_2, \ldots) : x_1 = 0\}$ or $\{(x_1, x_2, \ldots) : x_1 = 1\}$. Indeed, since this is a conformal field theory, we can act on the space with special conformal transformations which will in general turn planes into spheres.

Problem 5.1. *Show that a special conformal transformation with* $b^\mu = (1, \vec{0})$ *turns the plane* $x^\mu = (1/2, \vec{0})$ *into a sphere centered at the origin of radius one. Furthermore, show that reflection about the plane* $x_1 = 1/2$ *becomes inversion after the special conformal transformation.*

Thus another way of insisting on reflection positivity is to work with the cylindrical coordinate system from section 4 where $\tau = \log r$ and to claim

$$\langle \mathcal{T}(\mathcal{O})\mathcal{O}\rangle \geq 0 \ . \tag{5.7}$$

where $\mathcal{T}$ sends $\tau \to -\tau$ (or equivalently $r \to 1/r$) in all the insertions that make up $\mathcal{O}$. In a Lorentzian context, Wick rotating time $\tau \to it$, we can then sometimes go further and think of $\mathcal{T}(\mathcal{O})$ as a Hermitian conjugate $\mathcal{O}^\dagger$.

From our experience building up representations of the conformal algebra using P_μ and K_μ, we saw that P_μ functioned like a raising operator while K_μ was a lowering operator. Given this intuition, let us see whether there is some sense in which K_μ can be treated as a reflection (or Hermitian conjugate) of P_μ. We make the change of variables $x_\mu = e^\tau n_\mu$ and hence $\tau = \frac{1}{2}\log x^2$ and $n_\mu = x_\mu/\sqrt{x^\nu x_\nu}$. We find then that

$$\begin{aligned} iP_\mu &= \partial_\mu = \left(\frac{\partial \tau}{\partial x^\mu}\frac{\partial}{\partial \tau} + \frac{\partial n_\nu}{\partial x^\mu}\frac{\partial}{\partial n_\nu} \right) \\ &= e^{-\tau}\left(n_\mu \frac{\partial}{\partial \tau} + (\delta_{\mu\nu} - n_\mu n_\nu)\frac{\partial}{\partial n_\nu} \right) \ . \end{aligned} \tag{5.8}$$

Building off the previous result, we quickly see that for the special conformal transformation

$$iK_\mu = \left[x^2\partial_\mu - 2x_\mu (x\cdot\partial) \right] = e^\tau \left[-n_\mu \frac{\partial}{\partial \tau} + (\delta_{\mu\nu} - n_\mu n_\nu)\frac{\partial}{\partial n_\nu} \right] \ . \tag{5.9}$$

In other words $\mathcal{T}(iP_\mu) = iK_\mu$. We have swept a factor of -1 under the rug here by including some extra factors of i. This factor deserves a longer discussion that I would prefer not to get into here.

Consider now a primary state $|\phi_I\rangle$, pushing our insertions off to $\tau \to \pm\infty$. From reflection positivity follow a number of claims, two of which will be important for us:

$$\begin{aligned} -\langle \phi_I | K_\mu P_\nu | \phi_J \rangle &\geq 0 \ , & (5.10) \\ \langle \phi_I | K_\mu K_\nu P_\lambda P_\rho | \phi_J \rangle &\geq 0 \ , & (5.11) \end{aligned}$$

are non-negative as matrices (i.e. all the eigenvalues are zero or positive). Applying the commutation relation of translations with special conformal transformations to (5.10), along with the constraint that K_μ annihilates $|\phi_J\rangle$, we find that

$$-2i\langle\phi_I|(\delta_{\mu\nu}D - M_{\mu\nu})|\phi_J\rangle \geq 0 \ . \tag{5.12}$$

(Notice we have replaced $\eta_{\mu\nu}$ with $\delta_{\mu\nu}$ because we are working with a Euclidean theory, not a Lorentzian one.) For a scalar operator, $M_{\mu\nu}$ will annihilate $|\phi_J\rangle$, and we find the constraint that $\Delta \geq 0$. Comparing with (5.6), you may be confused because the scalar is supposed to be bounded below by $\frac{d-2}{2}$ while we just found the constraint $\Delta > 0$. In fact, $\Delta = 0$ must be allowed, as it corresponds to the identity operator. What happens more precisely is that there is a gap in the spectrum and the next allowable dimension is that of a free field, $\frac{d-2}{2}$. To see this, one has to consider (5.11):

Problem 5.2. *By studying $\langle\phi|K^2P^2|\phi\rangle$ for scalar primary ϕ, demonstrate that the conformal dimension must satisfy the quadratic constraint $\Delta(2(\Delta+1)-d) \geq 0$.*

A little bit of group theory allows one to analyze the general case of (5.12), which we will not do here. However, we know how to represent $M_{\mu\nu}$ for spinors and vectors from chapter 2, from which you can deduce the corresponding bounds (5.6):

Problem 5.3. *Use the explicit representation of $M_{\mu\nu}$ from problem 2.2 for spinors and vectors to show that Δ is bounded below by $\frac{d-1}{2}$ and $d-1$ respectively.*

A natural question is if any further constraints on the spectrum can be found by considering more complicated correlation functions involving K_μ and P_μ. The answer appears to be no.

One way to argue that three point function coefficients are real in CFT is to consider $\langle\mathcal{R}(\mathcal{O})\mathcal{O}\rangle$ where $\mathcal{O} = \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)$. By taking a reflection plane that is very far form the insertions, we expect the dominant contribution of this six point function to be of the form $\langle\mathcal{R}(\mathcal{O})\mathcal{O}\rangle \sim \langle\mathcal{R}(\mathcal{O})\rangle\langle\mathcal{O}\rangle$. For this quantity to be positive, the three point functions need to be real.

5.2 The Bootstrap

Now let us define

$$F_{\Delta,I}(u,v) \equiv \frac{v^\eta G_{\Delta,I}(u,v) - u^\eta G_{\Delta,I}(v,u)}{u^\eta - v^\eta} \ . \tag{5.13}$$

and imagine that we have found a candidate spectrum for the theory. We have some set, possibly infinite, of dimensions for scalar operators, some set of dimensions for vector operators, and so on. Now we design a linear operator $\mathcal{O}$ such that

$$\mathcal{O}(F_{\Delta,I}(u,v)) \ \ \geq \ \ 0 \ , \tag{5.14}$$

for every operator in the spectrum but $\mathcal{O}(1) < 0$. Then, because we know $c^2_{\Delta,I} > 0$, we can rule this spectrum out as possible data for a CFT. In fact, by cleverly choosing $\mathcal{O}$, it is possible to rule out whole families of possible CFTs.

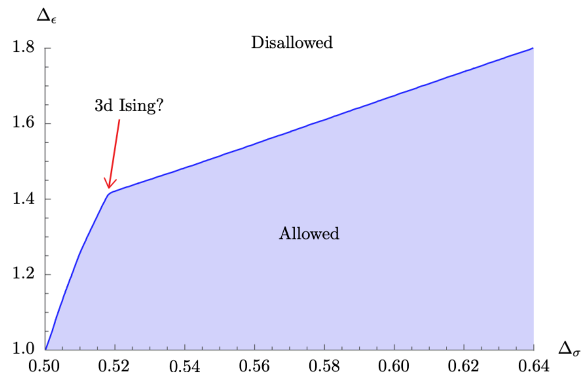

Figure 6: Upper bound on the dimension of Δ_ϵ of the lowest dimension scalar in the $\sigma \times \sigma$ OPE, where σ is a real scalar primary in a unitary 3d CFT with a $\mathbb{Z}_2$ symmetry. [[From Simmons-Duffin's TASI lectures]]

Let us consider a CFT with a scalar operator σ of dimension Δ_σ. The OPE of two such scalars will have the generic form

$$\sigma(x)\sigma(0) = \frac{1}{|x|^{2\Delta_\sigma}} \left(1 + C_{\sigma\sigma\epsilon}|x|^{\Delta_\epsilon}\epsilon(0) + \ldots\right) \ , \tag{5.15}$$

where $\epsilon(x)$ is the leading operator to appear in the OPE after the identity. In a free CFT, we anticipate that $\epsilon(x)$ will be the normal ordered product of $\sigma(x)$ with itself. In this case, $\Delta_\sigma = \frac{d-2}{2}$ and $\Delta_\epsilon = d-2$. But more generally, it is not obvious what Δ_ϵ should be. By applying the bootstrap procedure, we can determine an upper bound for Δ_ϵ as a function of Δ_σ. See fig. 6. Reassuringly, the point $(\frac{1}{2}, 1)$ lies on the bounding curve in $d = 3$. Moreover, there is a kink in the bounding curve close to the location of the 3d Ising model.

In fact, by imposing crossing symmetry on more than one four point function, one can often further pin down the data of interesting CFTs. For example, the most accurate data for the 3d Ising model at the critical point currently come from bootstrap bounds:[5]

$$(\Delta_\sigma, \Delta_\epsilon) = (0.518151(6), 1.41264(6)) \ . \tag{5.16}$$

One might ask if these results have some experimental relevance. Recall the 3d Ising model has Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} s_i \cdot s_j \ , \tag{5.17}$$

where $s_i = \pm 1$ and the sum is over nearest neighbors on a 3d cubic lattice. When we talk about the CFT associated with the Ising model, we mean the CFT that describes the behavior of the lattice model at the critical temperature, where it is on the border between an

[5] D. Simmons-Duffin, A Semidefinite Program Solver for the Conformal Bootstrap, JHEP 06 (2015) 174, arxiv.org/abs/1502.02033.

ordered low temperature and a disordered high temperature system. While I am not aware of a measurement of the critical exponents for Ising, there is one for a small generalization of Ising. We can promote $\vec{s}_i$ to n-component vectors of unit length. In the case $n = 2$, the associated CFT is believed to also describe helium along the line in the temperature-pressure plane that separates the superfluid from the ordinary fluid.

The analog of Δ_ϵ above for the $n = 2$ model was calculated from a bootstrap approach to be 1.51136(22).[6] However, the experiment (which needs to be done in space to avoid the effects of gravity) measured 1.50946(22). This discrepancy is eight standard deviations, which as far as I am aware, remains unexplained. My reading is that while it seems likely that the theoretical result is correct is far as it goes, the physics measured by the experiment may not be precisely that of a CFT. On that slightly unsatisfactory note, I conclude this discussion of CFT, leaving it to one of you to improve the story in the next retelling.

6 Spinors and Clifford Algebras

To learn SUSY, we must first master the formalism necessary to describe spinors and fermions. We can attribute much of this formalism to Dirac, who had the insight that the Dirac equation should be a kind of square root of the Klein-Gordon equation:

$$(\gamma^\mu \partial_\mu - m)\psi = 0 \ . \tag{6.1}$$

Acting on the left with $(\gamma^\mu \partial_\mu + m)$, one finds

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - m^2)\psi = 0 \ . \tag{6.2}$$

This second equation is equivalent to the Klein-Gordon equation,

$$(\partial^2 - m^2)\psi = 0 \ , \tag{6.3}$$

provided

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \ . \tag{6.4}$$

From this innocuous looking anti-commutation relation follows an intricate structure that depends sensitively on the space-time dimension – the Clifford algebra and its representations.[7] The matrix γ^μ has a vector index that we can lower and raise using the metric $\eta_{\mu\nu}$ and its inverse.

[6] Chester et al., Carving out OPE space and precise O(2) critical exponents, JHEP 06 (2020) 142, arxiv.org/abs/1912.03324.

[7] Clifford became a student at KCL in 1860, at the tender age of 15. He later was elected a fellow at Trinity College, Cambridge in 1868. After surviving a shipwreck along the Sicilian coast during a voyage to observe the solar eclipse of December 1870, he started work as a professor mathematics and mechanics at UCL. He suffered a pair of nervous breakdowns, perhaps due to overwork, and succumbed to tuburcolosis in 1879, at the age of 33. In the *Ethics of Belief*, he wrote "It is wrong always, everywhere, and for anyone, to believe anything upon insufficient evidence."

6.1 Clifford Algebras

Introducing fermions ψ requires also introducing a set of γ-matrices. The choice of ψ and the associated γ^μ furnish a representation of the Clifford algebra. Generically, we take the representation to be over the complex numbers. For now, we suppress the spinor indices $\alpha, \beta, \ldots$ on ψ and γ^μ. Thus when we write

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \tag{6.5}$$

there is an implicit identity matrix in spinor space $\mathrm{id}^\alpha{}_\beta$ on the right hand side.

Let us begin with the even dimensional case $d = 2k + 2$. We group the gamma matrices into $k + 1$ pairs of anti-commuting raising and lowering operators

$$\gamma^{0\pm} = \frac{1}{2}(\pm\gamma^0 + \gamma^1) \ , \tag{6.6}$$

$$\gamma^{a\pm} = \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1}) \ , \quad a = 1, \ldots, k \ . \tag{6.7}$$

Problem 6.1. *Show that these linear combinations satisfy the relations*

$$\begin{aligned} \{\gamma^{a+}, \gamma^{b-}\} &= \delta^{ab} \ , \\ \{\gamma^{a+}, \gamma^{b+}\} &= \{\gamma^{a-}, \gamma^{b-}\} = 0 \ . \end{aligned} \tag{6.8}$$

In particular, note that $(\gamma^{a+})^2 = 0 = (\gamma^{a-})^2$. By repeatedly acting with the $k + 1$ γ^{a-} on a spinor, we can eventually reach a lowest weight state ζ such that

$$\gamma^{a-}\zeta = 0 \text{ for all } a \ . \tag{6.9}$$

Starting from ζ and acting with the raising operators γ^{a+}, at most once each, we can obtain all of the $2^{k+1} = 2^{d/2}$ states in the representation. The states can be labeled $\mathbf{s} = (s_0, s_1, \ldots, s_k)$, where each of the $s_a = \pm\frac{1}{2}$:

$$\zeta^{(\mathbf{s})} \equiv (\gamma^{k+})^{s_k+1/2} \cdots (\gamma^{0+})^{s_0+1/2}\zeta \ . \tag{6.10}$$

The lowest weight state ζ corresponds to all $s_a = -\frac{1}{2}$.

Taking the $\zeta^{(\mathbf{s})}$ as a basis, we derive the matrix elements of γ^μ from the definitions and the anti-commutation relations. Starting in $d = 2$, we find

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{6.11}$$

The 2×2 matrices are chosen to obey the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ but additionally such that the states in the representation take the simple form where

$$\zeta^{(-1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \quad \zeta^{(+1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ . \tag{6.12}$$

Note that the 2×2 matrices that appear are related to two of the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{6.13}$$

In particular, $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$. This relation is not surprising since these matrices give a three dimensional Euclidean representation of the Clifford algebra

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \ . \tag{6.14}$$

Increasing d by two doubles the size of the γ-matrices. Given a representation Γ^μ in $2k$ dimensions, we can construct a representation γ^μ in $2k+2$ dimensions using the prescription,

$$\begin{aligned} \gamma^\mu &= \Gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\Gamma^\mu \otimes \sigma^3 \ , \quad \mu = 0, \ldots, d-3 \ , \\ \gamma^{d-2} &= \mathrm{id} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{id} \otimes \sigma^1 \ , \quad \gamma^{d-1} = \mathrm{id} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathrm{id} \otimes \sigma^2 \ . \end{aligned} \tag{6.15}$$

The 2×2 matrices that we add act on the index s_k, which newly appears in going from $2k$ to $2k+2$ dimensions. (In what follows, we will set $d = 2k+2$.)

The basis choice is not unique. There are many ways of constructing this $2^{d/2}$ dimensional representation of a Clifford algebra. We claim, however, that they are all equivalent up to an appropriate unitary transformation, $\gamma^\mu \to U\gamma^\mu U^{-1}$. In four dimensions, for instance, the construction above leads to the γ-matrices

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \ , \quad \gamma^1 = \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \ , \\ \gamma^2 &= \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \ , \quad \gamma^3 = \begin{pmatrix} 0 & -i\,\mathrm{id} \\ i\,\mathrm{id} & 0 \end{pmatrix} \ . \end{aligned} \tag{6.16}$$

A different, more popular choice of basis in four dimensions, often found in field theory text books, is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \ , \tag{6.17}$$

where $\sigma^\mu = (\mathrm{id}, \sigma^i)$ and $\bar\sigma^\mu = (-\,\mathrm{id}, \sigma^i)$.

Problem 6.2. *Demonstrate a unitary transformation U that relates the representations (6.16) and (6.17). See if you can choose a basis such that the 4d gamma matrices are purely real, a so-called "really real" representation.*

The representation $\zeta^{(\mathbf{s})}$ of the Clifford algebra is also a representation – the so-called Dirac spinor representation – of the Lorentz group. In an earlier exercise, we demonstrated that the Lorentz generators can be written as

$$M^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] \ . \tag{6.18}$$

The generators $M^{2a,2a+1}$ commute and can be simultaneously diagonalized to give the weights of the representation. (Each of the $M^{2a,2a+1}$ operators functions like a J_z angular momentum operator in quantum mechanics.) In terms of our raising and lowering operators, we have

$$S_a \equiv i^{\delta_{a,0}} M^{2a,2a+1} = \gamma^{a+}\gamma^{a-} - \frac{1}{2} \ . \tag{6.19}$$

In this way $\zeta^{(\mathbf{s})}$ is a simultaneous eigenstate of the S_a with eigenvalues s_a. The spinors $\zeta^{(\mathbf{s})}$ thus form the 2^{k+1} dimensional spinor representation of the Lorentz algebra $so(2k+1,1)$.

While the representation $\zeta^{(\mathbf{s})}$ is irreducible as a representation of the Clifford algebra, it is in general not irreducible as a representation of the Lorentz group. Because the Lorentz generator $M^{\mu\nu}$ is quadratic in the γ-matrices, it can never flip an odd number of spins when acting on $\zeta^{(\mathbf{s})}$. Thus the states with even and odd numbers of $-1/2$ spins will not mix under the action of $M^{\mu\nu}$, and the Dirac representation falls apart into two smaller representations in an even number of space-time dimensions.

In fact, we can construct a matrix, the analog of "gamma five" in four dimensions, to help perform this decomposition. The matrix detects the "chirality" of the state, i.e. the parity of the number of down spins, and commutes with $M^{\mu\nu}$. This matrix is a product of all the other gamma matrices:

$$\gamma \equiv i^{-k}\gamma^0\gamma^1\cdots\gamma^{d-1} \ . \tag{6.20}$$

Problem 6.3. *Show that in terms of the S_a operators, we can write*

$$\gamma = 2^{k+1}S_0S_1\cdots S_k \ . \tag{6.21}$$

As a result, it is clear that γ is diagonal in our $\zeta^{(\mathbf{s})}$ basis, taking the eigenvalue $+1$ when there are an even number of $-\frac{1}{2}$ spins and -1 for an odd number of $-\frac{1}{2}$ spins. The states with eigenvalue $+1$ form a Weyl representation of the Lorentz algebra, while the states with eigenvalue -1 form a second, inequivalent Weyl representation. The eigenvalue of γ is often called the *chirality* of the representation.

The matrix γ performs a second key function by allowing us to construct representations of the Clifford algebra in odd dimensions. We simply use γ as the d^{th} gamma matrix, as it satisfies the requisite anti-commutation relations with the other gamma matrices to give $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$. One interesting fact is that we could just as well take $-\gamma$ as the d^{th} gamma matrix, which gives a second inequivalent representation of the Clifford algebra in odd dimensions. (Note we cannot change only the sign of γ and leave the other γ_μ untouched under conjugation $\gamma \to U\gamma U^{-1}$.)

6.2 Majorana spinors

A subtle feature of representations of the Clifford algebra is the possibility of imposing a Majorana condition. A Majorana representation is a "real" (as opposed to complex representation), and thus has half the dimensionality of Dirac representation. The subtlety comes from the fact that we need to consider a more general reality condition than $\zeta^* = \zeta$. We need to allow for the fact that ζ^* is some linear operator B acting on ζ:

$$\zeta^* = B\zeta \ . \tag{6.22}$$

Indeed, under a unitary transformation $\zeta \to U\zeta$, and hence $B \to U^*BU^{-1}$. Thus even if we can find a particular basis where B is the identity, after acting by U, we will in general find some nontrivial B. Taking the conjugate of the definition (6.22) yields an additional consistency condition, $\zeta = B^*\zeta^* = B^*B\zeta$, implying $B^*B = \text{id}$.

As we saw earlier in an exercise, an infinitesimal Lorentz transformation is generated by something quadratic in the gamma matrices, $M_{\mu\nu} = -\frac{i}{2}\gamma_{\mu\nu}$. For this reality condition to make sense, we need it to be compatible with the Lorentz transformations:

$$\begin{aligned}
(\delta\zeta)^* &= (B\delta\zeta) \ , && (6.23)\\
-\frac{i}{2}\omega^{\mu\nu}(M_{\mu\nu}\zeta)^* &= \frac{i}{2}\omega^{\mu\nu}BM_{\mu\nu}\zeta \ , && (6.24)\\
(M_{\mu\nu}\zeta)^* &= -BM_{\mu\nu}\zeta \ . && (6.25)
\end{aligned}$$

On the right hand side, we can expand $BM_{\mu\nu}\zeta = BM_{\mu\nu}B^{-1}B\zeta$ while on the left $(M_{\mu\nu}\zeta)^* = M^*_{\mu\nu}\zeta^*$. Thus, the matrix B had better act as

$$BM_{\mu\nu}B^{-1} = -M^*_{\mu\nu} \tag{6.26}$$

on the Lorentz generators. On the individual gamma matrices, we are then allowed a sign ambiguity, $B\gamma_\mu B^{-1} = \pm\gamma^*_\mu$.

We will leave the general story as a problem and focus on three low dimensional cases of interest, $d = 2$, 3 and 4. In $d = 2$, the gamma matrices $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$ (6.11) are manifestly real. As a result, we can take $B = \text{id}$. The "gamma five" matrix $\gamma = \gamma^0\gamma^1 = \sigma^3$ is real and diagonal. While the original Dirac representation is two complex (or four real dimensional), we can reduce this representation into different types of smaller representations. There are Weyl representations with one complex (or two real) components. There are Majorana representations with two real components. Finally, because B is the identity in a basis where γ is diagonal, we can have Majorana-Weyl spinors with one real component.

In $d = 3$, the gamma matrices $\gamma^0 = i\sigma^2$, $\gamma^1 = \sigma^1$ and $\gamma^3 = \sigma^3$ are again all manifestly real, allowing for a Majorana representation with $B = \text{id}$. In odd dimensions, there are no Weyl representations.

In $d = 4$, for the basis (6.17), we can write $B = \gamma^2\gamma$. We know that B has the correct properties to impose a Majorana condition because

$$B\gamma^\mu B^{-1} = (\gamma^\mu)^* \ . \tag{6.27}$$

In the basis (6.17), "gamma five" is diagonal

$$\gamma = \begin{pmatrix} \text{id} & 0 \\ 0 & -\text{id} \end{pmatrix} \tag{6.28}$$

while B is not. Moreover, γ and B do not commute, implying that they cannot be simultaneously diagonalized. In other words, we cannot impose both a Majorana and a Weyl condition at the same time. We can have Majorana spinors or we can have Weyl spinors, but not both at the same time in four dimensions.

There is an elegant general story which we leave as a problem. Curiously, the representation theoretic structure has a periodicity modulo eight as a function of dimension. This periodicity turns out to be a rather deep feature of the Clifford algebra, with relations to other areas of mathematics, such as Bott periodicity.

Problem 6.4. *In $d = 2k+2$ dimensions, the matrices $\gamma^{\mu *}$ and $-\gamma^{\mu *}$ satisfy the same Clifford algebra as γ^μ and so must be related to γ^μ by a unitary similarity transformation. We would like to determine explicitly the form of this similarity transformation for the basis (6.15) and study its properties. Consider two candidate matrices*

$$B_1 = \gamma^3 \gamma^5 \cdots \gamma^{d-1} \ , \quad B_2 = \gamma B_1 \ . \tag{6.29}$$

a) *Show that*

$$B_1 \gamma^\mu B_1^{-1} = (-1)^k \gamma^{\mu *} \ , \quad B_2 \gamma^\mu B_2^{-1} = (-1)^{k+1} \gamma^{\mu *} \ , \tag{6.30}$$

and also that

$$B_i M_{\mu\nu} B_i^{-1} = -M^*_{\mu\nu} \tag{6.31}$$

for $i = 1$ and 2. As a result, the spinors ζ and $B_i^{-1}\zeta^*$ must transform in the same way under the Lorentz group.

b) *Show that*

$$B_1 \gamma B_1^{-1} = B_2 \gamma B_2^{-1} = (-1)^k \gamma^* \ . \tag{6.32}$$

As a result, both B matrices will change the eigenvalue of γ when k is odd and not when it is even. When ($d = 2 \mod 4$) each Weyl representation is its own conjugate, while when ($d = 0 \mod 4$), each Weyl representation is conjugate to the other.

c) *That ζ and $B_i^{-1}\zeta^*$ transform the same way under the Lorentz group allow us to impose the Majorana reality condition $\zeta^* = B\zeta$, provided $B^*B = \mathrm{id}$ as discussed above. Show that a Majorana condition is possible using B_1 only if $k = 0$ or 3 (mod 4) and using B_2 only if $k = 0$ or 1 (mod 4).*

d) *Extending to odd dimensions, show that a Majorana condition is possible only when $k = 0$ or 3 (mod 4).*

e) *Make sure that you understand the contents of Figure 7.*

f) *How do the details of the previous arguments change if we use a metric with mostly minus signature?*

Having completed the exercise above, one may ask if there are any other possible B's to consider which may satisfy the consistency conditions. If so, then $BM_{\mu\nu}B^{-1} = B'M_{\mu\nu}B'^{-1}$, which implies there is a linear operator $B^{-1}B'$ which commutes with all of the Lorentz generators. By Schur's Lemma, anything that commutes with all elements of an irreducible representation must be a multiple of the identity. In odd dimensions, where the Dirac representation is irreducible, there can be nothing else. In even dimensions, the Dirac representation splits into two Weyl representations, and $B^{-1}B'$ is not necessarily the identity. It is instead a linear combination of the identity with γ, the two operators which act like multiples of the identity when restricted to the Weyl representations. Indeed, in the exercise above, we had B_1 and B_2 in the even dimensional case, which differed by a factor of γ.

d	Majorana	Weyl	Majorana-Weyl	min. rep.
2	yes	self	yes	1
3	yes	-	-	2
4	yes	complex	-	4
5	-	-	-	8
6	-	self	-	8
7	-	-	-	16
8	yes	complex	-	16
9	yes	-	-	16
10=2+8	yes	self	yes	16
11=3+8	yes	-	-	32
12=4+8	yes	complex	-	64

Figure 7: Properties of spinor representations in various dimension. A dash indicates the condition cannot be imposed. "self" means the Weyl representation is self-conjugate under complex conjugation while "complex" indicates complex conjugation gives the other Weyl representation. The dimension of the smallest representation is given in the last column. The conditions – Majorana, Weyl, Majorana-Weyl – that can be imposed on the representations repeat as a function of the dimension modulo 8.

6.3 Spinor Inner Product

In addition to the annoying complexity that the reality condition for spinors should be generalized to $\psi^* = B\psi$, a second irritating feature about spinors is that $\psi^\dagger\psi$ is not a Lorentz scalar, as we now verify. Under an infinitesimal Lorentz transformation, we showed earlier that

$$\delta\psi_\alpha = \frac{1}{4}\omega^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\beta\psi_\beta \ . \tag{6.33}$$

It follows that the Hermitian conjugate spinor transforms as

$$\delta\psi^\dagger = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_\nu^\dagger\gamma_\mu^\dagger \tag{6.34}$$

From the iterative construction (6.15) of the gamma matrices, it is clear that

$$\gamma_0^\dagger = -\gamma_0 \ , \quad \gamma_i^\dagger = \gamma_i \ . \tag{6.35}$$

Problem 6.5. *Verify (6.35) for both odd and even dimensions.*

In fact, under a unitary similarity transformation $\gamma^\mu \to U\gamma^\mu U^{-1}$, this property is preserved, and we expect it to hold in general. As a result, we can write the Hermitian conjugation relation as

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \ , \tag{6.36}$$

and the transformation rule (6.34) can be written

$$\delta\psi^\dagger = -\frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_\nu\gamma_\mu\gamma_0 = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\mu\nu}\gamma_0 \ . \tag{6.37}$$

(While we have suppressed the spinor index, the structure $(\gamma_0\gamma_{\mu\nu}\gamma_0)_\alpha{}^\beta$ means it is most natural to write the Hermitian conjugate spinor with an upper index, $(\psi^\dagger)^\alpha$.) The sign and the additional factors of γ_0 will not cancel, and $\zeta^\dagger\psi$ has a nontrivial transformation under the Lorentz group. It is in other words not a scalar quantity. In the γ_0 sickness lies the cure, and we can define a modified conjugate spinor

$$\bar{\psi} \equiv \psi^\dagger\gamma_0 \ . \tag{6.38}$$

In this case, the infinitesimal Lorentz transformation becomes

$$\delta\bar{\psi} = -\frac{1}{4}\bar{\psi}\omega^{\mu\nu}\gamma_{\mu\nu} \tag{6.39}$$

and hence

$$\begin{aligned}\delta(\bar{\zeta}\psi) &= \delta\bar{\zeta}\psi + \bar{\zeta}\delta\psi \\ &= -\frac{1}{4}\bar{\zeta}\omega^{\mu\nu}\gamma_{\mu\nu}\psi + \frac{1}{4}\bar{\zeta}\omega^{\mu\nu}\gamma_{\mu\nu}\psi \\ &= 0 \ . \end{aligned} \tag{6.40}$$

Thus $\bar{\zeta}\psi$ is a Lorentz scalar.

From two spinors, we can construct other Lorentz covariant objects as well, such as vectors and anti-symmetric tensors:

$$\bar{\zeta}\gamma_\mu\psi \ , \quad \bar{\zeta}\gamma_{\mu\nu}\psi \ , \ldots \tag{6.41}$$

Problem 6.6. *Show that $v_\mu = \bar{\zeta}\gamma_\mu\psi$ is a vector, i.e. show that $\delta v_\mu = -\omega_\mu{}^\nu v_\nu$ under the transformation (6.33).*

We should next consider how this modified definition of spinor conjugation, $\bar{\psi} = \psi^\dagger\gamma_0$, interfaces with the Majorana condition $\psi^* = B\psi$:

$$\bar{\psi} = \psi^\dagger\gamma_0 = \psi^{T*}\gamma_0 = \psi^T B^T\gamma_0 \tag{6.42}$$

The combination $C \equiv B^T\gamma_0$ is often referred to as the charge conjugation matrix, and for Majorana spinors (in fact Dirac spinors as well), we can write the Lorentz covariant objects as $\zeta^T C\psi$, $\zeta^T C\gamma_\mu\psi$, $\zeta^T C\gamma_{\mu\nu}\psi$, etc. Before, we had the relation $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$. The equivalent condition that guarantees compatibility with the Lorentz group for Majorana spinors is

$$-\gamma_\mu^T = C\gamma_\mu C^{-1} \ . \tag{6.43}$$

Restoring indices, we can think of $C^{\alpha\beta}$ as a metric on spinor indices, such that $\bar{\psi}\zeta = \psi^T C\zeta = \psi_\alpha C^{\alpha\beta}\zeta_\beta$. The inverse metric is then $C^{-1}_{\alpha\beta}$ with lower indices, and we can raise indices via $\psi^\alpha = \psi_\beta C^{\beta\alpha}$.

Before closing this section, we should discuss some elementary spinor manipulations that will be useful later on in demonstrating supersymmetry. First, spinor fields are Grassman valued, which means they anticommute:

$$\psi_\alpha\zeta_\beta = -\zeta_\beta\psi_\alpha \ . \tag{6.44}$$

There is a sign ambiguity then in how to define complex conjugation. We make the choice

$$(\psi_\alpha \zeta_\beta)^* = \zeta_\beta^* \psi_\alpha^* \ , \tag{6.45}$$

analogous to the way Hermitian conjugation acts on matrices.

We will often need to perform various manipulations with Majorana spinors, the simplest of which is perhaps

$$(\bar\zeta \psi)^* = (\zeta_\alpha C^{\alpha\beta} \psi_\beta)^* = \psi_\beta C^{\alpha\beta} \zeta_\alpha = -\zeta_\alpha C^{\alpha\beta} \psi_\beta = -\bar\zeta \psi \tag{6.46}$$

where we have made use of the fact that we can work in a basis where ζ, ψ, and C are real, yielding the curious result that $\bar\zeta\psi$ is pure imaginary.[8]

Problem 6.7. *The Majorana Flip Relations. Show that in $d = 2$, 3 and 4,*

$$\lambda^T C \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_p} \chi = (-1)^p \chi^T C \gamma^{\mu_p} \cdots \gamma^{\mu_2} \gamma^{\mu_1} \lambda \ . \tag{6.47}$$

In these dimensions, which allow for Majorana spinors, if we impose that λ and χ are Majorana, then we can replace $\lambda^T C$ with $\bar\lambda$ and similarly for χ. How are these rules modified in $d = 2$ and 4 to incorporate a γ matrix?

6.4 Fierz re-arrangement

Consider the following list of gamma matrices and antisymmetric products of gamma matrices:

$$\gamma_\Gamma \in \{\mathrm{id}, \gamma, \gamma_\mu, \gamma_\mu \gamma, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma, \ldots\} \tag{6.48}$$

where $\Gamma = \mu\nu\lambda\cdots$ is a generalized index and $\gamma_{\mu\nu\lambda\cdots}$ is an antisymmetric product over the given indices with weight one, e.g.

$$\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \tag{6.49}$$

in the two index case. We would like to think about these matrices as vectors in a matrix valued inner product space, with inner product $\langle M_1, M_2 \rangle = \mathrm{tr}(M_1^\dagger M_2)$. Because of the relation $\gamma \sim \gamma_0 \gamma_1 \cdots \gamma_{d-1}$, these matrices are not all linearly independent. In fact they stop being linearly independent once the number of indices is larger than $d/2$.

Problem 6.8. *Convince yourself that the counting works out, that there are precisely enough linearly independent matrices in the list (6.48) to span a vector space that has dimension $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$, i.e. the size of a gamma matrix.*

[8] One way to change this property is to modify the definition of a barred spinor to include a factor of i (see for example Freedman and van Proeyen). Another is to work in a mostly minus convention for the metric (see for example Peskin and Schroeder).

Provided we restrict the number of indices, the list of vectors is actually orthogonal with respect to our inner product. A key observation required is that a single gamma matrix is traceless:

$$
\begin{aligned}
2\eta_{\mu\nu}\,\mathrm{tr}(\gamma_\lambda) &= \mathrm{tr}(\{\gamma_\mu,\gamma_\nu\}\gamma_\lambda) \\
&= \mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\nu\gamma_\mu\gamma_\lambda) \\
&= \mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\mu\gamma_\lambda\gamma_\nu) \\
&= \mathrm{tr}(\gamma_\mu\{\gamma_\nu,\gamma_\lambda\}) \\
&= 2\eta_{\nu\lambda}\,\mathrm{tr}(\gamma_\mu)\ .
\end{aligned}
\tag{6.50}
$$

Choosing $\mu = \nu \neq \lambda$ then implies $\mathrm{tr}(\gamma_\lambda) = 0$.

Problem 6.9. *Generalize this result to show that* $\mathrm{tr}(\gamma_{\mu_1\cdots\mu_n}) = 0$*, provided* $0 < n < d$*. From this tracelessness, argue that the list of vectors* $\gamma_\Gamma \in \{\mathrm{id}, \gamma_\mu, \gamma_\mu\gamma, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma, \ldots\}$ *is orthogonal, provided we restrict the indices such that they are linearly independent.*

A completeness relation for our basis set (6.48) is then

$$
\delta_\alpha^\beta\delta_\gamma^\delta = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'}(\gamma_\Gamma)_\gamma{}^\beta(\gamma_{\Gamma'})_\alpha{}^\delta
\tag{6.51}
$$

for some constants to be determined $c_{\Gamma\Gamma'}$ where Γ and Γ' are generalized indices that range over the list of independent elements in the list (6.48). To determine the $c_{\Gamma\Gamma'}$, we multiply both sides by $(\gamma_{\Gamma''})_\beta{}^\gamma$, and sum over β and γ:

$$
(\gamma_{\Gamma''})_\alpha{}^\delta = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'}\,\mathrm{tr}(\gamma_\Gamma\gamma_{\Gamma''})(\gamma_{\Gamma'})_\alpha{}^\delta\ ,
\tag{6.52}
$$

By orthogonality, $\mathrm{tr}(\gamma_\Gamma\gamma_{\Gamma''}) = 0$ unless $\Gamma = \Gamma''$ and the double sum reduces to a single sum

$$
(\gamma_{\Gamma''})_\alpha{}^\delta = \sum_{\Gamma'} c_{\Gamma''\Gamma'}\,\mathrm{tr}(\gamma_{\Gamma''}^2)(\gamma_{\Gamma'})_\alpha{}^\delta\ .
\tag{6.53}
$$

For this equality to hold, we must have that $c_{\Gamma''\Gamma'} = 0$ unless $\Gamma' = \Gamma''$. In the case of equality, we have further that

$$
c_{\Gamma\Gamma} = \frac{1}{\mathrm{tr}(\gamma_\Gamma^2)} = \pm_\Gamma \frac{1}{2^{\lfloor \frac{d}{2} \rfloor}}\ ,
\tag{6.54}
$$

where $\pm_\Gamma$ arises because $\gamma_\Gamma^2 = \pm 1$ and the power of 2 occurs as a dimension of the representation of the Clifford algebra $\mathrm{tr}(\mathrm{id}) = 2^{\lfloor \frac{d}{2} \rfloor}$.

We have gone through this abstract argument because we will frequently be in a situation in the future where we want to be able to shuffle spinor bilinears around, a manipulation of the form $(\bar\lambda\psi)(\bar\zeta\chi) \to (\bar\lambda\chi)(\bar\zeta\psi)$. Consider a slightly more general situation where we have only three spinors, two of which are contracted together. We will use our decomposition of the identity $\delta_\alpha^\beta\delta_\gamma^\delta$ in terms of the generalized gamma matrix (6.48):

$$
\begin{aligned}
(\bar\lambda\psi)\chi_\alpha &= \bar\lambda^\gamma\psi_\delta\chi_\beta\delta_\alpha^\beta\delta_\gamma^\delta \qquad (6.55)\\
&= -\sum_\Gamma c_{\Gamma\Gamma}\bar\lambda^\gamma(\gamma_\Gamma)_\gamma{}^\beta\chi_\beta(\gamma_\Gamma)_\alpha{}^\delta\psi_\delta \\
&= -\frac{1}{2^{\lfloor \frac{d}{2} \rfloor}}\sum_\Gamma \pm_\Gamma(\bar\lambda\gamma_\Gamma\chi)(\gamma_\Gamma\psi)_\alpha\ . \qquad (6.56)
\end{aligned}
$$

This swapping of ψ and χ in the contraction is called a Fierz re-arrangement identity.

Problem 6.10. *Show that in three dimensions, the Fierz re-arrangement identity is*

$$(\bar{\lambda}\psi)\chi_\alpha = -\frac{1}{2}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{2}(\bar{\lambda}\gamma_\mu\chi)(\gamma^\mu\psi)_\alpha \ . \tag{6.57}$$

Furthermore, show that in the special case $\lambda = \chi$ and the spinors are Majorana, this identity reduces to

$$(\bar{\lambda}\psi)\lambda_\alpha = -\frac{1}{2}(\bar{\lambda}\lambda)\psi_\alpha \ . \tag{6.58}$$

Problem 6.11. *There is yet another type of spinor representation, symplectic Majorana fermions. They can be useful for writing down actions with extended supersymmetry. These spinors ψ^i_α carry an extra index $i = 1$ or 2, and satisfy the following reality property:*

$$\bar{\psi}^i = \epsilon^{ij}\psi_j^T\tilde{C} \ . \tag{6.59}$$

The tensor $\epsilon^{12} = -\epsilon^{21} = 1$ and is zero otherwise. Construct $\tilde{C}$ using the B_1 and B_2 matrices from problem 6.4. In what dimensions are symplectic Majorana fermions allowed? In what dimensions can fermions be simultaneously symplectic Majorana and Weyl. (Note the language may be slightly confusing. Symplectic Majorana fermions are not also Majorana.)

6.5 Two Component Notation

In even dimensions, it is sometimes useful to work directly with the irreducible Weyl representations rather than the reducible Dirac representation. This practice is especially common in four dimensions and has its own formalism, sometimes called Weyl notation or two component notation. In Weyl notation, where γ is diagonal, one breaks the four component Dirac spinor into two, two-component pieces

$$\psi = \begin{pmatrix} \lambda_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix} \tag{6.60}$$

where $a, \dot{a} = 1, 2$ and uses the Clifford algebra in the basis (6.17). In this basis, λ has chirality $+1$ and $\bar{\chi}$ is chirality -1.

In much of the previous discussion, we almost always suppressed the spinor indices. However, in this two component language, writing down the indices often helps indicate what kinds of contractions are possible. Thus it has a very distinct look and feel to it. We will not use this notation in the rest of the module, but we present it here for completeness.

Let us reconsider some of our inner products in this two component notation. For the usual inner product,

$$\bar{\psi}\psi = \psi^\dagger\gamma_0\psi = \lambda^\dagger\bar{\chi} - \bar{\chi}^\dagger\lambda = \bar{\lambda}_{\dot{a}}\bar{\chi}^{\dot{a}} - \chi^a\lambda_a \tag{6.61}$$

where we define $\lambda^\dagger = \bar{\lambda}_{\dot{a}}$ and $\bar{\chi}^\dagger = \chi^a$. Note that $C = -i\gamma_0\gamma_2\gamma = -i\,\mathrm{diag}(\sigma^2, \sigma^2)$. For the inner product associated with Majorana spinors

$$\psi^T C\psi = \lambda^T(-i\sigma^2)\lambda + \bar{\chi}^T(-i\sigma^2)\bar{\chi} = \lambda_a\epsilon^{ab}\lambda_b + \bar{\chi}^{\dot{a}}\epsilon_{\dot{a}\dot{b}}\bar{\chi}^{\dot{b}} \ , \tag{6.62}$$

which allows us to think about the completely antisymmetric tensor ϵ_{ab} as a metric on these two component spinors, allowing us to raise and lower indices. We are treating

$$\epsilon_{ab} = \epsilon^{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{6.63}$$

There is actually a small problem at this point, because contracting ϵ^{ab} with two ϵ_{bc} with lower indices will give $-\epsilon_{ab}$. Let us therefore use $c^{ab} = \epsilon^{ab}$ and $c^{\dot{a}\dot{b}} = \epsilon^{\dot{a}\dot{b}}$, while $c_{ab} = -\epsilon_{ab}$ and $c_{\dot{a}\dot{b}} = -\epsilon_{\dot{a}\dot{b}}$. We then write the Majorana inner product in the form

$$\psi^T C\psi = \lambda_a c^{ab}\lambda_b - \bar{\chi}^{\dot{a}} c_{\dot{a}\dot{b}}\bar{\chi}^{\dot{b}} = \lambda^a\lambda_b - \bar{\chi}_{\dot{a}}\bar{\chi}^{\dot{a}} . \tag{6.64}$$

Let's do one more, an inner product involving a gamma matrix:

$$\bar{\psi}\gamma_\mu\psi = \psi^\dagger\gamma_0\gamma_\mu\psi = \bar{\lambda}_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\lambda_b - \chi_a(\sigma^\mu)^{a\dot{b}}\bar{\chi}_{\dot{b}} . \tag{6.65}$$

Finally, we write down the Majorana condition in two component notation. From the condition

$$\psi^* = B\psi \tag{6.66}$$

where $B = -i\gamma_2\gamma$, one finds that

$$\lambda^* = i\sigma^2\bar{\chi} , \quad \bar{\chi}^* = -i\sigma^2\lambda . \tag{6.67}$$

Adding back the indices, we see that $\lambda^*_{\dot{a}} = c_{\dot{a}\dot{b}}\bar{\chi}^{\dot{b}}$ and $\bar{\chi}^{*a} = c^{ab}\lambda_b$ or equivalently $\lambda^*_{\dot{a}} = \bar{\chi}_{\dot{a}}$ and $\bar{\chi}^{*a} = \lambda^a$.

As this two-component formalism is not central to our approach, we will leave the subject here after this brief and incomplete introduction.

7 Elementary Consequences of Supersymmetry

A generic supersymmetry algebra can be written as follows:

$$\begin{aligned} \{Q_\alpha, \bar{Q}^\beta\} &= 2(\gamma^\mu)_\alpha{}^\beta P_\mu , \\ [Q_\alpha, P_\mu] &= 0 , \\ [M_{\mu\nu}, Q_\alpha] &= \frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta , \end{aligned} \tag{7.1}$$

along with the Poincaré generators (2.8).[9] The most important relation is the first line, that the supercharges square to space-time translations. The second line means that the Q_α's are invariant under translations, while the third implies that the Q_α transform as space-time spinors, as they should given their index. This basic algebra typically comes with a Majorana

[9]There is some arbitrariness in the normalization of the Q's. We have chosen the two on the right hand side of the first line in order to write some supersymmetry variations later on in a simpler way, with fewer factors of two.

condition that $\bar{Q} = Q^T C$, and so will only be allowed in the dimensions that admit Majorana spinors.

Part of what makes supersymmetry so interesting is the variety of different algebras that can occur along with the intricate rules that determine when they can and cannot be constructed. Beyond the simple algebra above, one can construct so called N-extended algebras with more super charges where Q^I_α carries an additional index I that runs from one to N. There are also centrally extended algebras with additional "central elements" on the right hand side of the first relation. In certain curved manifolds with a high degree of symmetry, such as anti-de Sitter space, the underlying Poincaré symmetry can be replaced with a different bosonic algebra and then extended to a super Lie algebra.

We can gain much insight from the supersymmetry algebra alone. We begin with some elementary manipulations of the first line of (7.1), multiplying both sides by γ_0:

$$\{Q_\alpha, (Q^\dagger)^\beta\} = -2(\gamma_\mu \gamma_0)_\alpha{}^\beta P^\mu \tag{7.2}$$

Tracing over the spinor indices then yields

$$\mathrm{tr}(QQ^\dagger + Q^\dagger Q) = 2^{\lfloor \frac{d}{2} \rfloor + 1} P^0 \ , \tag{7.3}$$

where we have used the fact that $\mathrm{tr}(\gamma_{\mu\nu}) = 0$ and that $\mathrm{tr}(\mathrm{id}) = 2^{\lfloor \frac{d}{2} \rfloor}$. The momentum component is just the energy $P^0 = E$ and so we see that

$$E = \frac{1}{2^{\lfloor \frac{d}{2} \rfloor}} \,\mathrm{tr}(Q^\dagger Q) \ . \tag{7.4}$$

(In a "really real" representation, we can replace $Q^\dagger_\alpha$ with Q_α.)

The quantity $Q^\dagger Q$ is manifestly positive, and thus the energy in a SUSY theory is a positive definite quantity. States $|0\rangle$ with $E = 0$, if they exist, are the lowest energy, or vacuum states. Any such state must furthermore be annihilated by the supercharges

$$Q_\alpha |0\rangle = 0 \ , \tag{7.5}$$

and therefore preserve the supersymmetry (i.e. be invariant with respect to supersymmetry transformations).

The trace relation gives a simple diagnostic for spontaneous symmetry breaking – where the vacuum state breaks the symmetry although the action is invariant. If one finds that the vacuum state $|\Omega\rangle$ has positive energy, then the state breaks the supersymmetry $Q_\alpha|\Omega\rangle \neq 0$. Similarly if one finds that the vacuum state is not supersymmetric, $Q_\alpha|\Omega\rangle \neq 0$, then it must have positive energy.

Next, we look at the representations of the SUSY algebra. Since $[P_\mu, Q_\alpha] = 0$, it is also true that $[P^2, Q_\alpha] = 0$. As with the Poincaré group, we can use the eigenvalues of the Casimir P^2, i.e. the mass squared, to label representations of the SUSY algebra. All the members of a given irreducible representation will have the same value of m^2.[10] We pursue the same strategy that is used in classifying representations of the Poincaré algebra and treat

[10]There are interesting exceptions to this rule when the space-time is curved and the underlying Poincaré algebra is replaced with something else.

the massive and massless cases separately. It will be simpler to perform the analysis in a "really real" representation where $C = \gamma_0$ and $Q^* = Q$.

In the massive case, we can go to a rest frame where $P^\mu = (m, 0, \ldots)$. The anti-commutation relation of the supercharges reduces to

$$\{Q_\alpha, Q_\beta\} = 2m\delta_{\alpha\beta}. \tag{7.6}$$

After a rescaling $\tilde{Q}_\alpha = m^{-1/2} Q_\alpha$, we recover the familiar Clifford algebra

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = 2\delta_{\alpha\beta} \ , \tag{7.7}$$

but now in $\mathfrak{d} = 2^{\lfloor \frac{d}{2} \rfloor}$ Euclidean dimensions. As such, it must have $2^{\mathfrak{d}/2} = 2^{2^{\lfloor \frac{d}{2} \rfloor - 1}}$ states. As the number of these Q_α matrices is even, we can construct a "gamma five" matrix as well,

$$(-1)^F = i^{\#} \tilde{Q}_1 \tilde{Q}_2 \cdots \tilde{Q}_{\mathfrak{d}} \tag{7.8}$$

which anti-commutes with the other $\tilde{Q}_\alpha$. Previously, we interpreted the ± 1 eigenvalues of "gamma five" as chirality of the state, but here they determine whether the state is fermionic or bosonic. Let $|\pm\rangle$ be an eigenstate of $(-1)^F$. As Q_α itself is fermionic, acting with it on a state will swap the state's fermionic/bosonic nature:

$$(-1)^F Q_\alpha |\pm\rangle = -Q_\alpha (-1)^F |\pm\rangle = \mp Q_\alpha |\pm\rangle \ . \tag{7.9}$$

One more remarkable thing we can learn comes from the fact that $(-1)^F$ is traceless (see problem 6.9). The trace is also the sum of the eigenvalues, and so there must be an equal number of bosonic and fermionic states in the super multiplet (irreducible representation). The pain we endured in learning about Clifford algebras and spinors is paying off!

For massless particles, the best we can do is pick a frame where $P^\mu = (E, -E, 0, 0, \ldots)$, and the SUSY algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2E(\mathrm{id} + \gamma_{01})_{\alpha\beta} \ . \tag{7.10}$$

Problem 7.1. *Show that the matrix $\frac{1}{2}(\mathrm{id} + \gamma_{01})$ acts like a projector,*

$$\left(\frac{1}{2}(\mathrm{id} + \gamma_{01})\right)^2 = \frac{1}{2}(\mathrm{id} + \gamma_{01}) \ , \tag{7.11}$$

half of whose eigenvalues are equal to zero and the other half are equal to one.

Using the projector, we can choose a new basis where half of the Q_α commute with one another and the other half do not:

$$\{Q_{\alpha'}, Q_{\beta'}\} = 4E\delta_{\alpha'\beta'} \ , \quad \{Q_{\alpha''}, Q_{\beta''}\} = 0 \tag{7.12}$$

where the primed and double primed indices run over only $\frac{\mathfrak{d}}{2} = 2^{\lfloor \frac{d}{2} \rfloor - 1}$ values. Rescaling the nontrivial Q's a bit differently this time, $\tilde{Q}_{\alpha'} = (2E)^{-1/2} Q_{\alpha'}$, we find a Clifford algebra with half as many Q's as before. We can repeat the previous argument with this smaller algebra.

The massless supermultiplet has fewer states, $2^{\mathfrak{d}/4}$ instead of $2^{\mathfrak{d}/2}$. Half of these states are bosons and the other half fermions.

Finally we draw some conclusions from the commutator $[Q_\alpha, M_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta$ or equivalently the fact that Q_α has spin. Acting with Q_α thus must change the Lorentz group representation of the state. Let us focus on the 4d case, where massive states are labeled by their remaining $SO(3)$ spin quantum number while massless states are characterized by a helicity under the remaining $SO(2)$. We can think of Q_α as carrying spin (or helicity) one half. Using our quantum mechanics intuition, acting with Q_α should be like tensoring the underlying representation of the particle by a spin one half representation and should lead to new possible representations with angular momentum either larger or smaller by a quantized unit of 1/2.

In a supermultiplet, there will be a state with maximum spin $j_{\max}$ (or helicity in the massless case). The remaining states then have spins $j_{\max} - \frac{1}{2}$, $j_{\max} - 1$, etc. Acting with the appropriate lowering Q operator on the $j_{\max}$ state should lead to a new state with spin or helicity less by an amount one half, $j_{\max} - 1/2$.

An annoying complication is that if $j_{\max}$ is large enough, the multiplets tend to have more positive helicity states than negative and so are not CPT complete. The standard procedure to remedy the problem and obtain a theory that is CPT invariant is to add by hand a "mirror multiplet" with a corresponding lowest helicity state and use raising instead of lowering operators.

We can consider a few examples. In four dimensions, the smallest representation of a Clifford algebra is 4 dimensional. Focusing on massless states, we can then use two of the four Q_α's to create a multiplet, leading to one raising and one lowering operator and two states with helicities λ and $\lambda + \frac{1}{2}$. One of the states is fermionic and the other bosonic. Such a set of states is not CPT complete and needs to be supplemented with a mirror multiplet with helicities $-\lambda$ and $-\lambda - \frac{1}{2}$. One important example is the multiplet $\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ – really $\left(0, \frac{1}{2}\right)$ and its mirror – corresponding to a field theory with a complex scalar and a Majorana fermion. (Note the Dirac equation reduces the number of on-shell fermionic degrees of freedom from the size of the representation, four, down to two.) Another option is to have $\left(-1, -\frac{1}{2}, \frac{1}{2}, 1\right)$ corresponding to a gauge field and its superpartner, sometimes called a photino or gluino. A gauge field in 4d has two on-shell degrees of freedom, corresponding to two polarizations. The massive multiplets will be twice as large. One has $\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, which is the same as the massless $\left(0, \frac{1}{2}\right)$ multiplet along with its mirror. We will construct actions later in the course with precisely these particle contents.

In general, going to higher dimension forces us to consider representations with larger and larger spin. While the numbers of Q's grow exponentially, the number of polarization states for a particle with a given spin tends to grow as a power law, linearly for a gauge field for example, or quadratically for a graviton. Ten dimensions turns out to be the largest dimension with a multiplet with helicities less than or equal to one. In this dimension, the smallest spinor representation is 16 dimensional Majorana-Weyl. For a massless multiplet, we can use 8 of the corresponding supercharges to construct four pairs of raising and lowering operators. Acting with at most four lowering operators will change the spin by at most an amount 2, corresponding to the helicity difference between the two polarizations of a photon. (In more detail, the 16 possible states divide up between the 8 polarization states of a photon and the 8 physical degrees of freedom of a Majorana-Weyl fermion.) In other words, supersymmetric gauge theories (without gravity) must have $d \leq 10$.

Eleven dimensions is the largest dimension with a multiplet with helicities less than or equal to two. Now the smallest representation is a 32 dimensional Majorana spinor. We get eight pairs of raising and lowering operators, corresponding to a maximum helicity difference of 4, i.e. the difference in helicity between two polarizations of a graviton. In other words, supergravity theories (without higher spin fields) must have $d \leq 11$.[11]

7.1 Witten Index

We argued for something above that is not always true. Specifically, we claimed that $\mathrm{tr}(-1)^F = 0$, since it was just like the "gamma five" matrix we considered in our construction of the Clifford algebra. While it is true that the trace vanishes when acting on a supermultiplet with nonzero energy, for the vacuum there is no such constraint. Witten took advantage of this fact to write down his index

$$\mathcal{W} = \mathrm{tr}_H(-1)^F \tag{7.13}$$

where the trace is over the whole Hilbert space, not just the positive energy states. Because $\mathrm{tr}(-1)^F$ is nearly always zero, this index measures the difference between the number of fermionic and bosonic vacuum states

$$\mathcal{W} = \{\# \text{ of bosonic vacua}\} - \{\# \text{ of fermionic vacua}\} \,. \tag{7.14}$$

By definition, it is an integer. It cannot be continously varied and cannot receive corrections as coupling constants are varied. It provides perhaps the simplest example of a non-renormalization theorem in supersymmetry. In a generic quantum field theory, one could well imagine that varying parameters leads to a vacuum state becoming a nonzero energy state or vice versa. In a supersymmetric theory, however, the nonzero energy states are all paired – one fermion and one boson. Thus the vacuum states have to disappear or appear in pairs, such that the Witten index remains invariant. A nonzero Witten index must also mean that supersymmetry cannot be spontaneously broken. There must always be a few vacuum states left that cannot pair off and disappear.

7.2 The Goldstino

I wanted to elaborate a little bit more about spontaneous breaking of supersymmetry and how it compares to the usual case of the breaking of a spontaneous global symmetry. In the usual case of a continuous global symmetry, there is a charge Q that generates that symmetry and also a conserved current $J^\mu(x)$ that follows from Noether's Theorem. Conservation here

[11] The multiplet has 44 graviton states, 128 gravitino states, and 84 states associated with an antisymmetric three-form, for a grand total $2^8 = 256$ states. The number of degrees of freedom of a graviton map to the number of metric degrees of freedom. The metric starts out as a $d \times d$ symmetric matrix, but we can remove one row and column using diffeomorphism invariance (or the freedom to change variables). The remaining trace also drops out of the equations of motion, leading to $\frac{d(d-1)}{2} - 1 = \frac{d(d-3)}{2}$ degrees of freedom. The gravitino ψ_μ^α has a spinor and a vector index, but is "gamma traceless", $\gamma^\mu \psi_\mu^\alpha = 0$, leading to $16 \times 9 - 16 = 128$ on-shell degrees of freedom. An anti-symmetric three index tensor has $9 \times 8 \times 7/3! = 84$ on-shell degrees of freedom.

means that the divergence $\partial_\mu J^\mu(x) = 0$ vanishes. The conserved charge is then the integral of the charge density over a spatial slice of the theory in question:

$$Q = \int \mathrm{d}^{d-1}x \, J^0(x) \ . \tag{7.15}$$

Conservation of the current implies that Q is time independent:

$$\frac{dQ}{dt} = \int \mathrm{d}^{d-1}x \, \partial_0 J^0(x) = -\int \mathrm{d}^{d-1}x \, \partial_i J^i(x) = 0 \tag{7.16}$$

assuming that $J_i(x)$ falls off suitably fast at spatial infinity.

In the case of spontaneous symmetry breaking, even though the action and the path integral more generally respect the symmetry, the ground state does not. (The canonical example here is a complex scalar field ϕ in the presence of a Mexican hat potential $V(\phi) = |\phi|^4 - |\phi|^2$, where the vacuum does not respect the $U(1)$ symmetry.) If we think of Q as a generator of the symmetry and $|\Omega\rangle$ as a ground state, then equivalently,

$$Q|\Omega\rangle \neq 0 \ . \tag{7.17}$$

There is a local manifestation of this symmetry breaking. Instead of shifting the vacuum everywhere, we can shift it locally, using $J^0(x)$. This ability to shift the vacuum locally is reflected in the guaranteed existence of a massless particle, the Goldstone boson, which roughly, one can think of as being created from the vacuum by the action of $J^\mu(x)$. Equivalently, the inner produce $\langle\Omega|J^0(x)|\theta\rangle$ fails to vanish, where $|\theta\rangle$ is a state that contains a Goldstone boson.

There is an equivalent statement in the supersymmetric context. Now we have a supercharge Q_α which has a spinor index and generates the supersymmetry transformations. It can be written as a spatial integral over the time-like component of a supercurrent S^μ_α, which has both a spinor and a vector index:

$$Q_\alpha = \int \mathrm{d}^{d-1}x \, S^0_\alpha \ . \tag{7.18}$$

Spontaneous breaking of supersymmetry means that $Q_\alpha|\Omega\rangle \neq 0$. There is again a massless particle which represents this freedom to shift the vacuum locally. Now however, it must be fermionic. People call it the Goldstino, and roughly speaking it is created from the vacuum by the action of the supercurrent. More precisely, the inner product

$$\langle\Omega|S^\mu_\alpha|\psi\rangle \tag{7.19}$$

fails to vanish, where $|\psi\rangle$ is a state which contains a Goldstino.

Problem 7.2. *It is possible to have extended supersymmetry where the Q^I_α carry an extra index $I = 1, 2, \ldots \mathcal{N}$. Assuming Majorana fermions and forgetting about central charges, the first (and most important) line of the supersymmetry algebra is modified to*

$$\{Q^I_\alpha, Q^J_\beta\} = -2\delta^{IJ}(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \ . \tag{7.20}$$

Let us restrict to the four dimensional case where we can choose the Q^I_α to be $\mathcal{N}$ copies of a four dimensional Majorana-Weyl spinor representation.

a) *What is the typical size of a massive particle multiplet? of a massless one?*

b) *For a massless multiplet, what is the difference in helicity between the highest weight and lowest weight states? What is the largest $\mathcal{N}$ for which one can restrict to particles with spin less than or equal to one (i.e. gauge theories)? to particles with spin less than or equal to two (i.e. gravitational theories)?*

c) *For $\mathcal{N} = 2$, $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories, try to describe the particle content of some massless multiplets with small spin, i.e. less than or equal to two.*

Problem 7.3. SUSY Quantum Mechanics. *Suppose we have a quantum mechanical Hamiltonian that can be written in the form $H = \frac{1}{2}\{Q^\dagger, Q\}$ where*

$$Q = (P - iW'(x)) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $P = -i\partial_x$ is the usual momentum operator while $W(x)$ is a real function of the position x.

a) *Show that H has non-negative eigenvalues. Show also that zero energy states $H|\psi\rangle = 0$ must be annihilated by both Q and $Q^\dagger$, $Q|\psi\rangle = 0 = Q^\dagger|\psi\rangle$.*

b) *Given a zero energy ground state $H|\psi\rangle = 0$, deduce and solve any first order differential equations that $\psi(x) = \langle x|\psi\rangle$ must satisfy. What relationships, if any, can you deduce between the asymptotic behavior of $W(x)$ and the normalizability of $\psi(x)$?*

c) *Express H in terms in P and W.*

d) *Let $\psi(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$ be an eigenfunction of H with eigenvalue $E \neq 0$. Show that $Q^\dagger\psi(x)$ is also an eigenfunction. Compute its eigenvalue.*

e) *Consider the case $W(x) = \log(\cosh(x))$. What form does H take? What are the ground state(s)?*

8 Simple 4d SUSY Models: Wess-Zumino and Maxwell

The simplest four dimensional supersymmetric theory is often called the Wess-Zumino model. It has a Majorana Fermion $\psi(x)$ along with some scalar fields. Off-shell, $\psi(x)$ has four real components which are reduced to two real components on-shell by the Dirac equation. As we saw before, for supersymmetry, there must then be a pair of real scalar fields $A(x)$ and $B(x)$ as well.

Why does the Dirac equation reduce the number of degrees of freedom from 4 to 2? From a classical point of view, we associate a degree of freedom to the ability to choose the position and momentum of a particle. If the particle is described by a second order differential equation, those two quantities – position and momentum (or equivalently velocity) – are the integration constants of the differential equation. The Dirac equation, on the other hand, is not a single second order but a quadruplet (in 4d) of first order equations. In general, we can replace a single second order differential equation with a pair of first order equations, e.g. in place of

$$\phi''(x) = p(x)\phi'(x) + q(x)\phi(x) \ , \tag{8.1}$$

we could introduce $\pi(x) = \phi'(x)$ and write instead

$$\begin{aligned} \pi' &= p\pi + q\phi \ , & (8.2)\\ \phi' &= \pi \ . & (8.3) \end{aligned}$$

Going backward, we expect the four first order components that make up the Dirac equation should correspond to a pair of second order differential equations and thus to two degrees of freedom. Identifying which components of the spinor $\psi(x)$ correspond to "position" and which to "momentum" is unfortunately a bit ambiguous. The canonical commutation relation involves $\psi(x)$ with itself and yields no insight. Somehow, the components of $\psi(x)$ should be thought of as position and momentum at the same time, in some linear combination.

Following our nose, we test the following free theory for supersymmetry

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu A)(\partial^\mu A) - \frac{1}{2}(\partial_\mu B)(\partial^\mu B) - \frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi \ . \tag{8.4}$$

The bosonic part leads straightforwardly the the expected equations of motion $\Box A = 0 = \Box B$. Note we can replace the pair of real scalars with a complex scalar $\phi = A + iB$ and its conjugate $\phi^* = A - iB$, in which case the Lagrangian becomes

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu\phi^*)(\partial^\mu\phi) - \frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi \ . \tag{8.5}$$

The fermionic action may seem more mysterious. The factor of i should be thought of as combining with the ∂_μ to give the Hermitian generator of translations P_μ. The resulting expression is indeed real, as we can verify explicitly. We work in a "really real" representation where the γ^μ have real coefficients and $C = \gamma_0$:

$$\begin{aligned} \left(i\bar\psi\gamma^\mu\partial_\mu\psi\right)^* &= -i\left(\bar\psi\gamma^\mu\partial_\mu\psi\right)^* \\ &= -i\left(\psi^T C\gamma^\mu\partial_\mu\psi\right)^* \ . \end{aligned} \tag{8.6}$$

We use the result that $(\psi_\alpha\chi_\beta)^* = \chi^*_\beta\psi^*_\alpha = -\psi^*_\alpha\chi^*_\beta$, and further that $\psi^* = \psi$ in this "really real" basis for the gamma matrices. As the matrix $C\gamma^\mu$ is real, taking the complex conjugate of the expression in parentheses yields a minus sign showing that indeed

$$\left(i\bar\psi\gamma^\mu\partial_\mu\psi\right)^* = i\bar\psi\gamma^\mu\partial_\mu\psi \ . \tag{8.7}$$

Let us also verify that we get the correct equation of motion for the fermion. Varying the action with respect to ψ, we obtain

$$\begin{aligned} \delta\mathcal{L}_0 &= -\frac{i}{2}\delta\psi^T C\gamma^\mu\partial_\mu\psi - \frac{i}{2}\psi^T C\gamma^\mu\partial_\mu\delta\psi \\ &= -\frac{i}{2}\delta\psi^T C\gamma^\mu\partial_\mu\psi + \frac{i}{2}(\partial_\mu\psi^T)C\gamma^\mu\delta\psi - \frac{i}{2}\partial_\mu(\psi^T C\gamma_\mu\delta\psi) \ . \end{aligned} \tag{8.8}$$

Note that it's important here that the fermion is real. With a complex fermion, we should vary ψ and $\bar{\psi}$ independently, similar to what we would do with a complex scalar. We now use one of the Majorana flip relations to replace $(\partial_\mu \psi)^T C\gamma^\mu \delta\psi$ with $-\delta\psi^T C\gamma^\mu(\partial_\mu\psi)$. We also throw out the total derivative term, assuming that we can implement the appropriate boundary conditions. The result is the Dirac equation:

$$\delta\mathcal{L}_0 \;=\; -i\delta\psi^T C\gamma^\mu \partial_\mu \psi \tag{8.9}$$

To investigate supersymmetry, we will vary the action by an infinitesimal Grassman valued object ϵ which transforms as a Majorana spinor. This object ϵ we can think of in rough analogy to the parameter a^μ that we used in considering infinitesimal translations. While P_μ has engineering dimension one, the infinitesimal length a_μ must have engineering dimension -1. Similarly, Q as the square root of P will have engineering dimension 1/2 while ϵ has engineering dimension $-1/2$. To translate between δ and Q, we have

$$\begin{aligned} [\delta_1, \delta_2] &= [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] \\ &= \bar{\epsilon}_1^\alpha \bar{\epsilon}_2^\beta \{Q_\alpha, Q_\beta\} \ . \end{aligned} \tag{8.10}$$

Note that $\delta = \bar{\epsilon}Q = \bar{Q}\epsilon$ is bosonic and so it is natural to take a commutator.

The supersymmetric variation should rotate a scalar into a fermionic operator and a fermion into a scalar operator. By dimension counting, we should be able to relate the variation of a scalar, e.g. $[Q, \phi]$, directly to ψ. The free scalar has engineering dimension one, while the fermion has engineering dimension 3/2. A natural guess is

$$\begin{aligned} \delta\phi &= \bar{\epsilon}(a + b\gamma)\psi, \\ \delta\phi^* &= \bar{\epsilon}(-a^* + b^*\gamma)\psi \ , \end{aligned} \tag{8.11}$$

where a and b are constants. The $-a^*$ in the second line comes from the fact that $\bar{\epsilon}\psi$ is purely imaginary in our conventions while we find $+b^*$ because $\bar{\epsilon}\gamma\psi$ is real. If we vary ϕ twice, we would like to produce a total derivative acting on ϕ, i.e. an infinitesimal translation. A natural guess then for the variation of ψ is

$$\delta\psi = \frac{1}{2}\left((1 + c\gamma)\partial\!\!\!/\phi + (1 - c^*\gamma)\partial\!\!\!/\phi^*\right)\epsilon \ . \tag{8.12}$$

We have used the freedom to rescale ϵ to fix the coefficient of $\partial\!\!\!/\phi$ to be $(1 + c\gamma)$ for c an undetermined constant. The coefficient of $\partial\!\!\!/\phi^*$ is then fixed by the Majorana property.

Let us begin by seeing if the constants a, b, and c can be adjusted to make this infinitesimal transformation a symmetry of the action. (We must check that the variation of the action vanishes off-shell. Of course it will vanish on-shell, because that is how the equations of motion are derived in the first place.) The variation takes the form

$$\delta\mathcal{L}_0 \;=\; -\frac{1}{2}(\partial_\mu\delta\phi^*)(\partial^\mu\phi) - \frac{1}{2}(\partial_\mu\phi^*)(\partial^\mu\delta\phi) - i\delta\psi^T C\gamma^\mu\partial_\mu\psi \ , \tag{8.13}$$

where we have used (8.9). We need look only at the terms proportional to ϕ. The result for the terms proportional to ϕ^* will follow by complex conjugation:

$$\delta\mathcal{L}_0|_\phi \;=\; -\frac{1}{2}\left(-a^*\bar{\epsilon}\partial_\mu\psi + b^*\bar{\epsilon}\gamma\partial_\mu\psi\right)\partial^\mu\phi - \frac{i}{2}\left[(\partial\!\!\!/\phi + c\gamma\partial\!\!\!/\phi)\epsilon\right]^T C\gamma^\mu\partial_\mu\psi \ . \tag{8.14}$$

Now using that $-\gamma_\mu^T C = C\gamma_\mu$ (6.43) while $\gamma^T C = C\gamma$, we see that

$$\delta\mathcal{L}_0|_\phi \;=\; \frac{1}{2}\left[a^*\bar\epsilon\partial^\mu\psi - b^*\bar\epsilon\gamma\partial^\mu\psi + i\bar\epsilon\gamma^\mu\gamma^\nu\partial_\nu\psi + ic\bar\epsilon\gamma^\mu\gamma\gamma^\nu\partial_\nu\psi\right](\partial_\mu\phi) \tag{8.15}$$

Next, we write $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] = \eta^{\mu\nu} + \gamma^{\mu\nu}$ and remark that $\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\psi = \partial_\mu(\gamma^{\mu\nu}\phi\partial_\nu\psi)$ is a total derivative, allowing us to group terms:

$$\delta\mathcal{L}_0|_\phi \;=\; \frac{1}{2}\left[(a^*+i)\,\bar\epsilon\partial_\mu\psi + (-b^*-ic)\,\bar\epsilon\gamma\partial_\mu\psi\right](\partial^\mu\phi)\ . \tag{8.16}$$

For the variation to vanish, we thus require $a=i$ and $b=ic^*$.

Returning now to the issue of whether or not we are dealing with supersymmetry, we can see if we get something sensible for $[\delta_1,\delta_2]\phi$:

$$\begin{aligned}
\delta_1\delta_2\phi &= \delta_1\bar\epsilon_2(a+b\gamma)\psi \\
&= \frac{1}{2}\bar\epsilon_2(a+b\gamma)\left[(1+c\gamma)\partial\!\!\!/\phi + (1-c^*\gamma)\partial\!\!\!/\phi^*\right]\epsilon_1 \\
&= \frac{1}{2}(a+bc)\bar\epsilon_2\partial\!\!\!/\phi\epsilon_1 + \frac{1}{2}(ac+b)\bar\epsilon_2\gamma\partial\!\!\!/\phi\epsilon_1 \\
&\quad + \frac{1}{2}(a-bc^*)\bar\epsilon_2\partial\!\!\!/\phi^*\epsilon_1 + \frac{1}{2}(-ac^*+b)\bar\epsilon_2\gamma\partial\!\!\!/\phi^*\epsilon_1\ .
\end{aligned} \tag{8.17}$$

To simplify the commutator, we need the Majorana flip relations that $\bar\epsilon_2\gamma^\mu\epsilon_1 = -\bar\epsilon_1\gamma^\mu\epsilon_2$ along with $\bar\epsilon_2\gamma\gamma^\mu\epsilon_1 = \bar\epsilon_1\gamma\gamma^\mu\epsilon_2$:

$$[\delta_1,\delta_2]\phi = (a+bc)\bar\epsilon_2\partial\!\!\!/\phi\epsilon_1 + (a-bc^*)\bar\epsilon_2\partial\!\!\!/\phi^*\epsilon_1\ . \tag{8.18}$$

We argued before that $\bar\epsilon_2\gamma^\mu\epsilon_1$ has the correct transformation properties to be a Lorentz vector. Thus, if the second term can be made to vanish, we have found that $[\delta_1,\delta_2]$ produces an infinitesimal translation when acting on ϕ, as it should if we are discussing supersymmetry. In addition to $a=i$ and $b=ic^*$ that we found in demanding $\delta\mathcal{L}_0$ vanish, we now also find $a=bc^*$, allowing us to set $c^*=\pm1$ and $b=\pm i$. We will make the positive sign choice, leading to

$$[\delta_1,\delta_2]\phi = 2i\bar\epsilon_2\partial\!\!\!/\phi\epsilon_1\ . \tag{8.19}$$

The supersymmetry transformation can be cast in a more compact form using the projectors $\Pi_\pm \equiv \frac{1}{2}(1+\gamma)$:

$$\begin{aligned}
\delta\phi &= 2i\bar\epsilon\Pi_+\psi\ ,\quad \delta\phi^* = 2i\bar\epsilon\Pi_-\psi\ , && (8.20)\\
\delta\psi &= \Pi_+\partial\!\!\!/\phi\epsilon + \Pi_-\partial\!\!\!/\phi^*\epsilon\ . && (8.21)
\end{aligned}$$

These operators $\Pi_\pm$ project the fermions onto positive or negative chirality states, i.e. into the space of Weyl fermions. It perhaps makes sense that some kind of projector appears, as the fermion has more degrees of freedom than a single scalar.

Finally, we need to check that $[\delta_1,\delta_2]\psi$ is indeed an infinitesimal translation. We find that

$$\begin{aligned}
\delta_1\delta_2\psi &= 2i(\partial_\mu\bar\epsilon_1\Pi_+\psi)\gamma^\mu\Pi_-\epsilon_2 + 2i(\partial_\mu\bar\epsilon_1\Pi_-\psi)\gamma^\mu\Pi_+\epsilon_2 \\
&= i(\bar\epsilon_1\partial_\mu\psi)\gamma^\mu\epsilon_2 - i(\bar\epsilon_1\gamma\partial_\mu\psi)\gamma^\mu\gamma\epsilon_2
\end{aligned} \tag{8.22}$$

This doesn't yet look much like $\bar{\epsilon}_1\gamma^\mu\epsilon_2$ multiplying $\partial_\mu\psi$, in analogy to what we obtained for the scalar, but we can use our Fierz rearrangement identities. In 4d, we have the following independent gamma matrices:

$$\mathrm{id}\ ,\quad \gamma_\mu\ ,\quad \gamma\ ,\quad \gamma_\mu\gamma\ ,\quad \gamma_{\mu\nu}\ . \tag{8.23}$$

Note that there are $1+4+1+4+6=16$ of these matrices, which is indeed equal to 4×4, the size of gamma matrices for these Majorana fermions in 4d. The relevant Fierz identity is then

$$\begin{aligned}(\bar{\lambda}\rho)\chi &= -\frac{1}{4}(\bar{\lambda}\chi)\rho-\frac{1}{4}(\bar{\lambda}\gamma_\mu\chi)(\gamma^\mu\rho)-\frac{1}{4}(\bar{\lambda}\gamma\chi)(\gamma\rho)\\ &\quad+\frac{1}{4}(\bar{\lambda}\gamma_\mu\gamma\chi)(\gamma^\mu\gamma\rho)+\frac{1}{8}(\bar{\lambda}\gamma_{\mu\nu}\chi)(\gamma^{\mu\nu}\rho)\ .\end{aligned} \tag{8.24}$$

The extra factor of $1/2$ in the last term compensates for the overcounting from $\gamma_{\mu\nu}=-\gamma_{\nu\mu}$. We are interested in the special case $(\bar{\epsilon}_1\rho)\epsilon_2-(\bar{\epsilon}_2\rho)\epsilon_1$ where ρ is either $\partial_\mu\psi$ or $\gamma\partial_\mu\psi$. Because of the Majorana flip relations (6.47) supplemented by a couple of extra relations that also involve γ, it turns out only the terms that involve γ_μ and $\gamma_{\mu\nu}$ survive:

$$(\bar{\epsilon}_1\rho)\epsilon_2-(\bar{\epsilon}_2\rho)\epsilon_1=-\frac{1}{2}(\bar{\epsilon}_1\gamma_\mu\epsilon_2)(\gamma^\mu\rho)+\frac{1}{4}(\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2)(\gamma^{\mu\nu}\rho)\ . \tag{8.25}$$

Problem 8.1. *Show that $\psi^T C\gamma\lambda=\lambda^T C\gamma\psi$ and $\psi^T C\gamma\gamma_\mu\lambda=\lambda^T C\gamma\gamma_\mu\psi$.*

We find then the following

$$\begin{aligned}[\delta_1,\delta_2]\psi &= -\frac{i}{2}(\bar{\epsilon}_1\gamma_\mu\epsilon_2)\gamma^\lambda\gamma^\mu\partial_\lambda\psi+\frac{i}{8}(\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2)\gamma^\lambda\gamma^{\mu\nu}\partial_\lambda\psi\\ &\quad+\frac{i}{2}(\bar{\epsilon}_1\gamma_\mu\epsilon_2)\gamma^\lambda\gamma\gamma^\mu\gamma\partial_\lambda\psi-\frac{i}{8}(\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2)\gamma^\lambda\gamma\gamma^{\mu\nu}\gamma\partial_\lambda\psi\\ &= -i(\bar{\epsilon}_1\gamma_\mu\epsilon_2)\gamma^\lambda\gamma^\mu\partial_\lambda\psi\\ &= -2i(\bar{\epsilon}_1\gamma_\mu\epsilon_2)\partial^\mu\psi+i(\bar{\epsilon}_1\gamma_\mu\epsilon_2)\gamma^\mu\gamma^\lambda\partial_\lambda\psi\ .\end{aligned} \tag{8.26}$$

In proceeding from the first to the second line, we have used that γ anticommutes with γ^μ but commutes with $\gamma^{\mu\nu}$ and also that $\gamma^2=1$. In going from the second to the third equality, we used the anticommutation relations for γ^λ and γ^μ.

We haven't completely succeeded here. There is still the second term in the last line of (8.26), but notice that this second term is proportional to the equation of motion for the fermion, $\gamma^\mu\partial_\mu\psi=0$. What is going on here is that the supersymmetry algebra has failed to close off-shell. In order to get the required translation, we need to impose the equation of motion. We say that the supersymmetry algebra here closes on-shell. More formally, we can write

$$\begin{aligned}\bar{\epsilon}_1^\alpha\epsilon_{2\beta}\{Q_\alpha,\bar{Q}^\beta\} &= 2\bar{\epsilon}_1\gamma_\mu\epsilon_2 P^\mu\\ &= 2\bar{\epsilon}_1^\alpha\epsilon_{2\beta}(\gamma_\mu)_\alpha{}^\beta P^\mu\ .\end{aligned} \tag{8.27}$$

as expected from the first line of our original statement of the supersymmetry algebra (7.1) back in section 4.

In this particular case, there is an improved formalism where we can get the supersymmetry algebra to close off-shell as well, but it requires adding auxiliary fields, i.e. fields that do not carry dynamical degrees of freedom. In this case, we would need to add a complex scalar field traditionally called F. Then the degrees of freedom would balance off-shell – four bosonic and four fermionic.

Problem 8.2. *Consider the following improved Lagrangian*

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi + \frac{1}{2}F^* F \ , \tag{8.28}$$

along with the improved SUSY transformation rules

$$\begin{aligned} \delta\phi &= 2i\bar{\epsilon}\Pi_+\psi \ , \\ \delta\psi &= \gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + F\Pi_+\epsilon + F^*\Pi_-\epsilon \ , & (8.29) \\ \delta F &= 2i\bar{\epsilon}\Pi_-\gamma^\mu\partial_\mu\psi \ . & (8.30) \end{aligned}$$

a) Why doesn't F show up in $\delta\phi$? Why doesn't ϕ show up in δF?

b) Verify that the Lagrangian is invariant under these SUSY transformations.

c) Verify that the SUSY algebra closes off-shell, i.e. without imposing the equations of motion. This problem is rather lengthy, requiring examining $[\delta_1, \delta_2]$ acting on ϕ, ψ, and F.

Finding the appropriate auxiliary fields to close the SUSY algebra off-shell is in fact in general a difficult problem. In the case of $\mathcal{N} = 1$ and 2 supersymmetry, answers are usually known. For many cases with $\mathcal{N} = 4$ and 8 SUSY, the problem remains unsolved.

8.1 Interactions

We can add interactions to this model, but only in a rather limited fashion because of the constraints from supersymmetry. In the interest of simplicity, we will work with on-shell SUSY and no additional auxiliary fields. Consider the following interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -V(\phi, \phi^*) + U(\phi, \phi^*) i\bar{\psi}\Pi_+\psi + U(\phi, \phi^*)^* i\bar{\psi}\Pi_-\psi \ . \tag{8.31}$$

Note the second and third terms are complex conjugates of each other. Provided then that V is real, the interaction is a real quantity as well. (One reason to ignore a $(\bar{\psi}\psi)^2$ type interaction is that it is irrelevant in the language of the renormalization group.) This modified Lagrangian is not invariant under the original SUSY transformations, but it is under a minor modification of them,

$$\begin{aligned} \delta\phi &= 2i\bar{\epsilon}\Pi_+\psi \ , \quad \delta\phi^* = 2i\bar{\epsilon}\Pi_-\psi \ , \\ \delta\psi &= \gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + W(\phi,\phi^*)\Pi_-\epsilon + W(\phi,\phi^*)^*\Pi_+\epsilon \ . & (8.32) \end{aligned}$$

To verify SUSY of the new interacting Lagrangian, we start with the extra pieces that now do not cancel out in $\delta\mathcal{L}_0$ because of the modification of the supersymmetry transformations. From the derivation of the equation of motion for ψ, we can write the left-over piece as

$$\begin{aligned} \delta\mathcal{L}_0|_{\text{leftover}} &= -i\bar{\psi}\gamma^\mu\partial_\mu\left(W^*\Pi_+\epsilon + W\Pi_-\epsilon\right) & (8.33) \\ &= -i\bar{\psi}\left[(\partial W^*)(\partial\!\!\!/\phi)\Pi_+\epsilon + (\bar{\partial}W^*)(\partial\!\!\!/\phi^*)\Pi_+\epsilon + (\partial W)(\partial\!\!\!/\phi)\Pi_-\epsilon + (\bar{\partial}W)(\partial\!\!\!/\phi^*)\Pi_-\epsilon\right] \ . \end{aligned}$$

Next we consider the SUSY variation of the interactions, which we break up into terms that are linear and cubic in ψ: $\delta\mathcal{L}_{\rm int} = \delta_1\mathcal{L}_{\rm int} + \delta_3\mathcal{L}_{\rm int}$. The linear terms are as follows

$$\begin{aligned}
\delta_1\mathcal{L}_{\rm int} &= -(\partial V)(2i\bar{\epsilon}\Pi_+\psi) - (\bar{\partial}V)(2i\bar{\epsilon}\Pi_-\psi) \\
&\quad +iU(\gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + W^*\Pi_+\epsilon + W\Pi_-\epsilon)^T C\Pi_+\psi + c.c. \\
&\quad +iU\psi^T C\Pi_+(\gamma^\mu(\partial_\mu\phi)\Pi_-\epsilon + \gamma^\mu(\partial_\mu\phi^*)\Pi_+\epsilon + W^*\Pi_+\epsilon + W\Pi_-\epsilon) + c.c.
\end{aligned} \tag{8.34}$$

We use that $\Pi_+\gamma_\mu = \gamma_\mu\Pi_-$, $\gamma_\mu^T C = -C\gamma_\mu$ and $\gamma^T C = C\gamma$, along with projection conditions that $\Pi_\pm\Pi_\mp = 0$ and $\Pi_\pm^2 = \Pi_\pm$:

$$\delta_1\mathcal{L}_{\rm int} = -(\partial V)(2i\bar{\epsilon}\Pi_+\psi) - (\bar{\partial}V)(2i\bar{\epsilon}\Pi_-\psi) \tag{8.35}$$
$$-iU\bar{\epsilon}(\partial\!\!\!/\phi)\Pi_+\psi + iUW^*\bar{\epsilon}\Pi_+\psi + c.c.$$
$$+iU\bar{\psi}(\partial\!\!\!/\phi)\Pi_-\epsilon + iUW^*\bar{\psi}\Pi_+\epsilon + c.c. \tag{8.36}$$

Using the Majorana flip identities, this expression simplifies somewhat further

$$\begin{aligned}
\delta_1\mathcal{L}_{\rm int} &= -(\partial V)(2i\bar{\psi}\Pi_+\epsilon) - (\bar{\partial}V)(2i\bar{\psi}\Pi_-\epsilon) \\
&\quad +2iU\bar{\psi}(\partial\!\!\!/\phi)\Pi_-\epsilon + 2iUW^*\bar{\psi}\Pi_+\epsilon + c.c.
\end{aligned} \tag{8.37}$$

The combination $\delta\mathcal{L}_0 + \delta_1\mathcal{L}_{\rm int}$ has to vanish independently of $\delta_3\mathcal{L}_{\rm int}$ because of the differing numbers of fermions in the expressions. Pairing up terms, we find that the following expressions (and their complex conjugates) must vanish

$$\begin{aligned}
\partial W - 2U &= 0\ , \\
\bar{\partial}W &= 0\ , \\
\partial V - UW^* &= 0
\end{aligned} \tag{8.38}$$

The second equation implies the remarkable fact that W must be a holomorphic function of the fields, i.e. depend only on ϕ and not its complex conjugate ϕ^*. The first and third equations (along with their complex conjugates) can be assembled and used to solve for V as a function of W:

$$V = \frac{1}{2}WW^*\ . \tag{8.39}$$

(There are some integration constants which we have set to zero here.)

Problem 8.3. *Verify that $\delta\mathcal{L}_3 = 0$ as well, and so the action is supersymmetric. You will need some Fierz identities.*

Problem 8.4. *Verify that the SUSY variations (8.32) close on-shell.*

The holomorphic function $\mathcal{W}(\phi)$ defined such that

$$\frac{\partial\mathcal{W}}{\partial\phi} = W \tag{8.40}$$

is usually given the name superpotential. The choice of $\mathcal{W}$ determines all of the interactions in the Wess-Zumino model! Note this discussion can easily be promoted to a collection of scalar fields, ϕ_i, $i = 1, 2, \ldots N$, leading to a superpotential that is a holomorphic function of all of them $\mathcal{W}(\phi_i)$.

Consider for a moment a superpotential of the form

$$\mathcal{W} = \frac{1}{2} m\phi^2 + \frac{1}{3}\lambda\phi^3 \ . \tag{8.41}$$

We see that the quadratic term proportional to ϕ^2 will produce mass terms m^2 for the scalar ϕ and m for its superpartner ψ in the original Lagranigan. The cubic term will on the other hand lead to genuine interactions, a Yukawa and its complex conjugate of the schematic form $\phi\bar{\psi}\psi$ as well as a quartic $|\phi|^4$ potential for the scalar.

The holomorphic nature of $\mathcal{W}$ along with some global symmetries can greatly constrain the way in which $\mathcal{W}$ can be renormalized as a function of energy scale and also the way in which SUSY can be spontaneously broken. First consider a U(1) symmetry under which ϕ and $\Pi_+\psi$ have the same charge q, $\phi \to e^{i\alpha q}\phi$ and $\Pi_+\psi \to e^{i\alpha q}\Pi_+\psi$. Hence, ϕ^* and $\Pi_-\psi$ will have the opposite charge, $\phi^* \to e^{-i\alpha q}\phi^*$ and $\Pi_-\psi \to e^{-i\alpha q}\Pi_-\psi$. The correlation between ϕ and $\Pi_+\psi$ appears for consistency with the SUSY transformation rules (8.32). A more concise way of writing the transformation rule for the fermion is to use the gamma five matrix, $\psi \to e^{iq\alpha\gamma}\psi$.[12] By construction, the potential V as well as the Yukawas $U\bar{\psi}\Pi_+\psi$ and $U^*\bar{\psi}\Pi_-\psi$ will be inert under such a symmetry transformation provided $\mathcal{W}$ is inert under this U(1) as well.

There is the possibility of a more subtle global symmetry as well, under which the supercharge Q transforms. This symmetry is usually called R-charge, and the conventional normalization is that Π_+Q should have charge -1, $Q \to e^{-i\alpha\gamma}Q$. Hence, the R-charge of $\Pi_+\psi$ must be one less than the R-charge of ϕ for consistency with the SUSY transformation rules. If the superpotential itself $\mathcal{W}$ has R-charge 2, then the Lagrangian will again be invariant with respect to this global symmetry.

Coming back to the simple model (8.41), the first step in the renormalization argument is to assume that m and λ are not numbers but scalar fields in part of some larger supersymmetric field theory. Their only role for us, however, will be to take on expectation values $\langle m \rangle$ and $\langle \lambda \rangle$ that lead to masses and interactions of the dynamical field ϕ and its super partner ψ. Given their new interpretation as fields, we can restore a $U(1) \times U(1)_R$ symmetry to the theory. The relevant charge assignments for the individual fields such that the superpotential has charge zero and two respectively are

$$\begin{array}{c|cc} & U(1) & U(1)_R \\ \hline \phi & 1 & 1 \\ m & -2 & 0 \\ \lambda & -3 & -1 \end{array} \tag{8.42}$$

When λ is very small, the theory is nearly free, and we have good perturbative control over the behavior. However, as λ gets larger, so do the interactions, and much more complicated behavior can ensue in a generic quantum field theory. Through loop corrections one

[12] One might worry that such a transformation destroys the reality property of the Majorana spinor. Consider however a "really real" representation where $\psi^* = \psi$. In such a basis, γ is pure imaginary and hence $e^{iq\alpha\gamma}$ is purely real.

can generate additional interactions, for example ϕ^4. Non-polynomial and non-perturbative expressions like $e^{-\phi^2/\lambda}$ could appear as well. Here, however, supersymmetry and the U(1) symmetries make the rules much stricter. To respect the symmetries, the potential must be a holomorphic function with the scaling form

$$\mathcal{W} = m\phi^2 f\left(\frac{\lambda\phi}{m}\right) . \tag{8.43}$$

Without holomorphicity, we could satisfy the charge constraints much more easily by including the complex conjugate fields ϕ^*, m^*, and λ^*. In the limit where λ is very small, we can expand this function out as a power series involving only non-negative powers of λ. After all, the theory should be well-behaved with respect to λ in this nearly free limit:

$$\mathcal{W} = \sum_{n=0}^{\infty} g_n m^{1-n} \lambda^n \phi^{n+2} . \tag{8.44}$$

However, we also ought to be able to take a massless limit and expect the theory to be well-behaved. Thus we can rule out all terms with $n > 1$. The generic form of the super potential is then

$$\mathcal{W} = g_0 m\phi^2 + g_1 \lambda\phi^3 . \tag{8.45}$$

We can determine the constants g_0 and g_1 by matching to (8.41) in the weakly interacting limit $\lambda \to 0$. However, the constants g_0 and g_1 must be independent of λ, m and ϕ and so we have fixed $\mathcal{W}$ for all λ, and the super potential is not renormalized. There are some subtleties here which have to do with choice of renormalization group scheme and other subtleties associated with massless limits and Wilsonian RG, but we shall gloss over them.

This argument is easily generalizable to more complicated superpotentials. Each time we add a new coupling and new interaction term, e.g. $\lambda'\phi^4$, we also get a new U(1) symmetry to add to the mix, which constrains the renormalization of the new coupling. This argument can be further generalized to include gauge fields, which I hope we will have time to see later.

Problem 8.5. *Can you modify the argument above for a superpotential of the form* $\mathcal{W} = \frac{1}{2}m\phi^2 + \lambda\phi^r$ *for* r *some positive integer,* $r \geq 3$*?*

The form of the superpotential has consequences for spontaneous SUSY breaking as well. As we discussed earlier, to preserve SUSY, there must be a zero energy ground state available to the system. In the current language, the potential V must vanish or equivalently there must be a solution to the N unknowns $\langle\phi_i\rangle$ from the N constraints $\partial_{\phi_i}\mathcal{W} = 0$. For a generic holomorphic $\mathcal{W}$, for example a polynomial in ϕ_i, we expect to be able to satisfy these constraints without difficulty. Thus even if there are other local minima in V with $V > 0$ (where SUSY is broken) they will at best be metastable. The fields ϕ_i can tunnel out to the true vacuum eventually. If this process is long compared to the age of the universe, it might still be relevant for model building. Some phenomenologists have looked into this idea of meta-stable SUSY breaking because actually breaking SUSY spontaneously can be difficult.

In what generic situation then can we expect SUSY to be spontaneously broken? To be concrete, consider the following superpotential

$$\mathcal{W}(X_a, \phi_i) = \sum_{a=1}^{N} X_a f_a(\phi_i) \tag{8.46}$$

whee X_a and ϕ_i are complex scalar fields and $i = 1, \ldots, M$. Given our discussion of symmetries above, it is natural to associate the N X_a fields with an R-charge two and the ϕ_i with R-charge zero. The conditions for unbroken supersymmetry in the ground state are that

$$f_a(\phi_i) = 0 \quad \text{and} \quad \sum_a X_a \partial_{\phi_i} f_a = 0 \tag{8.47}$$

If $M > N$, then we can satisfy the N equations $f_a(\phi_i) = 0$ generically. The remaining M equations can be satisfied by choosing $X_a = 0$. However if $N > M$, then there will not generically be solutions to all $f_a(\phi_i) = 0$, and supersymmetry is broken in the vacuum.

The scalar potential takes the form

$$V = \sum_a |f_a|^2 + \sum_{i,a} |X_a \partial_i f_a|^2 \ . \tag{8.48}$$

If we choose the ϕ_i to minimize the first term, then the second term can be minimized by putting M linear constraints on the N X_a, giving a whole $N - M$ dimensional space of vacua, all with the same positive energy. As these vacua all break supersymmetry, we do not generically expect the situation to be stable with respect to quantum corrections. The potential V will be altered by these corrections, and likely only a single vacuum state will remain when the dust settles.

This model relates the concepts of spontaneous SUSY breaking and the spontaneous symmetry breaking of R-symmetry. Since at a generic point on the vacuum manifold $\langle X_a \rangle$ is not equal to zero, the R-symmetry is spontaneously broken. Indeed, the mantra is that for supersymmetry to be spontaneously broken, the R-symmetry must be spontaneously broken as well. This situation is bad news for phenomenology since we know SUSY is broken, but we have no evidence of a Goldstone boson from the spontaneously broken R-symmetry. There is a way out, however. What happens in many generic theories is that the superpotential is not generic. Quantum effects can give rise to precisely the right type of non-generic superpotential that break SUSY spontaneously without needing an R-symmetry.

8.2 Maxwell Field

Gauge symmetry is an important part of real world physics, in particular the Standard Model, and it is a critical part of these lectures to incorporate supersymmetry into gauge theories. Given the space we spent developing conformal field theory in these lectures, we have time now to deal with only the simplest gauge theory – a Maxwell field with a U(1) symmetry group. The lessons we learn in this simpler case are useful in dealing with more complicated non-abelian gauge theories although we will not have time for them in this class.

Before embarking, let us first do some simple counting on our fingers. A gauge theory is characterized by a massless vector field A_μ^a that transforms in the adjoint representation of the gauge group. Naively, A_μ should have the same number of space-time degrees of freedom as the number of dimensions, $\mu = 0, 1, \ldots, d-1$. However, this counting does not agree with our experience in four dimensions where the photon has just two polarizations. In Lorentz gauge $\partial_\mu A^\mu = 0$, the equation of motion for the photon is that of d massless scalar fields $\partial^2 A_\mu = 0$. However $\partial_\mu A^\mu = 0$ does not completely fix the gauge and we are free to perform a shift $A_\mu \to A_\mu + \partial_\mu \Lambda$ provided $\partial^2 \Lambda = 0$. This freedom allows us to remove one component of A_μ, say A_1. The gauge constraint $\partial_\mu A_\mu = 0$ then removes an additional degree of freedom. If we choose a reference frame where the photon is traveling in the 1 direction, then its momentum vector will be $p^\mu = (E, E, 0, 0, \ldots)$. In momentum space, the gauge constraint $p^\mu A_\mu = 0$ along with the freedom to shift $A_\mu \to A_\mu + p_\mu$, means the components A_2, A_3, ..., A_{d-1} are enough to specify the physical degrees of freedom.

For the simplest supersymmetry, we should then add the same number of fermionic degrees of freedom in the form of spin 1/2 fermions. [[Spin 3/2 particles are called gravitinos. A discussion of them would quickly lead us into supergravity which I want to avoid.]] So we should look at our table of fermions in various dimensions contained in figure 7 and see when the counting matches. Recall that the Dirac equation removes half of the degrees of freedom, and so we need to see when the numbers in the last column, divided by two, are equal to $d - 2$. The match happens precisely for $d = 3$, 4, 6 and 10. We can have supersymmetric gauge theories in other dimensions as well, but they will require adding fields in other representations of the Lorentz group, for example scalars.

We will focus on the four dimensional case in what follows. Our action here is constructed from a U(1) gauge field A_μ and its corresponding field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and a Majorana fermion λ where $\bar\lambda = \lambda^T C$:

$$S_{\rm SM} = -\int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar\lambda \gamma^\mu \partial_\mu \lambda \right) , \tag{8.49}$$

The theory is free, and perhaps not very interesting on its own. We could add charged matter fields to get a supersymmetric version of QED. Our interest here though is in the fact that it is supersymmetric:

$$\delta A_\mu = i \bar\epsilon \gamma_\mu \lambda , \tag{8.50}$$

$$\delta \lambda = -\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon . \tag{8.51}$$

Note that the transformation rules are consistent with naive engineering dimensions of the fields, where A_μ has dimension one and λ has dimension 3/2. We need to deal with the variation $\delta\bar\lambda$:

$$\begin{aligned} \delta\bar\lambda \gamma^\mu \partial_\mu \lambda &= \partial_\mu (\delta\bar\lambda \gamma^\mu \lambda) - (\partial_\mu \delta\bar\lambda) \gamma^\mu \lambda \\ &= \partial_\mu (\delta\bar\lambda \gamma^\mu \lambda) + \bar\lambda \gamma^\mu \partial_\mu \delta\lambda , \end{aligned} \tag{8.52}$$

integrating by parts and using one of the Majorana flip identities (6.47). Discarding the total derivative, the total SUSY variation reduces to

$$\begin{aligned} \delta S &= -\int d^4x \left(F^{\mu\nu} \partial_\mu \delta A_\nu + i \bar\lambda \gamma^\rho \partial_\rho \delta\lambda \right) \\ &= -\int d^4x \left(F^{\mu\nu} i \bar\epsilon \gamma_\nu \partial_\mu \lambda - \frac{i}{2} \bar\lambda \gamma^\rho \partial_\rho F_{\mu\nu} \gamma^{\mu\nu} \epsilon \right) . \end{aligned} \tag{8.53}$$

To proceed, we use again a Majorana flip identity, this time on $\bar{\epsilon}\gamma_\nu\partial_\mu\lambda$, and rewrite $\gamma^\rho\gamma^{\mu\nu} = \gamma^{\rho\mu\nu} + \eta^{\rho\mu}\gamma^\nu - \eta^{\rho\nu}\gamma^\mu$:

$$\delta S \;=\; -\int \mathrm{d}^4x \left(-iF^{\mu\nu}(\partial_\mu\bar{\lambda})\gamma_\nu\epsilon - \frac{i}{2}\bar{\lambda}(\partial_\rho F_{\mu\nu})(\gamma^{\rho\mu\nu} + 2\eta^{\rho\mu}\gamma^\nu)\epsilon \right) \;. \tag{8.54}$$

The combination $\gamma^{\rho\mu\nu}\partial_\rho F_{\mu\nu} = \gamma^{\rho\mu\nu}\partial_{[\rho}F_{\mu\nu]} = 0$ vanishes by a Bianchi identity, and the remaining bits combine to give a total derivative:

$$\delta S \;=\; \int \mathrm{d}^4x\,\partial_\mu\left(iF^{\mu\nu}\bar{\lambda}\gamma_\nu\epsilon\right) \;, \tag{8.55}$$

which can be discarded assuming SUSY preserving boundary conditions.

We next verify that the SUSY algebra closes in the proper way, consistent with (7.1). The simpler task is closure on the gauge field:

$$\begin{aligned} [\delta_1,\delta_2]A_\mu &= -i\bar{\epsilon}_2\gamma_\mu\left(\frac{1}{2}F^{\lambda\rho}\gamma_{\lambda\rho}\epsilon_1\right) - (1\leftrightarrow 2) \\ &= -i\bar{\epsilon}_2\left(\frac{1}{2}\gamma_{\mu\lambda\rho} + \eta_{\mu\lambda}\gamma_\rho\right)F^{\lambda\rho}\epsilon_1 - (1\leftrightarrow 2) \;. \end{aligned} \tag{8.56}$$

By the Majorana flip identities (6.47), the $\gamma_{\mu\nu\rho}$ term will cancel out of the commutator, leaving

$$\begin{aligned} [\delta_1,\delta_2]A_\mu &= -2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 F_{\mu\nu} \\ &= (2i\bar{\epsilon}_2\gamma^\nu\epsilon_1)\partial_\nu A_\mu - \partial_\mu(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 A_\nu) \end{aligned} \tag{8.57}$$

The first term is a translation and the second a gauge transformation. Thus the supersymmetry closes up to gauge transformations.

Closure on the fermions, as usual, is a more complicated story involving Fierz rearrangement identities. We find

$$\begin{aligned} [\delta_1,\delta_2]\lambda &= -2\partial_\mu\left(\frac{i}{2}\bar{\epsilon}_1\gamma_\nu\lambda\right)\gamma^{\mu\nu}\epsilon_2 - (1\leftrightarrow 2) \\ &= -i\gamma^{\mu\nu}(\bar{\epsilon}_1\gamma_\nu\partial_\mu\lambda)\epsilon_2 - (1\leftrightarrow 2) \;. \end{aligned} \tag{8.58}$$

We will need in particular the same 4d Fierz identity (8.24) that we used in verifying closure of the SUSY algebra for the Wess-Zumino model. We take $\lambda = \epsilon_1$, $\chi = \epsilon_2$, and $\rho = \gamma_\nu\partial_\mu\lambda$. From the Majorana flip identities (6.47), the only term on the right hand side of the Fierz identity that will contribute to the commutator are the ones that involve γ_μ and $\gamma_{\mu\nu}$. Hence we find

$$[\delta_1,\delta_2]\lambda \;=\; \frac{i}{2}(\bar{\epsilon}_1\gamma_\rho\epsilon_2)\gamma^{\mu\nu}\gamma^\rho\gamma_\nu\partial_\mu\lambda - \frac{i}{4}(\bar{\epsilon}_1\gamma^{\rho\sigma}\epsilon_2)\gamma^{\mu\nu}\gamma_{\rho\sigma}\gamma_\nu\partial_\mu\lambda \tag{8.59}$$

We now need to go through some rather tedious manipulations with the gamma matrices. The strategy here is to try to either get an infinitesimal translation, i.e. $\bar{\epsilon}_1\gamma_\mu\epsilon_2\partial^\mu\lambda$, or something that will vanish by the equations of motion, i.e. stuff times $\partial\!\!\!/\lambda$. Here we go for the simpler one:

$$\begin{aligned}
\gamma^{\mu\nu}\gamma^{\rho}\gamma_{\nu} &= -\gamma^{\mu\nu}\gamma_{\nu}\gamma^{\rho} + 2\gamma^{\mu\rho} \\
&= -3\gamma^{\mu}\gamma^{\rho} + 2\gamma^{\mu\rho} \\
&= -3\eta^{\mu\rho} - \gamma^{\mu\rho} \\
&= -4\eta^{\mu\rho} + \gamma^{\rho}\gamma^{\mu} \ .
\end{aligned} \tag{8.60}$$

And now for the more complicated one:

$$\begin{aligned}
\gamma^{\mu\nu}\gamma^{\rho\sigma}\gamma_{\nu} &= [\gamma^{\mu\nu}, \gamma^{\rho\sigma}]\gamma_{\nu} + \gamma^{\rho\sigma}\gamma^{\mu\nu}\gamma_{\nu} \\
&= 2(\eta^{\nu\rho}\gamma^{\mu\sigma} - \eta^{\mu\rho}\gamma^{\nu\sigma} + \eta^{\mu\sigma}\gamma^{\nu\rho} - \eta^{\nu\sigma}\gamma^{\mu\rho})\gamma_{\nu} + 3\gamma^{\rho\sigma}\gamma^{\mu} \\
&= 2\gamma^{\mu\sigma}\gamma^{\rho} + 6\eta^{\mu\rho}\gamma^{\sigma} - 6\eta^{\mu\sigma}\gamma^{\rho} - 2\gamma^{\mu\rho}\gamma^{\sigma} + 3\gamma^{\rho\sigma}\gamma^{\mu} \\
&= 2\gamma^{\mu\sigma\rho} + 2\gamma^{\mu}\eta^{\rho\sigma} + 4\eta^{\mu\rho}\gamma^{\sigma} - 2\gamma^{\mu\rho\sigma} - 2\gamma^{\mu}\eta^{\rho\sigma} - 4\eta^{\mu\sigma}\gamma^{\rho} + 3\gamma^{\rho\sigma}\gamma^{\mu} \\
&= 4\gamma^{\sigma\rho\mu} + 4\eta^{\mu\rho}\gamma^{\sigma} - 4\eta^{\mu\sigma}\gamma^{\rho} + 3\gamma^{\rho\sigma}\gamma^{\mu} \\
&= 4\gamma^{\sigma\rho}\gamma^{\mu} + 3\gamma^{\rho\sigma}\gamma^{\mu} \\
&= -\gamma^{\rho\sigma}\gamma^{\mu}
\end{aligned} \tag{8.61}$$

In the second line, we used the fact that $-\frac{i}{2}\gamma^{\mu\nu}$ are generators of the Lorentz algebra. Assembling the various pieces, we find for the commutator

$$[\delta_1, \delta_2]\lambda = -2i(\bar{\epsilon}_1\gamma^{\mu}\epsilon_2)\partial_{\mu}\lambda + \frac{i}{2}(\bar{\epsilon}_1\gamma_{\rho}\epsilon_2)\gamma^{\rho}\gamma^{\mu}\partial_{\mu}\lambda + \frac{i}{4}(\bar{\epsilon}_1\gamma_{\rho\sigma}\epsilon_2)\gamma^{\rho\sigma}\gamma^{\mu}\partial_{\mu}\lambda \ . \tag{8.62}$$

The first term is a translation. The second two vanish by the equations of motion. There is no gauge transformation piece here because λ does not transform under gauge transformations. Thus we have demonstrated that the SUSY algebra closes on-shell.

We did not expect the algebra to close off-shell. Fixing a gauge in 4d, the gauge field has 3 off-shell degrees of freedom while the Majorana fermion has 4. Thus we need one more bosonic degree of freedom to put together an off-shell formalism. The corresponding auxiliary field is often given the name D.

Problem 8.6. *How does the calculation of δS and $[\delta_1, \delta_2]$ above get modified in three dimensions?*

Problem 8.7. *Consider the following modification of the super Maxwell Lagrangian in four dimensions:*

$$\mathcal{L}_{\text{SYM}} = -\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\partial\!\!\!/\lambda - \frac{1}{2}D^2\right) \ .$$

We have added an auxiliary field D that will allow us to close the SUSY algebra off-shell. The SUSY transformation rules for these adjoint fields are

$$\begin{aligned}
\delta A_{\mu} &= i\bar{\epsilon}\gamma_{\mu}\lambda \ , \\
\delta\lambda &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon - i\gamma D\epsilon \ , \\
\delta D &= \bar{\epsilon}\gamma\partial\!\!\!/\lambda \ .
\end{aligned}$$

Verify that the action is supersymmetric and the SUSY algebra closes off-shell, up to gauge transformations.

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A Sources

I set the level and course material largely using these two sets of notes:

- L. F. Alday, "Conformal Field Theory", class notes from a set of lectures delivered at Oxford University,

  courses.maths.ox.ac.uk/node/view_material/5310

- N. Lambert, "Supersymmetry" (class notes for CM439Z/CMMS40 at King's College London) as well as "Supersymmetry and Gauge Theory" (class notes for 7CMMS41),

  nms.kcl.ac.uk/neil.lambert/

Here are further references on conformal field theory:

- P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Springer, 1997.

  The canonical reference for conformal field theory, also called "the yellow book". The early chapters cover CFT in general dimension and are useful for this course. The later chapters, which constitute most of the book, are devoted to CFT in $d = 2$.

- P. Ginsparg, "Applied Conformal Field Theory,"

  arxiv.org/abs/hep-th/9810828

  Another good reference, but again focused mostly on CFT in $d = 2$.

- S. Rychkov, "EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions,"

  arxiv.org/abs/1601.05000

  D. Simmons-Duffin, "TASI Lectures on the Conformal Bootstrap,"

  arix.org/abs/1602.07982

  Covers roughly the same material that is in this course, but targeted toward more advanced graduate students.

- J. Cardy, "Conformal Field Theory and Statistical Mechanics," Les Houches Lecture Notes, 2008

  www-thphys.physics.ox.ac.uk/people/JohnCardy/

In fact a variety of lecture notes are available from the home page of this master of conformal field theory.

Additional reference material for supersymmetry:

- J. Polchinski, *String Theory*, vol. 2, Appendix B.

  All you need to know about supersymmetry in 35 pages. The style is very dense, and you can spend hours working out the equations on each page.

- D. Z. Freedman and A. Van Proeyen, *Supergravity*.

  Supergravity does not start until page 185, and many of the early chapters duplicate material that we will cover in a nicer and more thorough fashion than we have time for.

- S. Weinberg, *Quantum Field Theory*, vol. 3.

  Technical, thorough, and index heavy. Weinberg uses four component fermions. An early chapter includes a proof of Coleman-Mandula. Another reasonable looking source for a proof is a Scholarpedia page,

  http://www.scholarpedia.org/article/Coleman-Mandula_theorem

  apparently written by Mandula himself.

- J. Wess and J. Bagger, *Supersymmetry and Supergravity*.

  The canonical reference. They use two component fermions. The book is easy to read, but one often wishes for more text and fewer equations. As equation heavy as it is, much of the technical detail is left to exercises.

- P. Argyres, *An Introduction to Global Supersymmetry*, 2001.

  A very nice set of lecture notes (essentially a text book) from a course that Phil Argyres taught at Cornell nearly 20 years ago. It was the canonical reference when I was a graduate student. It is available free from his website

  http://homepages.uc.edu/∼argyrepc/cu661-gr-SUSY/index.html

- K. Intriligator and N. Seiberg, "Lectures on Supersymmetric Gauge Theories and Electric-Magnetic Duality," arXiv:hep-th/9509066.

  A canonical reference for the low energy behavior of SQCD.